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TENSOR ANALYSIS

OF

NETWORKS

By

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SCHENECTADY, N. Y., U. S. A.

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INTRODUCTION

(*a*) During the last century it was discovered that the geometry of Euclid, which ruled supreme for 2000 years, is not the only true geometry and that it is possible to invent mathematically non-Euclidean spaces having any number of dimensions whose properties differ from those of our three-dimensional physical space. For the study of these new types of spaces a new mathematical discipline slowly evolved, which was shaped into a tool by Ricci and Levi-Civita. This tool is usually known today by the name "tensor analysis."

Great impetus was given to the application of this new mathematical tool to the study of *physical* problems by the advent of the theory of relativity in which Einstein showed that our physical universe itself is neither three-dimensional nor Euclidean. As tensor analysis became better known among physicists, it was employed with increasing frequency in the study of classical dynamical problems such as hydrodynamics, electromagnetic theory, elasticity, and lately in quantum dynamics. Since many engineering problems by the very nature of the variables used are inherently neither three-dimensional nor Euclidean, there is a need for an engineering tool that utilizes any number of variables in the most systematic manner. The application of tensorial methods to the solution of engineering problems arises from this need.

(*b*) The subject matter of this volume presents a new method of approach to the analysis and synthesis of networks that are most frequently encountered by the electrical engineer in his daily work. The method of approach is so formulated that it might serve as a first step to the systematic analysis and synthesis of rotating electrical machinery on the one hand and of transmission networks on the other. The latter in turn will serve as a second step in the study of radiation and of electronic phenomena from the point of view of the electrical engineer.

Analytical investigators usually build up a different, highly specialized method of reasoning for each particular field of electrical engineering in which they happen to be interested. As a result their work is usually a closed book to engineers specializing in other fields. It is hardly necessary to point out that the language, the terminology, and

the methods of reasoning of engineers specializing in synchronous machines, for instance, are utterly different from those specializing in multiwinding transformers or in vacuum tubes. Any information acquired in the study, say, of induction motors is of little use to the engineer in his attempt to study, say, the motion of an electric charge in a magnetic field.

In order to carry along the unified point of view throughout the range of interest of the engineer from network to field problems, from rectangular to curvilinear axes and from stationary to accelerated systems so that knowledge gained in one field of engineering might be useful also in an apparently different field of engineering, the writer abandons the beaten track followed by other writers both in the *analytical tool* used and in the *method of reasoning* followed. (Of course, the *results* arrived at are in all respects the same as those of other writers when the same assumptions are made in investigating the same problem.)

The *analytical tool* used in the quest for a unified point of view of the large variety of engineering structures is the same that is employed by physicists in their quest for a unified point of view of classical, relativistic, and quantum physics, namely "*tensor and spinor analysis*" (also known as the "absolute differential calculus"). The *method of reasoning* employed for the engineering work also follows as closely as possible that of modern theoretical physics, namely, the geometrical reasoning of a branch of geometry called "*differential geometry.*"

However, for engineering work it was found necessary to augment the concepts of differential geometry with the concepts of another branch of geometry, called "*topology*" or "analysis situs," that hitherto has not generally been employed in the study of physical phenomena. It was also necessary for engineering work to forge the concepts borrowed from these two apparently different fields into a single engineering tool. This fusion has been accomplished with the aid of tensorial methods.

It is emphasized that this book is not written by a mathematician and is not written for mathematicians. This book is written by an engineer for engineers who are interested in learning an *organized* method of attack to analyze and synthetize electrical networks. The method of tensor analysis is still a rapidly growing structure, and consequently there still exists among its exponents disagreement on notation and nomenclature so that a student has difficulty in deciding for himself what to take and what to leave for his own special purpose. The definitions and physical interpretations of all concepts are given here in a language that is thought to be best suited to the requirements

of the *engineer* who confronts such method of attack for the first time. This volume makes no pretense of possessing absolute mathematical rigor. Anyone interested in precise mathematical or geometrical definitions is referred to books given in the bibliography and written by mathematicians or mathematical physicists specializing in these fields. The aim of this volume is to get definite answers to definite engineering problems, the mathematical concepts serving only as an aid.

(*c*) *The keynote of this volume is "organization."* It undertakes the organization of the large variety of networks according to their basic properties and expected functions. By organization they become obedient to the command of the engineer as an army of well-disciplined soldiers responds to the control of the commanding general. *This organization is realized by the introduction of "groups of transformations" that control the unfolding of the analysis as the several ranks of officers direct the movements of the privates.*

Several aspects of the method of organization to be introduced here are not new to the electrical engineer. The *shorthand* procedure of denoting a collection of numbers by one symbol is used by the engineer whenever he solves a set of linear equations with the aid of *determinants* or manipulates them with the aid of *matrices*. Denoting the *complex number* $r + jx$ by one symbol Z is an analogous organization. Such shorthand procedures have been used in electrical engineering intermittently since the days of Kirchhoff. Lately Strecker and Feldtkeller and their followers have used systematically matrices with two rows and columns in the analysis of four-terminal networks; and Cauer and his followers have used matrices in their synthesis.

A further step in organization consists of denoting by a single symbol not a collection of numbers, but a *physical entity* actually existing in nature. *Vector analysis*, used by the electrical engineer since Maxwell, is such a type of organization. Since a physical entity may be measured from an infinite number of reference frames, each measurement giving a collection of numbers, a *single symbol now represents an infinite number of collections instead of one.* Vector analysis represents, however, a very limited type of organization, since it represents only physical entities existing in a *three-dimensional Euclidean space.* The concept of a "group of transformation" is also implied in vector analysis, relating the components of each physical entity along the various reference frames of a three-dimensional Euclidean space.

A still more advanced type of organization that is useful in physical problems requiring generalized coordinates employs new types of spaces, having more than three dimensions and having more complicated structures than a Euclidean space. These new types of spaces

are populated by new types of *physical (or geometrical) entities*, each denoted by a separate symbol. These spaces and the entities existing in them are created with the aid of "groups of transformations" so that in general there are as many basic types of spaces as there are "groups of transformations." The structure of the basic spaces depends on the type of entities that exist in them.

Tensor analysis is the systematic study of these generalized spaces and of the entities that may exist in them. From this point of view *tensor analysis may be considered as an extension and generalization of vector analysis from three- to n-dimensional spaces and from Euclidean to non-Euclidean spaces.* Of course, it is possible to disregard the geometrical picture entirely and to consider tensor analysis as the study of advanced types of *mathematical* entities.

These new types of spaces differ radically from the conventional Euclidean space; hence in their study the usual intuitive concepts of space must be discarded. The primitive definitions of vector analysis, as, for instance, "a vector is a quantity having magnitude and direction," have to be abandoned, as must also all other preconceived geometrical notions and definitions such as "magnitude," "direction," "parallelism." In studying tensor analysis the definition and manipulation of geometrical or physical entities have to be approached anew from the very foundations.

The organization does not stop at this point. The n-dimensional spaces can be generalized to infinite-dimensional spaces. Also instead of using only four- five-, that is integer-dimensional spaces, it is possible to use $\frac{4}{3}$- or 4.375- or π-dimensional spaces having all types of complicated structures. These spaces are used in the study of the more basic electrodynamical phenomena.

The Analytical Tool. (*a*) As befits a really powerful tool, various people see various advantages in the use of tensors, depending on their individual outlook and on their fields of endeavor. In applying tensors to the analysis and synthesis of the large variety of interrelated problems that confront the engineer, the following reflections may throw light on certain aspects of the tool that fit it to engineering application.

The use of tensor analysis in the solution of engineering problems may be compared to the use of a steel frame in erecting a skyscraper. Now, it is possible, at least theoretically, to erect a skyscraper a hundred stories high by simply placing brick upon brick on top of one another. History does record one such construction undertaken in the city of Babel, but it clouds in mystery the success of that pioneering enterprise.

Whatever advantages the use of steel frames offers for the construction

of buildings, analogous advantages are offered by the use of tensor and spinor) analysis in the solution of engineering problems. The foundations occupy less space, the erection of the building is speeded up, and it can stand more violent shocks from the vicissitudes of the elements. The engineer dares to design and build new types of structures—for new as well as for old uses—that he would not attempt had he no steel frames at his disposal. When the steel frame has been erected, it is possible to put in the bricks only on the sixtieth story and to furnish that particular floor alone, leaving the lower fifty-nine stories uncompleted, a feat that would be impossible without the steel frame. Similar unfinished steps may be left in the analysis of engineering problems where only the needed part may be investigated in detail. It is not necessary to carry in each particular problem the group of equations page after page, keeping their physical analysis in mind at the same time; all that may be left in a crude state in the form of a few symbols that act as the needed framework supporting the detailed parts. At any time it is possible to add additional floors to the already finished building or subtract any part and change it according to the new needs without disturbing the remaining structure.

In addition to allowing the engineer to build *skyscrapers*, the use of a steel frame allows the engineer to fabricate buildings by *mass production*. The same steel frame may be used for a great variety of buildings by arranging the brickwork and the partitions according to the taste and needs of the various tenants. Similarly it is found that it is possible to set up in the language of tensor analysis equations analogous to a steel frame that represent the performance or characteristics of a large variety of networks or rotating machines or transmission systems. Once these tensor equations have been established, it is possible to find the equations of performance or characteristics of any *one particular* network or machine or transmission system by a *routine* substitution of particular constants.

(*b*) This versatility of tensors enables the engineer in the study of, say, a large variety of rotating machines to select one whose structure is quite simple and to study the properties and equations of this particular simple machine only. If the engineer, with the aid of tensor concepts, learns the method of analysis and the physical phenomena taking place in this particular machine, *he learns at the same time the physics, the method of analysis and solution of a large variety of machines, without learning a new trick for each particular machine* as is necessary with other methods of attack given in all textbooks on machinery.

These two particular characteristics of the tensorial reasoning, namely: (1) the ability to introduce analytical *skyscrapers* and (2) the

ability to introduce *mass production* into the analysis and synthesis of engineering problems, are emphasized in this volume. The first characteristic of the method of tensors enables the engineer to attack and solve problems that he otherwise would not be inclined to attack because of the mechanical difficulties or could not attack because of the difficulties in visualizing the physics of the problem. The second characteristic enables the engineer to utilize the reasonings and results of one problem in the solution of many other problems, by storing away whole or partial results of one investigation in the form of tensors and expanding or combining them again in various manners in several new investigations.

This temporary storage and reutilization of the results of previous investigations is analogous to the storage of standardized parts, like the frames and laminations and shafts and bearings of rotating machines, in the stockroom, and their immediate recombination into complete machines with various specifications as the orders come in, without building each part of the machine anew when ordered. With the method of tensor analysis at his disposal the engineer can combine analogously his tensors that he built up previously and stored away into the new tensors needed, without repeating the whole analysis each time a new problem comes up.

(*c*) Just as architects using steel frames, similarly engineers using tensors, must keep in mind that once the steel frame has been erected, it is still necessary to put in the brickwork and to furnish the finished building and to find tenants. There are a certain number of windows and doors, partitions and stairs that have to be installed, whether or not the steel frame is used. Of course, mass-production methods can also be used in supplying these additional fixtures. Similarly in any engineering problem there are an absolute minimum number of additions, multiplications, divisions, finding of roots, etc., that cannot be avoided by any organized method of attack. Of course the steel frame facilitates the more systematic use of calculating machines and it enables the engineer to delegate a large part of his work to computers.

Also it must be kept in mind that it would rarely be good engineering to use a steel frame in constructing a one-car garage. No general rule can be given to determine the lower limit for the use of a steel frame, or for the use of tensors.

(*d*) A very important advantage in erecting a steel frame first in the study of physical phenomena is that *the same steel frame—the same tensor equation—is valid for several different types of physical phenomena.* Most of the tensor equations of hydrodynamics, or electrodynamics,

or optics, or elasticity, all have the same form; they differ from one another only by the brickwork and the furnishings. The tensor equation of any accelerated electrical machine is identical with the tensor equation of an accelerated electric charge viewed from any curvilinear reference frame, and the passage from any rotating machine to any other machine or from one frame to any other frame involves only a *routine* transformation.

The number of the type of girders used in the steel frame of nature—the number of tensors—is quite limited. The discovery of a new type of tensor or spinor existing in a physical phenomenon is analogous to the discovery of a new building-block in the structure of the atom.

This volume is an introduction to the theory and use of analytical steel frames in engineering. It deals with the simplest type of framework using only straight girders that have to be erected in organizing the *sets of linear, algebraic equations* that occur in the study of *asymmetrical, active networks having lumped constants.* As bricks it uses only additions, multiplications, and divisions. Differentiations (with their curved steel girders) are introduced only occasionally to establish points of contact with the differential equations to be organized in another volume, but those sections may be left out without disturbing the development.

The Method of Reasoning. (*a*) To those who are interested in how the method of reasoning used in two such divergent branches of abstract geometries as "topology" and "differential geometry" can be used in the analysis of practical engineering problems, the following remarks may be of some value.

Roughly speaking, "differential geometry" studies certain special properties of curves and other configurations drawn in flat or curved two-, three-, or more dimensional spaces. The main point of interest is the study of those properties of curves that are independent of the reference frames that happened to be assumed in the space. Tensor analysis serves as a powerful analytical tool in such studies where the reference frames are varied.

"Topology" deals with more general properties of curves drawn in a set of *interconnected* n-dimensional spaces. Such an interconnected structure is, for instance, the surface of a cube in which six two-dimensional planes, twelve one-dimensional lines, and eight zero-dimensional points are interconnected into one system. In the general case each of the spaces may be quite distorted. Here it is also possible to assume *on the same structure* various reference frames for the study of curves drawn, say upon the cube.

(*b*) Now *the types of problems allowed by the freedom of selecting the reference frames arbitrarily* **on the same complicated structure of spaces** (*that is covered in most textbooks and publications*) *are not sufficiently general to be of real value to the engineer.* This is perhaps the main reason why the method of these sciences has not been applied as yet to engineering. This lack of generality may be seen from the following considerations.

The engineer deals with a collection of zero-, one-, two-, and three-dimensional structures interconnected in innumerable ways. For instance, a transmission system contains two-, one-, and zero-dimensional structures in the form of rotating machines, transmission lines, and the junctions between the lines and machines. Similarly a bridge contains an assembly of piers, plates, girders, and their junctions; or a reciprocating engine contains among other things pistons, crankshafts, rods, and bearings. The superposition of electromagnetic phenomena upon the whole transmission system (or the superposition of stresses upon the bridge or motion upon the engine) is analogous to superimposing curves upon the interconnected spaces of various dimensions. Hence *the properties and equations of the superimposed curves are identical to those of the superimposed electrical or mechanical phenomena*, and the results of one investigation can be applied with a simple reinterpretation of symbols to the others.

However, the engineer changes the reference frame on some particular structure only occasionally. What the engineer does most often is to combine the component one-, two-, and three-dimensional structures in all imaginable manners to build a large variety of new structures on which he again superimposes the same type of forces. And *the engineer is interested chiefly in finding out how this large variety of new structures responds to applied forces in order to select the most suitable structure to accomplish some desired ends and not how one structure appears from various points of view.*

(*c*) This more general problem of the engineer appearing in the central foreground of this volume may be formulated from a geometrical point of view as follows:

Let there be a collection of zero-, one,- two-, and three-dimensional spaces. These spaces may be flat, curved, distorted, and so on. *Let this collection of spaces be interconnected and interlinked with one another in a large variety of manners analogously to the large variety of engineering structures, and let upon each of these structures certain types of configurations be drawn* (say the shortest line between any two points on the structure) analogously to the physical phenomenon superimposed upon the engineering structure.

Of course, each of these configurations can be expressed in the form of an equation, when some arbitrary reference frames are assumed on each structure. *One of the problems to be investigated is to find an easy way to establish the equations of the configurations drawn* **on all the different types of structures** *provided the equation of the configuration drawn on one particular structure is known.* Geometrically speaking, the problem is to set up a *correspondence* between the configurations drawn upon the different types of structures (or to find the "group" of the correspondence). In the language of the engineer the problem is to establish *in a routine manner* the equation of performance of all the different types of structures, if that of one of the structures is known. *The structures may contain different number and types of spaces.*

The equations of performance are changed in a routine manner with the aid of a group of transformations (to be called the "connection tensor") representing the manner of interconnection of the various spaces. By using this method the engineer needs to establish from purely physical considerations the equations of only *one* system (the so-called "primitive system"), while the equations of all the other systems follow automatically, without starting their study all over again.

In conjunction with this newly developed process of changing over from a reference frame *on one structure* to some arbitrary reference frame *on another structure*, also the customary process of changing over from one reference frame to another *on the same structure* occurs quite often as a special case. Of course on each structure it is possible to assume numerous actual or hypothetical reference frames.

(*d*) This volume undertakes the study of structures in which only zero- and one-dimensional spaces are interconnected. Owing to the simplicity of the component spaces not much differential geometry enters into the study and most of the concepts are borrowed from topology.

It is interesting to note that the foundations of topology were laid originally by Kirchhoff in his investigations of the flow of electricity through networks. However, the science of electricity and the science of topology soon drifted apart until, after a century of slow separation, hardly any traces of one can be found in the other. Of equal interest is the fact that both *modern* tensor analysis and *modern* differential geometry also owe much of their development to the study of the motion of an electrified particle in an electromagnetic field, requiring a four- or a five-dimensional space with a complicated structure for its geometrical analogy. Small wonder that the concepts and methods of tensor analysis, topology, and differential geometry, and the science

of electricity by their very nature go hand in hand, each influencing the other.

The Point of View. Although most of the applications of tensor analysis in theoretical physics pertain to *field* problems, this volume undertakes a new type of application to *circuit* problems. (The difference between field and circuit problems is more or less analogous to the difference between problems of differential geometry and those of topology.)

One other point of view assumed here must be especially emphasized. The main purpose of tensor analysis and its application to other sciences is usually to establish *all-inclusive equations, broad conclusions* valid for as many varieties of cases as possible. However, for engineering work that point of view is insufficient. *In engineering work it is also necessary to get specific answers to specific problems* in the shortest possible time. Hence in this volume every effort has been made to develop this phase of the tensorial method of attack, which of course is implied in tensor analysis but has been neglected by other writers owing to their lack of interest in specific problems. Among the many textbooks on tensor analysis quoted in the bibliography, there are very few which work out even one specific problem with the aid of the equations developed in them and for that reason these textbooks contain only occasionally a set of functions arranged in a row or a square or a cube, in which the present book abounds.

In order to facilitate the solution of *specific* problems several new concepts and points of view have been introduced, such, for example, as "primitive" systems, reduction formulas, "compound" networks, rules for the quick manipulation of sets of numbers, and generalization postulates.

The examples covered in this volume are quite elementary and do not represent all the types of network problems that can be attacked by tensorial methods. This volume hardly scratches the surface of the vast possibilities that can be brought into existence by proper "organization" introduced into the multitude of methods of attack on engineering problems. Filter networks, for instance, which are not even touched upon in this volume, though they have been given excellent treatment by organized methods, still offer a fertile field for a new method of attack with the aid of tensorial concepts.

There is nothing final about the notation, terminology, and definitions arrived at in this volume. They should serve only as starting points for new departures or as stepping stones for deeper penetration. It is strongly felt by the writer that beyond the footpaths cleared in

this volume there lies a vast and uncharted territory which awaits its explorers.

Suggestions for Reading. The greater part of this volume may be read by anyone who has a knowledge of elementary mathematics and some acquaintance with Ohm's and Kirchhoff's laws. The volume has been written with several types of readers in mind. Engineers who are interested exclusively in getting a quick and organized answer to their problems may leave out several of the chapters and many scattered sections that expound geometrical or other not strictly engineering viewpoints, such as Chapters VII, VIII, XVII, and XVIII. Many of the chapters deal with special engineering subjects such as windings, tube-circuits, etc., that may be left out by those who are not interested in that particular topic. Such chapters are XII, XV, XX, XXI, XXIII. Those who are not interested in engineering problems, but only in the manner of application of tensorial concepts, may read Chapters I–XI, XIII, XIV, XVI–XVIII, and the first part of XIX and XXI. Many of the earlier sections that may be left out at the first reading (since they are needed only in the more advanced work) are denoted by an asterisk.

This volume covers in detail the subject matter of the first two parts of a series of articles running currently in the *General Electric Review*. The serial, of which so far seventeen parts have appeared, began in the April, 1935, issue and is entitled "The Application of Tensors to the Analysis of Rotating Electrical Machinery." Some of the material from the other parts that are of introductory nature are also included and enlarged here. It is hoped that eventually other parts of the *General Electric Review* serial may be published in detail as time and opportunity permit.

Acknowledgments. The writer wishes to acknowledge with thanks the indefatigable support of Mr. P. L. Alger, who early recognized the possible utility of tensors as a new tool in engineering, encouraged its development and application in daily engineering work, and facilitated the publication of some of the results.

This opportunity is taken also to express appreciation for the cooperation of all concerned in the publication of the serial on tensors appearing in the *General Electric Review*. The writer is especially indebted to its Executive Editor, Mr. E. C. Sanders, for his vision and courage in undertaking and pursuing the publication of the serial, and to its Manager, Mr. C. H. Lang, for his continued support. Thanks are due to Mr. J. E. Glick, Assistant Editor, who so conscientiously and painstakingly edited the manuscript, and to Mr. W. B.

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GABRIEL KRON

SCHENECTADY, N. Y.
September, 1938

TENSOR ANALYSIS OF NETWORKS

CHAPTER I

THE ALGEBRA OF N-WAY MATRICES

I. A NEED FOR "ORGANIZATION" IN ATTACKING ENGINEERING PROBLEMS

(*a*) In engineering work a symbol like A may represent different types of quantities. Among others the following may be mentioned:

1. It may represent a *constant* quantity like $A = 5$, or $A = 3.14159$, or $A = 3 + 4j$.

2. It may stand for a *variable* quantity as $A = x$, or for a *function* of the variables, as $A = \cos x$, or $A = x^2 + 3x + 4$.

3. It may represent a *linear operator* as $A = d/dt = p$, or $A = p^2 + p$, or $A = 1 =$ Heaviside unit function, etc.

In all these cases the single symbol A stands for a single quantity, and consequently *in all engineering works as many symbols are used in general as there are quantities in the problem under consideration.* Because of the large variety of symbols needed to represent even simple types of engineering structures and the mental discipline necessary to keep the role of each symbol in memory throughout the whole analysis, a need arises for an analytical tool that dispenses with the large variety of symbols and keeps only an *absolute minimum* number throughout the analysis of the engineering problem.

(*b*) The problems of the engineer are fundamentally the same as those of the physicist; both express physical phenomena in mathematical symbols. Generally speaking, the physicist endeavors to reduce natural phenomena to their simplest possible form, usually expressible by a few, mostly one equation, introducing only as many mathematical symbols as there are corresponding physical concepts. That is, the physicist sets up an equation for, say, the conduction of electricity between *two* electrodes, or for an electromagnetic wave traveling along a *single* conductor, or for the electromotive force generated in a *single* conductor moving in a magnetic field, or for the passage of light through a lens, etc. Once the equation for the phenomenon is set up, the physicist's role has ended.

This is where the engineer's role begins. The engineer takes a two-electrode tube and adds several additional electrodes; and for good

measure he connects them to different types of networks; or he builds transmission networks covering whole continents; or he takes *several* moving conductors and constructs a large variety of complex rotating electrical machines; or he combines a series of lenses into an optical instrument, and so on.

That is, *the engineer generalizes the one-, two-, or three-dimensional problem of the physicist to k dimensions.* And that is where his difficulty originates. It is recognized that the engineer introduces no additional physical phenomena in his systems, but he introduces additional *interrelations* between the various components in such a way that the complications increase with an increase in the number of components. Most engineering problems require not discovery of new laws, but ingenuity in organizing interrelated phenomena, whose laws are already known for each component part of the system considered separately. For instance, the law of motion of a conductor through a magnetic field is known, and combining *several* conductors into a rotating electrical machine requires only an *organized method of analysis* and not discovery of new laws. The law valid for the motion of the single conductor must necessarily be valid also for the motion of any complicated network consisting of any number of conductors. The problem is how to express this fact in practical calculations.

In order to organize the large variety of engineering problems into the absolute minimum number of standardized types in which the physicist has expressed them, it is necessary to introduce new points of view, new symbols, new mental and physical concepts. *What is needed for the unified point of view is not additional mathematics, but " organization " of the already employed mathematics.*

II. SETS

In order to manipulate the large number of quantities that occur in engineering problems *the first logical step in organization consists of allowing the single symbol A to represent not one quantity (number, function, linear operator, etc.; as shown above, but a whole set of quantities that play analogous physical roles.* For instance, A may represent all the currents, i^a, i^b, i^c, $i^d \cdots$ flowing in all the meshes of a network instead of representing just one of the currents.

Accordingly the single symbol A may represent the following " sets " of quantities:

1. A set of k quantities, a, b, c, arranged in a *row*, called "one dimensional set " or " one-way set " as

$$A = \boxed{\begin{array}{c|c|c|c|c|c} a & b & c & d & e & f \end{array}} \qquad 1.1$$

2. A set of k^2 quantities arranged in a *square*, called " two-dimensional set " or " two-way set " as

$$A = \begin{array}{|c|c|c|c|} \hline a & b & c & d \\ \hline e & f & g & h \\ \hline i & j & k & l \\ \hline m & n & p & q \\ \hline \end{array} \tag{1.2}$$

3. A set of k^3 quantities arranged in a *cube*, called " three-dimensional set " or " three-way set " as shown in Fig. 1.1.

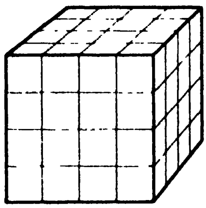


FIG. 1.1.—A Three-way Set

In general these sets are called " *n*-way sets."

III. *n*-WAY MATRICES

(*a*) In order that the final answer should be the same, irrespective of whether the original large number of quantities are used in the analysis or the symbols representing them, ***it is necessary to manipulate these sets of various dimensions, that is, to add them, multiply, differentiate, integrate them, etc.***

It is possible to devise all types of rules for the manipulation of sets, depending on the problem at hand. In physical and geometrical problems a particular type of manipulation has been found quite useful that will be given here. *When a set of quantities obeys the particular rules of manipulation to be shown presently*, the set is called an " ***n-way matrix*** " or an " ***n-dimensional matrix*** " or shortly an " ***n-matrix.***"

(*b*) In order to add, multiply, etc., *n*-way matrices of various dimensions with ease, ***in these pages a distinguishing mark or index will be attached to each row and column or layer of an n-matrix*** as shown in the following:

1. Examples of one-way matrices, or 1-matrices, are

$$\begin{array}{cccccccccc} & a & b & c & d & e & f & g & h & i \\ A = & \boxed{7} & \boxed{1} & \boxed{-9} & \boxed{0} & \boxed{0} & \boxed{2} & \boxed{7} & \boxed{8} & \boxed{-3} \end{array}$$

$$B = \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left[\begin{array}{c|c|c|c|c|c} 2+3j & 0 & 4-7j & 8j & 3 & +2j \end{array}\right] \end{array}$$

$$C = \begin{array}{c} \begin{array}{ccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m \end{array} \\ \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_{11} & A_{12} & A_{13} \end{array}\right] \end{array} \qquad 1.3$$

$$D = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \left[\begin{array}{c|c} xy + 3x^2 \cos y & 1 - y(2x - 3\sqrt{y}) \end{array}\right] \end{array}$$

$$E = \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left[\begin{array}{c|c|c|c|c|c} 3+4j & 0 & \sin\theta & -\tan\theta & 5 & \cos\theta \end{array}\right] \end{array}$$

A 1-matrix is also called a " linear matrix."

The components may be real or complex numbers or any other types of quantities. The number k of the components may be anything from two to infinity.

When the number of components is infinite, special methods have to be used, and consequently they will not be touched upon in this treatise.

2. Example of two-way matrices or 2-matrices are:

$$A = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 2 & 3 & 7 & -8 & -5 \\ b & 0 & 1 & 5 & 4 & 1 \\ c & 5 & -7 & 8 & -9 & 0 \\ d & 3 & 6 & 4 & 2 & 7 \\ e & 4 & 8 & 2 & 9 & 5 \end{array} \qquad B = \begin{array}{c|ccc} & a & b & c \\ \hline a & 3+7j & 0 & -2j \\ b & 9 & 5-j & 8 \\ c & -15 & 0 & j \end{array}$$

$$C = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 0 \\ b & 0 & \sin\theta & \cos\theta & 0 \\ c & 0 & -\cos\theta & \sin\theta & 0 \\ d & 0 & 0 & 0 & 1 \end{array} \qquad D = \begin{array}{c|ccc} & a & b & c \\ \hline a & x^2 & y & 3xa \\ b & zy & 0 & z \\ c & -5 & x^2y^2 & x^2y \\ d & xyz & x & 0 \end{array} \qquad 1.4$$

A two-way matrix is also called shortly a " matrix."

2-Matrices in which the components are arranged in a rectangle instead of a square also occur.

Two important matrices are:

1. The "*unit-matrix*" containing unity in its main diagonal components and zero elsewhere, as

$$I = \begin{array}{c|ccccc|} & a & b & c & d & e \\ \hline a & 1 & 0 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 1 & 0 \\ e & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array} \qquad 1.5$$

It is also called the "Kronecker's delta" and is denoted by δ.

2. The "*null-matrix*" containing zero in all components as

$$0 = \begin{array}{c|ccccc|} & a & b & c & d & e \\ \hline a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad 1.6$$

3. Example of a three-way matrix or 3-matrix with $4^3 = 64$ components is the cube of Fig. 1.2*a*.

On paper the components of a 3-matrix may be represented by splitting it into k 2-matrices, each containing k^2 quantities since $k^3 = k \times k^2$ (Fig. 1.2*b*). There are three possible ways of splitting a 3-matrix into 2-matrices.

The representation of *n*-matrices of more than three dimensions is shown later. A single quantity like 5 may be called a "zero-dimensional matrix" or "zero-way matrix" or "0-matrix."

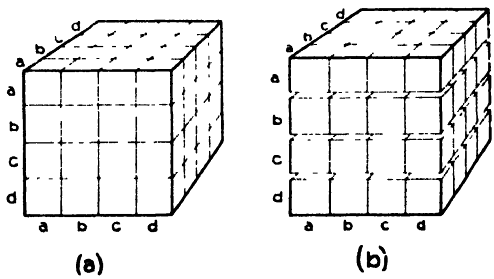


FIG. 1.2.—Splitting of a 3-Matrix into a Set of k 2-Matrices

It is emphasized that the indices or marks shown outside the heavy lines do not belong to the n-matrices. Strictly speaking, they should

not have been introduced at this point. They are introduced here only in anticipation of further developments in Chapter III.

IV. A PHYSICAL EXAMPLE FOR THE OCCURRENCE OF n-MATRICES

(*a*) For instance, let four stationary coils a, b, c, and d be given with mutual inductance between them, as shown in Fig. 1.3. The self-impedance of winding a is Z_{aa}, the mutual inductance between winding b and c is Z_{bc}, etc.

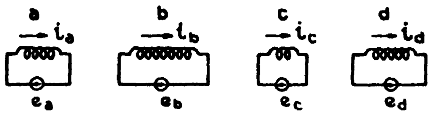


FIG. 1.3.—A Network with Four Meshes

The instantaneous voltages impressed upon each of them can be arranged in a row forming a 1-matrix

$$e = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \boxed{\begin{array}{c|c|c|c} e_a & e_b & e_c & e_d \end{array}} \end{array} \qquad 1.7$$

The instantaneous currents flowing in each of them can be also arranged in a row forming another 1-matrix

$$i = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \boxed{\begin{array}{c|c|c|c} i_a & i_b & i_c & i_d \end{array}} \end{array} \qquad 1.8$$

The self- and mutual impedance between the coils can be arranged in a square forming a 2-matrix

$$z = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline a & Z_{aa} & Z_{ab} & Z_{ac} & Z_{ad} \\ \hline b & Z_{ba} & Z_{bb} & Z_{bc} & Z_{bd} \\ \hline c & Z_{ca} & Z_{cb} & Z_{cc} & Z_{cd} \\ \hline d & Z_{da} & Z_{db} & Z_{dc} & Z_{dd} \end{array} \qquad 1.9$$

The diagonal components are the self-impedances; the others the mutual inductances.

(*b*) Instead of saying that there are four voltages e_a, e_b, e_c and e_d impressed on the network containing sixteen impedances and four currents, from now on it will be said that one voltage e is impressed on the network, whose impedance is z and in which one current i flows.

That is, with any stationary network containing any number of impedances at least three concepts may be associated, which define their performance:

1. The voltage matrix e representing all the impressed voltages.

2. The current matrix i representing all the currents flowing in the various coils.

3. The impedance matrix z representing all the self- and mutual impedances of the various coils.

With the simplest d-c. circuit containing one resistance also three concepts are associated, e, i, and z.

If the coils move or another point of view is introduced, these three concepts are insufficient to represent their performance and additional n-matrices have to be defined.

V. NOTATION

(*a*) Two types of notation will be used throughout this book to represent n-matrices:

"*Direct notation,*" in which each n-matrix is represented by one symbol, called the *base letter,* irrespective of its number of dimensions. It is customary to print these symbols in bold-face type as **e**, **i**, **z**, to differentiate them from ordinary letters e, i, z representing one quantity. In writing, a bar may be placed over them as $\bar{e}$, $\bar{i}$, $\bar{z}$. Of course, if it is certain that no misunderstanding may arise, the bold-face type or the bars may be dispensed with.

"*Index notation,*" in which each n-matrix again is represented by one symbol A, the base letter, but in addition indices are also attached to it, representing the dimensions in which its components are arranged. In particular:

1. A 1-matrix has one index, as A_α.
2. A 2-matrix has two indices, as $A_{\alpha\beta}$.
3. A 3-matrix has three indices, as $A_{\alpha\beta\gamma}$.
4. A 0-matrix has no index, as A.

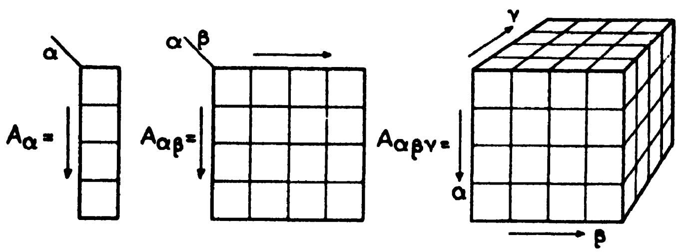


FIG. 1.4.—The Arrangement of Indices

In general, the base letter A, representing the n-matrix, has as many indices as the number of dimensions in which its components are arranged.

(*b*) It will be agreed that, in representing an n-matrix by several indices as $A_{\alpha\beta\gamma}$, in general (Fig. 1.4):

1. The first index represents the rows.
2. The second index represents the columns.
3. The third index represents the layers, parallel to the plane of the paper.

However, as long as the indices are correctly attached to the arrows, the order of representation, that is, whether the first index is a row or a column, is not important.

VI. "FIXED" AND "VARIABLE" INDICES

(*a*) Just as every coil in Fig. 1.3 has a definite name a, b, c, and d attached to it in order to handle the coils *individually*, similarly every row, every column, and every layer in an n-matrix has a distinctive name attached to it, as has been shown. These individual names are called "*fixed*" indices and are written alongside the row, column, and layer.

In order to handle all the coils *collectively*, in addition to the "fixed" indices $a, b, c, d \cdots$ another set of indices is also introduced in the index notation that *represent all the fixed indices.* Such collective names are called "*variable*" indices and will be denoted, say, by Greek letters $\alpha, \beta, \gamma \cdots$. *That is, the variable index α assumes all fixed values $a, b, c, d \cdots$ in succession* as do similarly β or γ. For instance, A_α represents *all* the components of the 1-matrix A, while A_b represents only one component, namely, the second one in the row.

For a 2-matrix the two variable indices are placed in the upper left-hand corner in their proper place alongside a slant line, as shown in Fig. 1.4. For a 3-matrix three arrows are drawn along the three edges at the cube, then alongside each arrow a variable index is placed as shown in Fig. 1.4.

(*b*) When all the indices are variable, as $A_{\alpha\beta}$, then all the components of the n-matrix are represented by it. When, however, one or more of the indices are fixed indices, as $A_{c\beta}$ or $A_{\alpha d\gamma}$, then a particular row or column or layer has been selected out of the n-matrix, as illustrated in Fig. 1.5.

For instance, $A_{\alpha d\gamma}$ represents a 2-matrix cut out of a 3-matrix. The existence of three indices indicates that originally A is a 3-matrix. The two variable indices α and γ show that a 2-matrix is cut out and that this 2-matrix is perpendicular to the plane of the paper (the variable indices being the first and the third index). The fixed index d shows that the 2-matrix is the last of the four 2-matrices.

Individual components are represented by giving their fixed indices

as $A_b = 5$ or $A_{bd} = 7$, etc., showing that 7 belongs to row b and column d.

When direct notation is used, then the variable indices are not introduced. The fixed indices a, b, c, d, however, still do exist and are

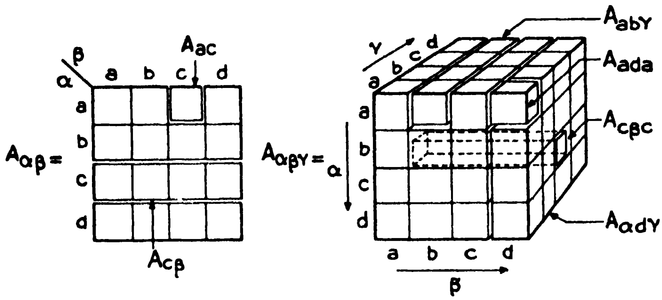


FIG. 1.5.—Representation of Parts of an n-Matrix

printed in bold-face type as **a**, **b**, **c**, and **d**, alongside the components. Hence a 1-matrix and a 2-matrix are written respectively as

$$\mathbf{e} = \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad 1.10$$

$$\mathbf{Z} = \begin{array}{c|c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \hline \mathbf{a} & 4 & 1 & 3 & 2 \\ \hline \mathbf{b} & 5 & 7 & 6 & 8 \\ \hline \mathbf{c} & 9 & 8 & 5 & 6 \\ \hline \mathbf{d} & 5 & 4 & 3 & 2 \\ \hline \end{array} \qquad 1.11$$

showing the fixed indices in bold-face type but the variable indices are left out. Partial n-matrices as shown in Fig. 1.5 may be represented in direct notation only with special symbolism in each case.

Hence, generally speaking, the difference between direct and index notations consists of omitting the variable indices in direct notation. In place of the variable indices, bold-face type is used to differentiate them from ordinary quantities.

VII. REPRESENTATION OF n-MATRICES OF HIGHER DIMENSIONS*

(a) With the aid of fixed and variable indices a 4-matrix $A_{\alpha\beta\gamma\delta}$, representing k^4 quantities, can be represented graphically as k cubes

* This section may be left out at first reading.

(since $k^4 = k \times k^3$), if the last variable index is replaced in succession by the fixed indices a, b, c, d, as shown in Fig. 1.6.

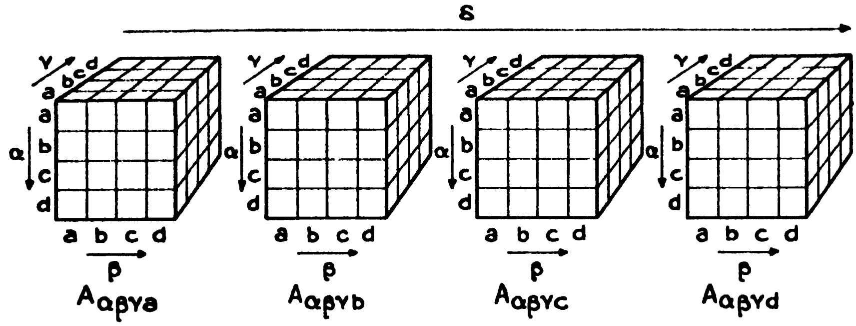


FIG. 1.6.—Representation of a 4-Matrix $A_{\alpha\beta\gamma\delta}$ as a Set of k 3-Matrices

Since each of the cubes can be represented on paper as k 2-matrices, $A_{\alpha\beta\gamma\delta}$ can be represented on paper as k^2 2-matrices ($k^4 = k^2 \times k^2$) as shown in Fig. 1.7.

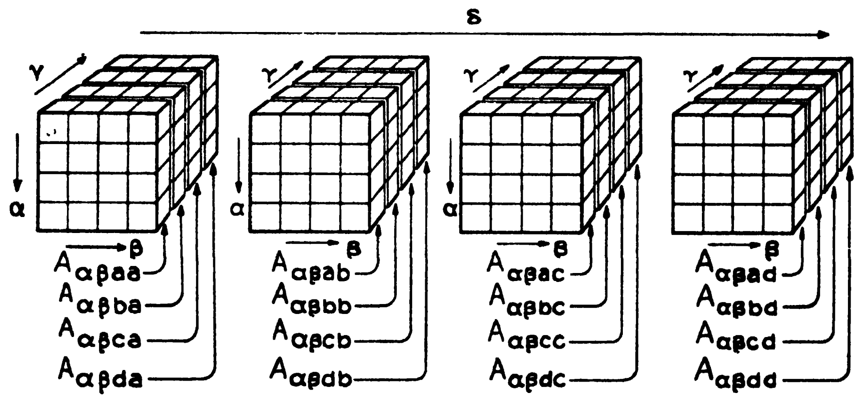


FIG. 1.7.—Representation of a 4-Matrix $A_{\alpha\beta\gamma\delta}$ as a Set of k^2 2-Matrices

Similarly $A_{\alpha\beta\gamma\delta\epsilon}$, a 5-matrix, may be represented graphically as k^2 cubes ($k^5 = k^2 \times k^3$) as shown in Fig. 1.8.

On paper it is represented as k^3 2-matrices by dividing each cube into k 2-matrices.

In this treatise all n-matrices with n greater than two will be represented on paper as being composed of 2-matrices, that is, $A_{\alpha\beta\gamma}$ will be represented by k 2-matrices, $A_{\alpha\beta\gamma\delta}$ by k^2 2-matrices, etc.

(*b*) Of course an n-matrix like $A_{\alpha\beta\gamma}$ may be represented on paper not only as k 2-matrices, but also as k^2 1-matrices, or as k^3 0-matrices,

by simply writing down in succession the various components until all components are covered.

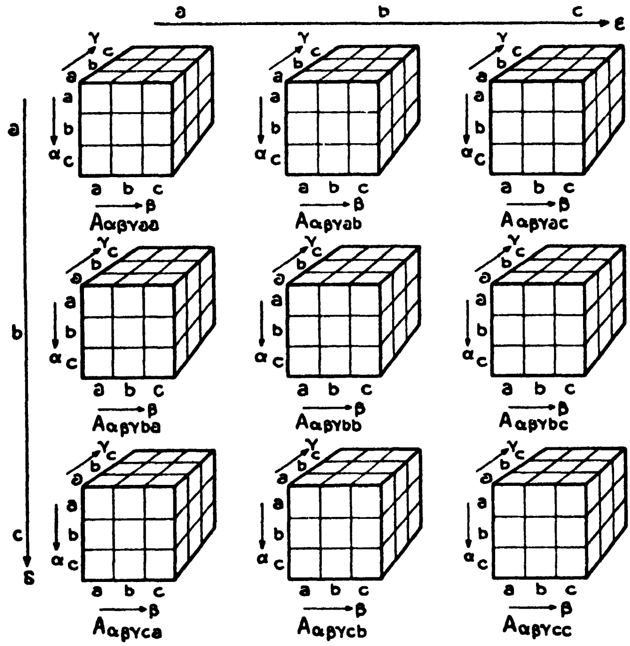


FIG. 1.8.—Representation of a 5-Matrix $A_{\alpha\beta\gamma\delta\epsilon}$ as a Set of k^2 3-Matrices

For instance the 3-matrix $A_{\alpha\beta\gamma}$ can be represented as *two* 2-matrices $A_{\alpha\beta a}$ and $A_{\alpha\beta b}$, as shown in Fig. 1.9,

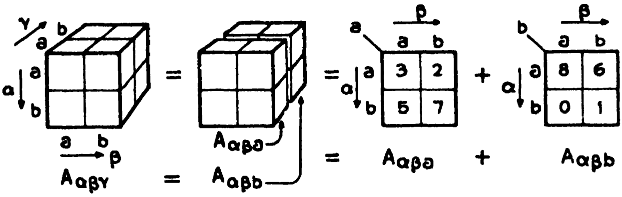


FIG. 1.9.—Representation of a 3-Matrix $A_{\alpha\beta\gamma}$ as a Set of k 2-Matrices

or it can be represented as k^2 = eight 0-matrices as

$$\begin{array}{c|c|c|c} A_{aaa} = 3 & A_{aba} = 2 & A_{aab} = 8 & A_{abb} = 6 \\ A_{baa} = 5 & A_{bba} = 7 & A_{bab} = 0 & A_{bbb} = 1 \end{array}$$

The 1-matrix A_α

$$A_\alpha = \begin{array}{c|c|c|c|} \alpha & a & b & c & d \\ \hline & \cos\theta & \sin\theta & 0 & 1 \\ \hline \end{array}$$

may be represented as $k = 4$ quantities (0-matrices) as

$$A_a = \cos\theta, \quad A_b = \sin\theta, \quad A_c = 0, \quad A_d = 1$$

It often happens that each component of a 1-matrix or a 2-matrix, etc., involves such a long expression that they must be represented on paper in terms of a set of k or k^2 quantities.

In books on tensor analysis each n-matrix is represented on paper by showing each of its components separately, that is, as a collection of 0-matrices. The aim of these books is not to work out numerical problems, but to prove some geometrical or physical theorem, hence a practical representation of n-matrices is of no importance in them. However, in engineering work the solving of numerous problems is of prime importance, and the usual representation of n-matrices was found to be impractical.

The method of representation of an n-matrix like $A_{\alpha\beta\gamma}$ in a cube or in k 2-matrices or in k^3 numbers is a question of taste. The breaking up of n-matrices into 2-*matrices* and representing them so on paper was found *by experience* to be the most practical procedure for quick or routine solution of engineering problems. Other forms of representations may also be introduced. However, the method of representing n-matrices on paper has nothing to do with the concepts and reasonings to be introduced later.

VIII. MATRIX NOTATION

(a) The main purpose of this book is to express sets of numbers along certain reference axes or circuits that vary from problem to problem. Hence, it is absolutely necessary to identify each circuit or reference axis by a distinctive name or sign, the so-called " fixed " index. However, n-way matrices are also used extensively in problems where they have no connections at all with physical reference frames. In such cases the fixed indices become superfluous and may be left out, so that, for example, the components of a 2-matrix may be written as:

$$Z = \begin{array}{|c|c|c|c|} \hline A & B & C & D \\ \hline E & F & G & H \\ \hline I & J & K & L \\ \hline M & N & O & P \\ \hline \end{array} \quad \text{or} \quad Z = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{bmatrix} \qquad 1.12$$

without giving names to the various rows and columns. As this notation is used in matrix analysis, it will be called here the " matrix notation."

To avoid any confusion, the fixed indices representing the various reference axes will always be shown in this book, but they may be omitted at pleasure wherever no misunderstanding may arise.

Summarizing, n-way matrices are represented:

1. In index notation as $Z_{\alpha\beta}$.
2. In direct notation as $\boldsymbol{Z}$.
3. In matrix notation as $\boldsymbol{Z}$ or $[Z]$, etc.

The *components* of an n-way matrix are represented:

1. In index notation by showing both variable and fixed indices.
2. In direct notation by showing only fixed indices.
3. In matrix notation by showing neither fixed nor variable indices.

In this book the matrix notation will be used only sparingly.

(*b*) It is impossible to draw attention emphatically enough to the fact that in the analysis of problems the reader should not confuse " *notation* " with " *method of reasoning.*" The mere fact that a set of numbers is arranged in a square and is denoted by one symbol does not make that square a " tensor " and it does not make the analysis used " tensor analysis."

The expression "tensor analysis" is used, as will be shown later, for a "*method of attack,*" a special "*point of view*" in the reasoning followed in analyzing problems, and it has decidedly nothing whatever to do with the notation employed. Leading writers on tensor analysis have found the index notation the most convenient for their purpose, but it does not follow that one is not allowed to use any other notation one's fancy dictates. Nor does it follow that, if one uses index notation, that makes his reasoning "tensor analysis."

IX. THE MANIPULATION OF n-WAY MATRICES

(*a*) *The rules of manipulation of n-way matrices, namely, their addition, multiplication, etc., have been originally so defined that the final answer is the same as if the ordinary quantities themselves (forming their components) had been manipulated in the usual manner.*

No new concepts are involved in these rules of manipulation to be shown presently; *these empirical rules merely speed up the mechanical labor of computation.* It is possible to invent still speedier rules, or even mechanical devices that perform these manipulations. These

rules have been devised to push the mechanical labor into the background to make room for the *concepts* to be introduced.

(*b*) The following manipulations will be considered in more or less detail: 1. Addition. 2. Multiplication. 3. Division. 4. Differentiation. 5. Integration.

In connection with each operation, the *equality* sign occurs between two n-matrices.

Two n-matrices of the same dimensions are *equal* if their corresponding components are equal. For instance,

$$\mathbf{A} = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} & \multicolumn{1}{c}{\mathrm{d}} \\ \hline 2 & 4 & -3 & 0 \\ \hline \end{array} \quad \text{and} \quad \mathbf{B} = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} & \multicolumn{1}{c}{\mathrm{d}} \\ \hline 2 & 4 & -3 & 0 \\ \hline \end{array}$$

are equal, that is $\mathbf{A} = \mathbf{B}$, since each of their components is equal in corresponding order.

X. ADDITION

(*a*) *Two n-way matrices of the same dimensions are added together by adding corresponding components.*

The sum of two 1-matrices is

$$\mathbf{A} = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} & \multicolumn{1}{c}{\mathrm{d}} \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \qquad \mathbf{B} = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} & \multicolumn{1}{c}{\mathrm{d}} \\ \hline -2 & 3 & 0 & 5 \\ \hline \end{array} \tag{1.13}$$

$$\mathbf{A} + \mathbf{B} = \mathbf{C} = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} & \multicolumn{1}{c}{\mathrm{d}} \\ \hline 1-2 & 2+3 & 3+0 & 4+5 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} & \multicolumn{1}{c}{\mathrm{d}} \\ \hline -1 & 5 & 3 & 9 \\ \hline \end{array} \tag{1.14}$$

As another example

$$\mathbf{A} = \begin{array}{|c|c|c|} \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} \\ \hline x^2 & 2+3j & 0 \\ \hline \end{array} \qquad \mathbf{B} = \begin{array}{|c|c|c|} \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} \\ \hline x & 4-7j & 3 \\ \hline \end{array} \tag{1.15}$$

$$\mathbf{A} - \mathbf{B} = \mathbf{C} = \begin{array}{|c|c|c|} \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} \\ \hline x^2 - x & -2+10j & -3 \\ \hline \end{array} \tag{1.16}$$

The sum of two 2-*matrices* is

$$\mathbf{A} = \begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} & \multicolumn{1}{c}{\mathrm{d}} \\ \cline{2-5} \mathrm{a} & 6 & 5 & -7 & 4 \\ \cline{2-5} \mathrm{b} & -8 & 1 & -9 & 5 \\ \cline{2-5} \mathrm{c} & -4 & 7 & 8 & 3 \\ \cline{2-5} \mathrm{d} & 2 & 0 & 6 & 9 \\ \cline{2-5} \end{array} \qquad \mathbf{B} = \begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{b}} & \multicolumn{1}{c}{\mathrm{c}} & \multicolumn{1}{c}{\mathrm{d}} \\ \cline{2-5} \mathrm{a} & 6 & -4 & 9 & 2 \\ \cline{2-5} \mathrm{b} & 1 & 8 & 7 & 3 \\ \cline{2-5} \mathrm{c} & 5 & -2 & 4 & -5 \\ \cline{2-5} \mathrm{d} & 7 & 3 & 6 & 1 \\ \cline{2-5} \end{array} \tag{1.17}$$

$$\mathbf{A}+\mathbf{B}=\mathbf{C}=\begin{array}{c|cccc|}
 & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\hline
\mathbf{a} & 6+6 & 5-4 & -7+9 & 4+2 \\
\mathbf{b} & -8+1 & 1+8 & -9+7 & 5+3 \\
\mathbf{c} & -4+5 & 7-2 & 8+4 & 3-5 \\
\mathbf{d} & 2+7 & 0+3 & 6+6 & 9+1 \\
\hline
\end{array}
=\begin{array}{|cccc|}
\mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\hline
12 & 1 & 2 & 6 \\
-7 & 9 & -2 & 8 \\
1 & 5 & 12 & -2 \\
9 & 3 & 12 & 10 \\
\hline
\end{array} \qquad 1.18$$

(*b*) *Two* n-*matrices of different dimensions cannot be added.*

(*c*) The occasion will often arise to add two 1-matrices or two 2-matrices, etc., which have, however, *different fixed indices.* For example, let

$$\mathbf{A}=\begin{array}{|c|c|c|c|}
\mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\
\hline
2 & -3 & 5 & 1 \\
\hline
\end{array} \quad 1.19 \qquad \mathbf{B}=\begin{array}{|c|c|c|}
\mathbf{e} & \mathbf{f} & \mathbf{g} \\
\hline
-7 & 4 & 3 \\
\hline
\end{array} \quad 1.20$$

The question is, what is their sum?

In this book it will be assumed that *in such cases each of the* 1-*matrices* **A** *and* **B** *have seven components* and seven fixed indices **a, b, c, d, e, f,** and **g.** However, the components not shown are zero. That is, it will be assumed that the complete representations of the above 1-matrices are

$$\begin{array}{l}
\mathbf{A}=\begin{array}{|c|c|c|c|c|c|c|}
\mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} & \mathbf{g} \\
\hline
2 & -3 & 5 & 1 & 0 & 0 & 0 \\
\hline
\end{array} \\
\\
\mathbf{B}=\begin{array}{|c|c|c|c|c|c|c|}
\mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} & \mathbf{g} \\
\hline
0 & 0 & 0 & 0 & -7 & 4 & 3 \\
\hline
\end{array}
\end{array} \qquad 1.21$$

Hence their sum is

$$\mathbf{A}+\mathbf{B}=\mathbf{C}=\begin{array}{|c|c|c|c|c|c|c|}
\mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} & \mathbf{g} \\
\hline
2 & -3 & 5 & 1 & -7 & 4 & 3 \\
\hline
\end{array} \qquad 1.22$$

In the case of two 2-matrices with different fixed indices such as

$$\mathbf{A}=\begin{array}{c|ccc|}
 & \mathbf{a} & \mathbf{b} & \mathbf{c} \\
\hline
\mathbf{a} & 8 & 9 & 2 \\
\mathbf{b} & 1 & 3 & -7 \\
\mathbf{c} & -5 & 6 & 4 \\
\hline
\end{array} \qquad \mathbf{B}=\begin{array}{c|cc|}
 & \mathbf{d} & \mathbf{e} \\
\hline
\mathbf{d} & 3 & -2 \\
\mathbf{e} & 1 & 5 \\
\hline
\end{array} \qquad 1.23$$

they are assumed to stand for 2-matrices each having in the present example five rows and columns with axes **a, b, c, d,** and **e**

$$
\mathbf{A} = \begin{array}{c|ccccc|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} \\ \hline \mathbf{a} & 8 & 9 & 2 & 0 & 0 \\ \mathbf{b} & 1 & 3 & -7 & 0 & 0 \\ \mathbf{c} & -5 & 6 & 4 & 0 & 0 \\ \mathbf{d} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{e} & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad \mathbf{B} = \begin{array}{c|ccccc|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} \\ \hline \mathbf{a} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{b} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{c} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{d} & 0 & 0 & 0 & 3 & -2 \\ \mathbf{e} & 0 & 0 & 0 & 1 & 5 \\ \hline \end{array} \qquad 1.24
$$

Hence their sum is

$$
\mathbf{A} + \mathbf{B} = \mathbf{C} = \begin{array}{c|ccccc|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} \\ \hline \mathbf{a} & 8 & 9 & 2 & 0 & 0 \\ \mathbf{b} & 1 & 3 & -7 & 0 & 0 \\ \mathbf{c} & -5 & 6 & 4 & 0 & 0 \\ \mathbf{d} & 0 & 0 & 0 & 3 & -2 \\ \mathbf{e} & 0 & 0 & 0 & 1 & 5 \\ \hline \end{array} \qquad 1.25
$$

This convention is introduced in this volume because of the special types of problems analyzed.

(*d*) It may also occur that of two given n-matrices some of the fixed indices are the same and some are different. In such cases again the above procedure is followed; that is, *it is assumed that the components along the missing indices are zero* and so they are replaced before the manipulation. For instance, let the two 2-matrices be added:

$$
\mathbf{A} = \begin{array}{c|cccc|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{e} \\ \hline \mathbf{a} & 8 & -5 & 3 & 4 \\ \mathbf{b} & 6 & 2 & -7 & 6 \\ \mathbf{c} & -2 & 4 & 5 & 1 \\ \mathbf{e} & 3 & 1 & 8 & 9 \\ \hline \end{array} \qquad \mathbf{B} = \begin{array}{c|cccc|} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \hline \mathbf{c} & 4 & 6 & 3 & -7 \\ \mathbf{d} & 1 & 2 & 7 & 2 \\ \mathbf{e} & 3 & -4 & 9 & 5 \\ \mathbf{f} & 8 & 5 & 6 & -4 \\ \hline \end{array} \qquad 1.26
$$

They actually stand for two 2-matrices, each having six fixed indices **a, b, c, d, e, f.**

$$
\mathbf{A} = \begin{array}{c|cccccc|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \hline \mathbf{a} & 8 & -5 & 3 & 0 & 4 & 0 \\ \mathbf{b} & 6 & 2 & -7 & 0 & 6 & 0 \\ \mathbf{c} & -2 & 4 & 5 & 0 & 1 & 0 \\ \mathbf{d} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{e} & 3 & 1 & 8 & 0 & 9 & 0 \\ \mathbf{f} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad \mathbf{B} = \begin{array}{c|cccccc|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \hline \mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{b} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{c} & 0 & 0 & 4 & 6 & 3 & -7 \\ \mathbf{d} & 0 & 0 & 1 & 2 & 7 & 2 \\ \mathbf{e} & 0 & 0 & 3 & -4 & 9 & 5 \\ \mathbf{f} & 0 & 0 & 8 & 5 & 6 & -4 \\ \hline \end{array} \qquad 1.27
$$

Hence their sum is

$$\mathbf{A}+\mathbf{B}=\mathbf{C}=\begin{array}{c|cccccc|}
 & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \hline
\mathbf{a} & 8 & -5 & 3 & 0 & 4 & 0 \\
\mathbf{b} & 6 & 2 & -7 & 0 & 6 & 0 \\
\mathbf{c} & -2 & 4 & 9 & 6 & 4 & -7 \\
\mathbf{d} & 0 & 0 & 1 & 2 & 7 & 2 \\
\mathbf{e} & 3 & 1 & 11 & -4 & 18 & 5 \\
\mathbf{f} & 0 & 0 & 8 & 5 & 6 & -4 \\ \hline
\end{array} \qquad 1.28$$

Each time the components of an n-matrix are represented on paper, it is advisable that *the fixed indices should be written alongside their respective row, column, or layer,* and should not be left out. If no misunderstanding may arise, then they may be left out. However, it must be kept in mind that *the indices do not form part of the* n-*matrix.*

(e) In index notation the addition (or subtraction) of two n-matrices is represented as follows:

$$A_\alpha - B_\alpha = C_\alpha \qquad 1.29$$

$$A_{\alpha\beta\gamma} + B_{\alpha\beta\gamma} = C_{\alpha\beta\gamma} \qquad 1.30$$

That is, the variable indices remain unchanged (showing that the fixed indices are also unchanged); only the base letters are changed.

XI. MULTIPLICATION OF 1-MATRICES

(a) *In order to learn how to multiply two* n-*matrices of various dimensions, it is sufficient to remember how to multiply two* 1-*matrices.*

Two 1-*matrices are multiplied together by multiplying corresponding components and then adding them. Their product is a single quantity.* For instance, if

$$\mathbf{e}=\begin{array}{|c|c|c|c|}
\multicolumn{1}{c}{\mathbf{a}} & \multicolumn{1}{c}{\mathbf{b}} & \multicolumn{1}{c}{\mathbf{c}} & \multicolumn{1}{c}{\mathbf{d}} \\ \hline
2 & 3 & 4 & 5 \\ \hline
\end{array} \quad 1.31 \qquad \mathbf{i}=\begin{array}{|c|c|c|c|}
\multicolumn{1}{c}{\mathbf{a}} & \multicolumn{1}{c}{\mathbf{b}} & \multicolumn{1}{c}{\mathbf{c}} & \multicolumn{1}{c}{\mathbf{d}} \\ \hline
1 & 4 & 2 & 3 \\ \hline
\end{array} \quad 1.32$$

then their product is

$$\mathbf{e}\cdot\mathbf{i} = (2\times 1)+(3\times 4)+(4\times 2)+(5\times 3) =$$
$$= 2+12+8+15 = 37 \qquad 1.33$$

If **e** represents the instantaneous voltages impressed on the four coils of Fig. 1.3 and **i** represents the four instantaneous currents flowing in them, then *their product* $\mathbf{e}\cdot\mathbf{i}$ *gives the instantaneous power input to the whole system,* the power being a single quantity.

(*b*) *In index notation the product of two* 1-*matrices* e_α *and* i_α *is represented as a summation*

$$\mathbf{e}\cdot\mathbf{i} = \sum_{\alpha=a}^{d} e_\alpha i_\alpha = P \tag{1.34}$$

where the variable index α assumes all the fixed indices in succession. This summation is equivalent to the above rule. That is, if

$$e_\alpha = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \boxed{\begin{array}{c|c|c|c} 2 & 3 & 4 & 5 \end{array}} \end{array} \quad 1.35 \qquad i_\alpha = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \boxed{\begin{array}{c|c|c|c} 1 & 4 & 2 & 3 \end{array}} \end{array} \quad 1.36$$

$$\text{then} \sum_{\alpha=a}^{d} e_\alpha i_\alpha = e_a i_a + e_b i_b + e_c i_c + e_d i_d =$$

$$= (2 \times 1) + (3 \times 4) + (4 \times 2) + (5 \times 3) =$$

$$= 2 + 12 + 8 + 15 = 37 = P \tag{1.37}$$

It should be noted that to the right of the summation sign both variable indices must be denoted by the same letter α or β that appears under the summation sign. That is,

$$\mathbf{e}\cdot\mathbf{i} = \sum_{\alpha} e_\alpha i_\alpha = \sum_{\beta} e_\beta i_\beta = \sum_{\alpha} i_\alpha e_\alpha = P \tag{1.38}$$

XII. THE PRODUCT OF 2-MATRICES BY THE ARROW RULE

(*a*) *A* 2-*matrix is multiplied by a* 1-*matrix by dividing the* 2-*matrix into* 1-*matrices, then multiplying each of the latter in succession by the given* 1-*matrix.*

Since a 2-matrix can be divided into 1-matrices in two different ways, an arrow will show the direction in which the 2-matrix should be divided into 1-matrices. For instance, let the 2-matrix **z** and the 1-matrix **i** be given as

$$\mathbf{z} = \begin{array}{c|ccc} & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{a} & 3 & 4 & 2 \\ \mathrm{b} & 9 & 1 & 5 \\ \mathrm{c} & 6 & 7 & 8 \end{array} \;\; \xrightarrow{\qquad} \qquad \mathbf{i} = \begin{array}{ccc} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline 2 & 3 & 4 \end{array} \;\; \xrightarrow{\qquad} \tag{1.39}$$

Their product $\mathbf{z} \cdot \mathbf{i}$ is found by dividing $\mathbf{z}$ into horizontal rows, as

$$\mathbf{z} = \begin{array}{c|ccc} & a & b & c \\ \hline a & [\,3 & 4 & 2\,] \\ b & [\,9 & 1 & 5\,] \\ c & [\,6 & 7 & 8\,] \end{array} \qquad \mathbf{i} = \begin{array}{ccc} a & b & c \\ \hline [\,2 & 3 & 4\,] \end{array} \tag{1.40}$$

then multiplying each horizontal row by the given 1-matrix, as

$$\begin{aligned} & (3 \times 2) + (4 \times 3) + (2 \times 4) = 6 + 12 + 8 = 26 \\ \mathbf{z} \cdot \mathbf{i} = \ & (9 \times 2) + (1 \times 3) + (5 \times 4) = 18 + 3 + 20 = 41 \\ & (6 \times 2) + (7 \times 3) + (8 \times 4) = 12 + 21 + 32 = 65 \end{aligned} \tag{1.41}$$

Each product gives a single number, altogether three numbers, that can be arranged *in their original order* to form a 1-matrix as:

$$\mathbf{z} \cdot \mathbf{i} = \mathbf{e} = \begin{array}{ccc} a & b & c \\ \hline [\,26 & 41 & 65\,] \end{array} \tag{1.42}$$

That is, *the product of a 2-matrix and a 1-matrix is a 1-matrix.*

Of course, in actual calculation it is not necessary to redraw the 2-matrix into a set of 1-matrices. *It is sufficient to draw an arrow in the direction in which the 2-matrix is imagined to be split up.*

(*b*) *Two 2-matrices are multiplied together by splitting up each into 1-matrices, then multiplying each 1-matrix of the first with each 1-matrix of the second in succession.* An arrow should show again the directions of the split-up. For example, let the two 2-matrices $\mathbf{A}$ and $\mathbf{B}$ be multiplied together

$$\mathbf{A} = \begin{array}{c|ccc} & a & b & c \\ \hline a & 3 & 4 & 7 \\ b & 1 & 6 & 8 \\ c & 9 & 2 & 5 \end{array} \qquad \mathbf{B} = \begin{array}{c|ccc} & a & b & c \\ \hline a & 7 & 5 & 2 \\ b & 1 & 3 & 4 \\ c & 9 & 8 & 6 \end{array} \tag{1.43}$$

Splitting them up along the arrows, there results:

$$\mathbf{A} = \begin{array}{c|ccc} & a & b & c \\ \hline a & [\,3 & 4 & 7\,] \\ b & [\,1 & 6 & 8\,] \\ c & [\,9 & 2 & 5\,] \end{array} \qquad \mathbf{B} = \begin{array}{c|ccc} & a & b & c \\ \hline a & \begin{bmatrix} 7 \\ 1 \\ 9 \end{bmatrix} & \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix} & \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \end{array} \tag{1.44}$$

Now, let every horizontal row of **A** be multiplied by each vertical column of **B** in succession, ***each product giving one number.*** Altogether there will be $3 \times 3 = 9$ numbers, that are to be arranged in a 2-matrix ***in the same order as they are produced.***

For instance, the product of the *second* horizontal row and the *third* vertical column is placed into the *second* row and *third* column of the resultant 2-matrix as

$$\begin{bmatrix} \; & \; & \; \\ 1 & 6 & 8 \\ \; & \; & \; \end{bmatrix} \cdot \begin{bmatrix} \; & \; & 2 \\ \; & \; & 4 \\ \; & \; & 6 \end{bmatrix} = \begin{bmatrix} \; & \; & \; \\ \; & \; & \boxed{74} \\ \; & \; & \; \end{bmatrix} \tag{1.45}$$

since $1 \times 2 + 6 \times 4 + 8 \times 6 = 2 + 24 + 48 = 74$.

Hence, multiplying all the 1-matrices in succession and placing their product in their proper order, the resultant product matrix is

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 3 & 4 & 7 \\ 1 & 6 & 8 \\ 9 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 7 & 5 & 2 \\ 1 & 3 & 4 \\ 9 & 8 & 6 \end{bmatrix} = \mathbf{C} = \begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{array} \\ \begin{bmatrix} 88 & 83 & 64 \\ 85 & 87 & 74 \\ 110 & 91 & 56 \end{bmatrix} \end{array} \tag{1.46}$$

For instance, the product of the *third* row and the *second* column is

$$(9 \times 5) + (2 \times 3) + (5 \times 8) = 45 + 6 + 40 = 91$$

and this number is placed into the *third* row and the *second* column of the resultant 2-matrix **C**.

As another example:

$$\begin{bmatrix} 2 + 3j & 3 \\ 2j & 0 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 + j \\ 0 & -j \end{bmatrix} = \begin{bmatrix} -8 - 12j & -1 + 2j \\ -8j & -2 + 2j \end{bmatrix} \tag{1.47}$$

For instance, the product of the first row and the first column is

$$(2 + 3j)(-4) + (3)(0) = -8 - 12j + 0 = -8 - 12j.$$

(*c*) When the products of 1-matrices and 2-matrices are represented in direct notation as $\mathbf{Z} \cdot \mathbf{i}$ or $\mathbf{i} \cdot \mathbf{Z}$ or $\mathbf{Z}_1 \cdot \mathbf{Z}_2$ etc., requiring two arrows alongside them, *the first arrow is always horizontal, the second arrow is always vertical.* Of course, the matrices are placed on the paper (or

are imagined to be placed) in the same order as they appear in the given expression. *Their order cannot be interchanged*; for instance:

$$\mathbf{e} \cdot \mathbf{Y} = \begin{array}{|c|c|} \hline a & b \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline c & d \\ \hline e & f \\ \hline \end{array} = \mathbf{i} = \begin{array}{|c|c|} \hline ac + be & aj + bf \\ \hline \end{array} \qquad 1.48$$

$$\mathbf{A} \cdot \mathbf{B} = \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline e & f \\ \hline g & h \\ \hline \end{array} = \mathbf{C} = \begin{array}{|c|c|} \hline ae + bg & af + bh \\ \hline ce + dg & cf + dh \\ \hline \end{array} \qquad 1.49$$

$$\mathbf{Z} \cdot \mathbf{i} = \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \cdot \begin{array}{|c|} \hline e \\ \hline f \\ \hline \end{array} = \mathbf{C} = \begin{array}{|c|} \hline ae + bf \\ \hline ce + df \\ \hline \end{array} \qquad 1.50$$

With a 1-matrix it is immaterial whether its components are placed in a horizontal row or in a vertical column.

(*d*) *Since, in the product of two n-matrices* **A** $\cdot$ **B**, *their order cannot be interchanged as* **B** $\cdot$ **A**, *that is, since*

$$\boxed{\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}} \qquad 1.51$$

the algebra of n-way matrices is called a " non-commutative algebra."

For instance, if, in the above product of two matrices **A** $\cdot$ **B**, their order is interchanged as **B** $\cdot$ **A**, the resultant 2-matrix **C′** is different from the former **C** given in equation 1.49

$$\mathbf{B} \cdot \mathbf{A} = \begin{array}{|c|c|} \hline e & f \\ \hline g & h \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} = \mathbf{C}' = \begin{array}{|c|c|} \hline ea + fc & eb + fd \\ \hline ga + hc & gb + hd \\ \hline \end{array} \qquad 1.52$$

XIII. THE PRODUCT OF 2-MATRICES BY THE SUMMATION RULE

(*a*) *In index notation the product of matrices is again represented by a summation* as

$$\mathbf{A} \cdot \mathbf{B} = \sum_{\beta} A_{\alpha\beta} \, B_{\beta\gamma} = C_{\alpha\gamma} = \mathbf{C} \qquad 1.53$$

the summation being equivalent to the arrow rule of multiplication given above.

In the index notation it follows, from the position of the indices to be summed up, just in what direction the individual matrices are to be split up.

In order to show that performing the indicated summation gives the same answer as the above arrow rule, let the following example be worked out in detail. Let

$$A_{\alpha\beta} = \begin{array}{c|cc} \alpha \backslash \beta & 1 & 2 \\ \hline 1 & a & b \\ 2 & c & d \end{array} \qquad B_{\beta\gamma} = \begin{array}{c|cc} \beta \backslash \gamma & 1 & 2 \\ \hline 1 & e & f \\ 2 & g & h \end{array} \tag{1.54}$$

then in the product $\sum_{\beta} A_{\alpha\beta}B_{\beta\gamma} = C_{\alpha\gamma}$ the indices α and γ, that are not to be summed up, may assume $2 + 2 = 4$ different values in succession:

1. $\alpha = 1, \gamma = 1$. Then allowing β to assume its fixed values:

$$\sum_{\beta} A_{1\beta}B_{\beta 1} = A_{11}B_{11} + A_{12}B_{21} = ae + bg = C_{11}$$

2. $\alpha = 1, \gamma = 2$. Then

$$\sum_{\beta} A_{1\beta}B_{\beta 2} = A_{11}B_{12} + A_{12}B_{22} = af + bh = C_{12}$$

3. $\alpha = 2, \gamma = 1$. Then

$$\sum_{\beta} A_{2\beta}B_{\beta 1} = A_{21}B_{11} + A_{22}B_{21} = ce + dg = C_{21}$$

4. $\alpha = 2, \gamma = 2$. Then

$$\sum_{\beta} A_{2\beta}B_{\beta 2} = A_{21}B_{12} + A_{22}B_{22} = cf + dh = C_{22}$$

Thereby all summations are covered. The final result is

$$C_{\alpha\gamma} = \begin{array}{c|cc} \alpha \backslash \gamma & 1 & 2 \\ \hline 1 & C_{11} & C_{12} \\ 2 & C_{21} & C_{22} \end{array} = \begin{array}{c|cc} \alpha \backslash \gamma & 1 & 2 \\ \hline 1 & ae + bg & af + bh \\ 2 & ce + dg & cf + dh \end{array} \tag{1.55}$$

The process of summation gives exactly the same answer as the arrow rule previously given. However, the summation procedure is awkward and slow, and it should be used only in case of doubt.

(*b*) From the last examples it follows that the summation is equivalent to *drawing the arrows along the two indices to be summed up.* The other two indices appear again in the final matrix.

The two identical indices β are called "dummy" indices, and the remaining indices α and γ are called "free" indices.

Since in a term the dummy indices occur twice, while the free indices occur only once, it is superfluous to add a summation sign Σ and write under it the dummy index for the third time. *It is an accepted convention to leave out the summation sign* and to write

$$\sum_{\beta} A_{\alpha\beta}B_{\beta\gamma} = A_{\alpha\beta}B_{\beta\gamma} \tag{1.56}$$

since there cannot be any doubt that the summation sign (that was left out) refers to β and not to α or γ. For instance

$$\sum_{\alpha} A_{\alpha\beta}B_{\alpha} = A_{\alpha\beta}B_{\alpha} \quad \text{and} \quad \sum_{\beta} A_{\alpha\beta}B_{\beta} = A_{\alpha\beta}B_{\beta} \tag{1.57}$$

The convention of leaving out the summation sign is also known as the "Einstein convention," since he introduced it first. This summation convention is not used in general in the theory of n-way matrices. It is introduced here only in anticipation of further developments.

It is emphasized that the introduction or removal of the summation sign is a question of individual taste and has no bearing whatever upon the concepts or ideas to be introduced later. The same remark applies to the procedure of performing the multiplication by drawing arrows and thereby multiplying 1-matrices together or by actually summing up the various components as suggested by the notation. The final answer is the same in either case.

(*c*) It should be noted that in index notation there are four different ways of multiplying two matrices together as shown by the dummy indices, namely,

$$A_{\alpha\beta}B_{\beta\gamma};\ A_{\alpha\beta}B_{\gamma\beta};\ A_{\alpha\beta}B_{\alpha\gamma};\ A_{\alpha\beta}B_{\gamma\alpha}$$

In direct notation only the first and the last products can be represented as $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{B}\cdot\mathbf{A}$. Later on, a notation will be introduced to represent the other two products.

Just which indices are summed up (or in which order the matrices are written) depends on the problem at hand.

(*d*) Later on 2-matrices with special components will occur (containing mostly zero components) whose multiplication may be performed much more quickly by following some special rule in each special case, instead of the arrow rule. These special rules, of course, are valid only in special cases; the arrow rule always gives the right answer.

XIV. THE PRODUCT OF ANY TWO n-MATRICES *

(*a*) Following the reasoning of the previous sections, *two n-matrices of various dimensions are multiplied together by splitting each into 1-matrices, then multiplying each 1-matrix of the first with each 1-matrix of the second, each product giving a single quantity.* The resulting quantities are arranged into a new n-matrix in their proper order. *The dummy indices represent the directions along which the given n-matrices are to be divided into 1-matrices.*

Before the n-matrices are split up into 1-matrices, first they should be split up into 2-matrices in order to represent them on paper. Then each 2-matrix is split up figuratively into 1-matrices along the dummy index by drawing arrows along it, and finally the 1-matrices are multiplied together. *That is, the multiplication of n-matrices of any dimension may be reduced to the multiplication of the 2-matrices* they are composed of.

(*b*) For instance, let the product

$$\sum_{\beta} A_{\alpha\beta\gamma} B_{\delta\beta} = A_{\alpha\beta\gamma} B_{\delta\beta} = C_{\alpha\gamma\delta} \qquad 1.58$$

be found, where $A_{\alpha\beta\gamma}$ is a 3-matrix and $B_{\delta\beta}$ is a 2-matrix. The dummy index is β, hence $A_{\alpha\beta\gamma}$ and $B_{\delta\beta}$ are to be split into 1-matrices along the β directions, as shown in Fig. 1.10.

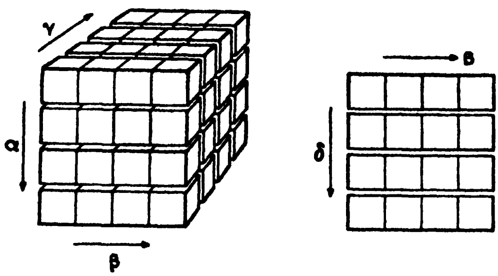


FIG. 1.10.—Splitting a 3-Matrix and a 2-Matrix into 1-Matrices along the Dummy Index β

The free indices are α, γ, and δ; hence the resultant products are to be arranged in a 3-matrix $C_{\alpha\gamma\delta}$.

To perform the multiplications quickly on paper, *it is advantageous to divide the 3-matrix first into k matrices along a free index*, say in a direction parallel to the plane of the paper, along γ. Afterward each of the k matrices $A_{\alpha\beta a}, A_{\alpha\beta b}, A_{\alpha\beta c} \cdots$ should be multiplied by the given matrix $B_{\delta\beta}$ by dividing each of them into 1-matrices along the direction of the dummy index β, giving n product-matrices $C_{\alpha a\delta}, C_{\alpha b\delta}, C_{\alpha c\delta} \cdots$. The latter matrices are equivalent to splitting up of the resultant 3-matrix $C_{\alpha\gamma\delta}$ along γ.

* This section may be left out at the first reading.

As an example let only two fixed indices a and b exist. Then the three matrices to be multiplied together are shown in Fig. 1.11.

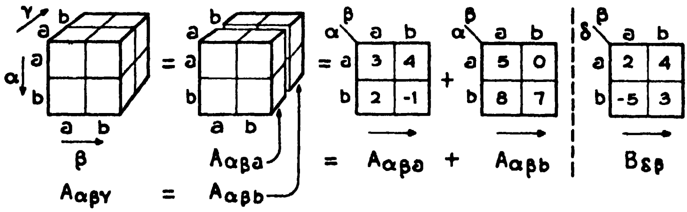


FIG. 1.11.—Multiplication of a 3-Matrix $A_{\alpha\beta\gamma}$ with a 2-Matrix $B_{\delta\beta}$

Let arrows be drawn along the direction β on each matrix to show that the matrices are assumed to be split into 1-matrices along those directions.

1. The product of the first matrix $A_{\alpha\beta a}$ and $B_{\delta\beta}$ gives $C_{\alpha a\delta}$:

$$\underset{A_{\alpha\beta a}}{\begin{array}{c|cc} \alpha\backslash\beta & a & b \\ \hline a & 3 & 4 \\ b & 2 & -1 \end{array}} \quad \underset{B_{\delta\beta}}{\begin{array}{c|cc} \delta\backslash\beta & a & b \\ \hline a & 2 & 4 \\ b & -5 & 3 \end{array}} =$$

$$= \begin{array}{|c|c|} \hline 3\times 2+4\times 4 & 3\times(-5)+4\times 3 \\ \hline 2 \quad 2+(-1)\times 4 & 2\times(-5)+(-1)(3) \\ \hline \end{array} = \begin{array}{c|cc} \alpha\backslash\delta & a & b \\ \hline a & 22 & -3 \\ b & 0 & -13 \end{array} = C_{\alpha a\delta} \tag{1.59}$$

2. The product of the second matrix $A_{\alpha\beta b}$ and $B_{\delta\beta}$ gives $C_{\alpha b\delta}$:

$$\underset{A_{\alpha\beta b}}{\begin{array}{c|cc} \alpha\backslash\beta & a & b \\ \hline a & 5 & 0 \\ b & 8 & 7 \end{array}} \quad \underset{B_{\delta\beta}}{\begin{array}{c|cc} \delta\backslash\beta & a & b \\ \hline & 2 & 4 \\ & -5 & 3 \end{array}} =$$

$$= \begin{array}{|c|c|} \hline 5\times 2+0\times 4 & 5\times 4+0\times 3 \\ \hline 8\times 2+7\times 4 & 8\times(-5)+7\times 3 \\ \hline \end{array} = \begin{array}{c|cc} \alpha\backslash\delta & a & b \\ \hline a & 10 & 20 \\ b & 44 & -19 \end{array} = C_{\alpha b\delta} \tag{1.60}$$

The two resultant matrices $C_{\alpha a\delta}$ and $C_{\alpha b\delta}$ form the 3-matrix $C_{\alpha\gamma\delta}$ shown in Fig. 1.12.

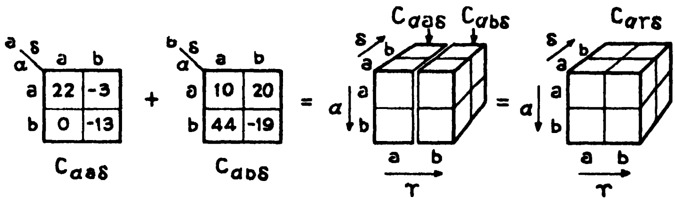


FIG. 1.12.—Constructing the Resultant 3-Matrix $C_{\alpha\gamma\delta}$

(*c*) Similar procedure follows in multiplying, say, $A_{\alpha\beta\gamma\delta}$ with $B_{\epsilon\gamma\sigma}$ along the dummy index γ. In that case $A_{\alpha\beta\gamma\delta}$ is divided into k^2 matrices and $B_{\epsilon\gamma\sigma}$ into k matrices. Then the matrices are multiplied together along γ.

It is emphasized that any number of special procedures may be developed to perform the multiplications, depending on the special problem at hand. In many problems most of the components are zero, hence it is practicable to represent each n-matrix as composed of single numbers and sum up the numbers as indicated by the dummy indices.

The free indices are the original indices without the two dummy indices. *Hence the dimension of the product n-matrix is the sum of the original ones minus two.* For example, the product of a 3-matrix and a 2-matrix is $(3 + 2) - 2 = 3$, a 3-matrix; the product of a 2-matrix and a 1-matrix is $(2 + 1) - 2 = 1$, a 1-matrix; the product of two 1-matrices is $(1 + 1) - 2 = 0$, a 0-matrix.

(*d*) *Any n-matrix is multiplied by a single quantity by multiplying each of its components by the quantity.* Its dimension is not changed. To illustrate:

$$3A_{\alpha\beta} = 3 \times \begin{array}{c|cc} \alpha \diagdown \beta & a & b \\ \hline a & 1 & -2 \\ b & 4 & 5 \end{array} = \begin{array}{c|cc} \alpha \diagdown \beta & a & b \\ \hline a & 3 & -6 \\ b & 12 & 15 \end{array} \qquad 1.61$$

(*e*) In direct notation the products of 3-matrices and n-matrices of higher dimension need special symbols that are not considered here. Generally speaking, the direct notation has been abandoned by practically all writers on tensor analysis. It is used in these pages only for simple problems, and even then only to serve as a stepping stone until the reader gets used to the index notation.

It is possible to have more than one set of dummy indices such as $A_{\alpha\beta\gamma}B_{\beta\gamma}$. The treatment of such general products will be taken up later.

If the components of the n-matrices to be multiplied together do not contain linear operators such as $p = d/dt$, then their order in index notation may be interchanged. That is, $A_\beta B_{\beta\gamma}$ may be written as $B_{\beta\gamma}A_\beta$.

XV. DETERMINANTS

(*a*) In order to learn how to divide by a 2-matrix, it is necessary to learn what a determinant of a 2-matrix is.

With every 2-matrix (a set of k^2 numbers) there is associated a single number, called the "*determinant*" of the 2-matrix. It is formed from the components of the 2-matrix by certain multiplications and additions. No other n-matrix has a determinant.

When the matrix has only *two* rows and columns, its determinant is defined as follows:

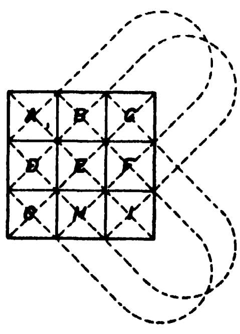
$$\mathbf{Z} = \begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} \qquad \text{Determinant of } \mathbf{Z} = |\mathbf{Z}| = AD - CB \qquad 1.62$$

For instance,

$$\mathbf{Z} = \begin{array}{|c|c|}\hline 2 & -3 \\ \hline 4 & 5 \\ \hline\end{array} \qquad \begin{aligned} \text{Det.} = |\mathbf{Z}| &= 2 \times 5 - 4 \times (-3) \\ &= 10 + 12 = 22 \end{aligned} \qquad 1.63$$

When the matrix has *three* rows and columns its determinant is defined by the following scheme

$$\mathbf{Z} = \begin{array}{|c|c|c|}\hline A & B & C \\ \hline D & E & F \\ \hline G & H & I \\ \hline\end{array} \qquad \text{Det.} =$$



$$\text{Det.} = AEI + BFG + CHD \\ - GEC - DBI - AFH \qquad 1.64$$

For instance,

$$\mathbf{Z} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 2 & 8 & 4 \\ \hline \end{array} \qquad \text{Det.} = 1\times5\times4+2\times6\times2+3\times8\times4 - 2\times5\times3-4\times2\times4-1\times6\times8$$

$$= 20 + 24 + 96 - 30 - 32 - 48 = 140 - 110 = 30 \qquad 1.65$$

Later, in Chapter X, a labor-saving method will be shown that in most problems eliminates the need of calculating the determinant of matrices having *more than three* rows.

(*b*) With each component of a matrix there is associated a number, called the "minor" of the component. *The minor of any component is found by striking out the row aud column in which the component lies, then calculating the determinant of the remaining matrix.*

For instance, the minor of 3 in the following matrix is 22

$$\mathbf{Z} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 2 & 8 & 4 \\ \hline \end{array} \quad \text{Minor of } 3 = \begin{array}{|c|c|c|}\hline & & \\ \hline 4 & 5 & \\ \hline 2 & 8 & \\ \hline \end{array} = 4 \times 8 - 2 \times 5 = 22 \qquad 1.66$$

The minor of 5 is -2, that is,

$$\begin{array}{|c|c|c|}\hline 1 & & 3 \\ \hline & & \\ \hline 2 & & 4 \\ \hline \end{array} = 1 \times 4 - 2 \times 3 = -2 \qquad 1.67$$

(*c*) When a matrix has more than three rows and columns, its determinant may be found in several ways. One way of finding the determinant is the following:

1. Find the minors of each component lying in the first row (or column).
2. Multiply every other minor by minus one. The result of multiplying the minors by $\pm$ 1 is called "cofactor."
3. Multiply each cofactor by the corresponding component of the first row (or column) whose cofactor it happens to be.
4. Add these products.

The result is the determinant. This method requires the calculation of several determinants, each having one less row and column than the desired determinant.

An extensive literature is available on the theory and calculation of determinants, which should be consulted for further information.

XVI. DIVISION WITH 2-MATRICES

(*a*) *Only a 2-matrix (and a single number) can be used as a divisor.* Division with other n-matrices is not defined. Division by a 2-matrix $\mathbf{Z} = Z_{\alpha\beta}$ is represented as a multiplication by its "inverse" $\mathbf{Z}^{-1} = (Z_{\alpha\beta})^{-1}$. Hence, generally speaking, division does not exist in matrix algebra. Its only trace is the "inverse" of a 2-matrix, $\mathbf{Z}^{-1}$, provided the determinant of the 2-matrix is not zero.

The calculation of the "inverse" of a 2-matrix is equivalent of solving a set of n linear equations with n unknowns. Hence the two procedures follow parallel.

(*b*) The inverse of a matrix is found by the following steps:

1. Interchange rows and columns.
2. Replace each component by its minor.
3. Multiply every other minor with minus one, starting with plus one in the upper left-hand corner, as shown in the following scheme.

+	−	+		−
−	+	−		+
+	−	+		−
−	+	−		+

1.68

After this multiplication the result is called a "*cofactor.*"

4. Divide each resultant component by the determinant of the whole matrix.

The calculation of the inverse of a matrix is time-consuming, and, when the matrix has more than, say, four rows and columns, in general its inverse should be calculated only if the components are known numbers. If the components of the matrix Z are algebraic symbols, its inverse should be denoted if possible only symbolically, as $\mathbf{Z}^{-1}$, and each numerical case of inverse should be calculated separately. However, in many problems most components of the matrix are zero, in which case it is practical to calculate the inverse in terms of algebraic symbols.

Later, a labor-saving device will be shown to calculate the inverse of matrices with large numbers of rows and columns.

(*c*) As an example let the inverse of

$$\mathbf{Z} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 2 & 8 & 4 \\ \hline \end{array} \qquad 1.69$$

be calculated. Its determinant is $1 \times 5 \times 4 + 2 \times 6 \times 2 + 3 \times 8 \times 4 - 2 \times 5 \times 3 - 4 \times 2 \times 4 - 1 \times 8 \times 6 = 30$.

1. Interchanging rows and columns gives

1	4	2
2	5	8
3	6	4

2. Replacing each component by its minor gives

−28	−16	−3
4	− 2	−6
22	4	−3

3. Changing the sign of every other component gives

−28	16	−3
−4	−2	6
22	−4	−3

4. Dividing each component by 30 (the determinant) gives

$$Z^{-1} = \begin{array}{|c|c|c|} \hline -14/15 & 8/15 & -1/10 \\ \hline -\ 2/15 & -1/15 & 1/15 \\ \hline 11/15 & -2/15 & -1/10 \\ \hline \end{array} \qquad 1.70$$

(*d*) *The product of a 2-matrix* Z *and its inverse* Z^{-1} *is always the "unit" matrix.* That is

$$\boxed{Z \cdot Z^{-1} = I} \quad \text{or} \quad \boxed{Z^{-1} \cdot Z = I} \qquad 1.71$$

This fact may serve as a check on the correctness of the inverse calculation. For the above example

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 2 & 8 & 4 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline -14/15 & 8/15 & -1/10 \\ \hline -\ 2/15 & -1/15 & 1/15 \\ \hline 11/15 & -2/15 & -1/10 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array} \qquad 1.72$$

(*e*) *A matrix that contains only diagonal components is called a "diagonal matrix."* Its inverse is found by taking the inverse of each of its components. For instance

$$Z = \begin{array}{|c|c|c|} \hline A & 0 & 0 \\ \hline 0 & B & 0 \\ \hline 0 & 0 & C \\ \hline \end{array} \qquad Z^{-1} = \begin{array}{|c|c|c|} \hline 1/A & 0 & 0 \\ \hline 0 & 1/B & 0 \\ \hline 0 & 0 & 1/C \\ \hline \end{array} \qquad 1.73$$

Its inverse matrix is also a diagonal matrix.

In index notation the inverse of a matrix $A_{\alpha\beta}$ is denoted by a different base letter $B_{\beta\alpha}$, *with its indices interchanged.*

XVII. DIFFERENTIATION *

(*a*) An *n*-matrix is differentiated with respect to a single number by differentiating each of its components separately. The dimension of the *n*-matrix does not change.

For instance, let a 2-matrix be given whose components are functions of θ

$$Z_{\alpha\beta} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c|c|c|c|} \multicolumn{1}{c}{\alpha \backslash \beta} & \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} \\ \hline & 1 & 0 & 0 \\ \hline & 0 & \cos\theta & -\sin\theta \\ \hline & 0 & \sin\theta & \cos\theta \\ \hline \end{array} \qquad 1.74$$

Differentiating each component with respect to θ

$$\frac{\partial Z_{\alpha\beta}}{\partial\theta} = A_{\alpha\beta} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c|c|c|c|} \multicolumn{1}{c}{\alpha \backslash \beta} & \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} \\ \hline & 0 & 0 & 0 \\ \hline & 0 & -\sin\theta & -\cos\theta \\ \hline & 0 & \cos\theta & -\sin\theta \\ \hline \end{array} \qquad 1.75$$

(*b*) An *n*-matrix is differentiated with respect to a 1-matrix by differentiating each component of the *n*-matrix with respect to each component of the 1-matrix. Since each component of the *n*-matrix becomes a 1-matrix after differentiation, *its dimension increases by one.* That is, a 2-matrix becomes a 3-matrix, and so on.

* This section may be left out at the first reading.

For instance, let the n-matrix to be differentiated be

$$e_\alpha = \begin{array}{c|c|c|c|} \alpha & a & b & c \\ \hline & \cos x_m & 3 & \sin x_k \\ \hline \end{array} \qquad 1.76$$

and the 1-matrix

$$x_\beta = \begin{array}{c|c|c|c|} \beta & m & n & k \\ \hline & x_m & x_n & x_k \\ \hline \end{array} \qquad 1.77$$

That is, let $\partial e_\alpha / \partial x_\beta = A_{\alpha\beta}$ be found.

Differentiating each component of e_α:

1. With respect to $x_m = \begin{array}{c|c|c|c|} \alpha & a & b & c \\ \hline m & -\sin x_m & 0 & 0 \\ \hline \end{array}$

2. With respect to $x_n = \begin{array}{c|c|c|c|} \alpha & a & b & c \\ \hline n & 0 & 0 & 0 \\ \hline \end{array}$

3. With respect to $x_k = \begin{array}{c|c|c|c|} \alpha & a & b & c \\ \hline k & 0 & 0 & \cos x_k \\ \hline \end{array}$

Hence the resultant n-matrix is

$$\frac{\partial e_\alpha}{\partial x_\beta} = A_{\alpha\beta} = \begin{array}{c|c|c|c|} {}_\beta \diagdown {}^\alpha & a & b & c \\ \hline m & -\sin x_m & 0 & 0 \\ \hline n & 0 & 0 & 0 \\ \hline k & 0 & 0 & \cos x_k \\ \hline \end{array} \qquad 1.78$$

(*c*) *In general, any n-matrix is differentiated with respect to any other n-matrix by differentiating each component of the first with respect to each component of the second one. The dimension of the resultant n-matrix is the sum of the dimensions of the other two.* For instance,

$$\frac{\partial A_{\alpha\beta\gamma}}{\partial B_{\delta\epsilon}} = C_{\alpha\beta\gamma\delta\epsilon} \quad \text{or} \quad \frac{\partial A}{\partial x_\alpha} = B_\alpha \qquad 1.79$$

In direct notation, differentiation is denoted as $\partial \mathbf{e} / \partial \mathbf{x} = \mathbf{A}$. The notation does not give any clue, however, about the dimensions of $\mathbf{e}$, $\mathbf{x}$, or $\mathbf{A}$.

XVIII. INTEGRATION *

An n-matrix is integrated with respect to a single number by integrating each of its components. For instance, if

$$A_\alpha = \begin{array}{c|c|c|c} \alpha & a & b & c \\ \hline & 2 & \sin\theta & \cos\theta \end{array} \qquad 1.80$$

$$\int A_\alpha\, d\theta = B_\alpha = \begin{array}{c|c|c|c} \alpha & a & b & c \\ \hline & 2\theta + A & -\cos\theta + B & \sin\theta + C \end{array} \qquad 1.81$$

A 1-matrix is integrated with respect to another 1-matrix by integrating each component of the first with respect to the *corresponding* component of the second one, then adding them as indicated by the dummy indices. For instance, if

$$A_\alpha = \begin{array}{c|c|c|c} \alpha & a & b & c \\ \hline & \cos x_a & 3 & \sin x_c \end{array} \qquad 1.82$$

$$dx_\alpha = \begin{array}{c|c|c|c} \alpha & a & b & c \\ \hline & dx_a & dx_b & dx_c \end{array}$$

$$\int A_\alpha dx_\alpha = \int A_a dx_a + \int A_b dx_b + \int A_c dx_c =$$

$$= \int \cos x_a dx_a + \int 3 dx_b + \int \sin x_c dx_c =$$

$$= (\sin x_a + A) + (3x_b + B) - (\cos x_c + C) \qquad 1.84$$

A detailed analysis of these and also of other cases will be taken up later.

* This section may be left out at the first reading.

CHAPTER II

THE FIRST GENERALIZATION POSTULATE

I. OCCURRENCE OF *N*-WAY MATRICES

(*a*) n-Matrices may arise in many types of mathematical studies. They may occur in algebraic, differential, or integral equations, in particular when there are:

1. Several equations with several variables.
2. One equation with several variables.
3. One equation with one variable, in which the variable or its coefficient have special forms.

There is a very extensive literature available on the use of n-matrices, especially in the *solution* of differential and integral equations.

(*b*) One of the purposes of introducing n-matrices is to reduce *during the analysis* the number of equations and the number of symbols to as few a number as possible, usually to one. Such a reduction of the number of equations and the number of symbols greatly facilitates the organization, the manipulation, and the solution of the problem, since all unessential details (the particular values of the various *components* of n-matrices) are eliminated from view when one symbol, say $\mathbf{Z}$, is used in place of n^2 or n^3 components throughout the whole analysis.

Once the final answer is reached in terms of n-matrices, it is necessary in most engineering work to replace the n-matrices by their components, thereby to enlarge the symbolic expressions to the usual form, and then to perform the indicated numerical calculations *in a routine manner* by slide rule or by some mechanical calculating device (network analyzer, differential analyzer), and so on.

With the use of n-matrices the saving of labor occurs not in the number of *final* additions, multiplications, differentiations, or integrations that have to be performed to get a numerical answer but in the reduction of the amount of *intermediary steps* that are needed to arrive to an answer. *That is, the use of* n-*matrices saves in that portion of the work that requires thinking and reasoning on the part of the engineer,*

but it does not save in that portion of the work that can be performed by a machine.

(*c*) ***It cannot be sufficiently emphasized, however, that the saving of mental and physical labor is of secondary importance compared with far more important considerations*** that influence the very foundations of the mental images and mathematical symbolism that have been built up to represent *physical* phenomena as they appear to the senses and instruments. These basic considerations will be successively pointed out as the introduction of new and more advanced concepts continues throughout the treatise.

II. N-MATRICES OF STATIONARY NETWORKS WITH LUMPED CONSTANTS

Let, in Fig. 2.1, a *stationary* network be given where the various branches contain resistances, self- and mutual inductances, capacities, some of them zero. Let various *instantaneous* voltages be impressed, and let currents flow in the various circuits. Let the meshes be called a, b, c, and d respectively.

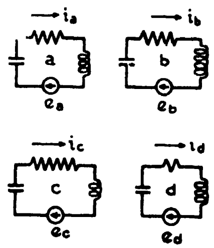


FIG. 2.1.—Stationary Network

In connection with the stationary network the following n-matrices may be established:

1. The values of the various impressed *voltages* may be arranged in a row, representing the "*impressed voltage matrix*"

$$\mathbf{e} = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\mathbf{a}} & \multicolumn{1}{c}{\mathbf{b}} & \multicolumn{1}{c}{\mathbf{c}} & \multicolumn{1}{c}{\mathbf{d}} \\ \hline e_a & e_b & e_c & e_d \\ \hline \end{array} \qquad e_\alpha = \begin{array}{c|c|c|c|c|} \alpha & \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \cline{2-5} & e_a & e_b & e_c & e_d \\ \cline{2-5} \end{array}$$

The components e_a, e_b may represent instantaneous values; or steady a-c. values, or d-c. voltages, or any time function. That is, e_a may be 5, or $2 + 3j$, or $2 \sin \omega t$, or 21, where 1 is the Heaviside unit function, etc.

2. The values of the various *currents* may be arranged in a row, representing the "*current matrix*"

$$\mathbf{i} = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\mathbf{a}} & \multicolumn{1}{c}{\mathbf{b}} & \multicolumn{1}{c}{\mathbf{c}} & \multicolumn{1}{c}{\mathbf{d}} \\ \hline i_a & i_b & i_c & i_d \\ \hline \end{array} \qquad i_\alpha = \begin{array}{c|c|c|c|c|} \alpha & \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \cline{2-5} & i_a & i_b & i_c & i_c \\ \cline{2-5} \end{array}$$

The components i_a, i_b may represent a-c., d-c., instantaneous, etc., values.

3. The self- and mutual *inductances* of the coils may be arranged in a square, the "*inductance matrix*"

$$\mathbf{L} = \begin{array}{cc} & \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{array} \\ \begin{array}{c} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{array} & \left[\begin{array}{cccc} L_{aa} & M_{ab} & M_{ac} & M_{ad} \\ M_{ab} & L_{bb} & M_{bc} & M_{bd} \\ M_{ac} & M_{bc} & L_{cc} & M_{cd} \\ M_{ad} & M_{bd} & M_{cd} & L_{dd} \end{array}\right] \end{array}
\qquad
L_{\alpha\beta} = \begin{array}{cc} \alpha \backslash \beta & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \left[\begin{array}{cccc} M_{aa} & M_{ab} & M_{ac} & M_{ad} \\ M_{ab} & L_{bb} & M_{bc} & M_{bd} \\ M_{ac} & M_{bc} & L_{cc} & M_{cd} \\ M_{ad} & M_{bd} & M_{cd} & L_{dd} \end{array}\right] \end{array}$$

The self-inductances are arranged in the main diagonal line, the mutual inductances on both sides of the line. For instance, in row b and column c the mutual inductance between coils b and c, namely, M_{bc}, is written. When the mutual inductance between any two coils, say c and d, is the same as that between d and c, that is, if $M_{cd} = M_{dc}$, then *the inductance matrix is symmetrical* with respect to the main diagonal line.

It is assumed that, when the self-inductance of a coil or the mutual inductance of two coils are measured, *all the other coils are open-circuited.*

When other reference frames are introduced in place of the actual circuits, the inductance matrix of stationary networks may become asymmetrical.

4. The various *resistances* may be arranged in a square, the "*resistance matrix*" $R_{\alpha\beta}$ containing in the present case no mutual resistances. Similarly the *elastances* (the inverse of the capacities) may be arranged in a square, the "*elastance matrix*," $S_{\alpha\beta}$, containing in the present case no mutual elastances, as

$$R_{\alpha\beta} = \begin{array}{cc} \alpha \backslash \beta & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \left[\begin{array}{cccc} R_{aa} & 0 & 0 & 0 \\ 0 & R_{bb} & 0 & 0 \\ 0 & 0 & R_{cc} & 0 \\ 0 & 0 & 0 & R_{dd} \end{array}\right] \end{array}
\qquad
S_{\alpha\beta} = \begin{array}{cc} \alpha \backslash \beta & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \left[\begin{array}{cccc} S_{aa} & 0 & 0 & 0 \\ 0 & S_{bb} & 0 & 0 \\ 0 & 0 & S_{cc} & 0 \\ 0 & 0 & 0 & S_{dd} \end{array}\right] \end{array}$$

5. The resistances, inductances, and elastances of the various coils may be combined into a single concept, the *impedance* of a coil $Z = R + Lp + S/p$ where $p = d/dt$. Similarly, the matrices **R**, **L**,

and S may be combined into one matrix, the "*impedance matrix*" **Z**

$$\mathbf{Z} = \mathbf{R} + \mathbf{L}p + \mathbf{S}/p \quad \Big| \quad Z_{\alpha\beta} = R_{\alpha\beta} + L_{\alpha\beta}p + S_{\alpha\beta}/p$$

$$\mathbf{Z} = \begin{array}{c|cccc} & \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} \\ \hline \mathrm{a} & Z_{aa} & Z_{ab} & Z_{ac} & Z_{ad} \\ \mathrm{b} & Z_{ab} & Z_{bb} & Z_{bc} & Z_{bd} \\ \mathrm{c} & Z_{ac} & Z_{bc} & Z_{cc} & Z_{cd} \\ \mathrm{d} & Z_{ad} & Z_{bd} & Z_{cd} & Z_{dd} \end{array}$$

where $Z_{aa} = R_{aa} + L_{aa}p + S_{aa}/p$, and so on, the components representing the various self- and mutual impedances. It is usually a symmetrical matrix for stationary networks, although not in all reference frames.

When the coils have *lumped* constants, are *stationary*, and have *single-frequency* voltages impressed upon them, then the impedance matrix **Z** need not be separated into its component matrices. In the present volume these conditions will usually be satisfied, hence the impedance matrix will be treated as a unit.

III. THE ORDER OF n-MATRICES IN DIRECT NOTATION

(*a*) The index notation supplies a very flexible and smoothly working apparatus to indicate the order and manner of manipulation of n-matrices. When the indices are discarded, as in direct notation, the role of the indices has to be replaced by special symbols and marks such as dots, crosses, stars, subscripts, etc. In simple cases a few symbols are sufficient, but as the complexity of problems increases, the number of special symbols increases correspondingly until the symbolism breaks down under its own weight. *The "index" notation may be looked upon as a type of "direct" notation in which the indices themselves take the role of the special symbols.*

One special symbol hitherto introduced in direct notation was the "dot." Now another special symbol is introduced, a subscript t as $\mathbf{A}_t$, called the "transpose" of **A**. (The symbol $\mathbf{A}^{-1}$ to represent the "inverse" of a matrix is used also in conjunction with the index notation.)

(*b*) It was shown in Section XIII, Chapter I, that in direct notation the products $A_{\alpha\beta}B_{\gamma\beta}$ and $A_{\alpha\beta}B_{\alpha\gamma}$ cannot be represented. In order to do that an additional symbol is introduced.

The "transpose" of a matrix **A** *is found by interchanging rows and columns.* The transpose of **A** is denoted as $\mathbf{A}_t$, so that

$$\mathbf{A} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & -2 & 3 & 0 \\ b & 7 & 8 & -3 & 5 \\ c & 3 & -4 & 6 & 1 \\ d & 9 & 0 & 5 & 8 \end{array} \qquad \mathbf{A}_t = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 7 & 3 & 9 \\ b & -2 & 8 & -4 & 0 \\ c & 3 & -3 & 6 & 5 \\ d & 0 & 5 & 1 & 8 \end{array}$$

Hence the above products are written as

$$A_{\alpha\beta}B_{\gamma\beta} = \mathbf{A}\cdot\mathbf{B}_t \quad \text{and} \quad A_{\alpha\beta}B_{\alpha\gamma} = \mathbf{A}_t\cdot\mathbf{B} \qquad 2.1$$

since in index notation if $\mathbf{A} = A_{\alpha\beta}$ then $\mathbf{A}_t = A_{\beta\alpha}$.

Taking the transpose of a transposed matrix, the original matrix is reestablished, that is

$$(\mathbf{A}_t)_t = \mathbf{A} \qquad 2.2$$

Similarly taking the inverse of an inverse matrix, the original is reestablished. That is

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A} \qquad 2.3$$

(*c*) *In index notation the order of n-matrices in a term is immaterial.* For instance, $A_{\alpha\beta}e_\beta$ may be written as $e_\beta A_{\alpha\beta}$, and $\Gamma_{\alpha\beta\gamma}e_\beta i_\gamma$ may be written as $i_\gamma e_\beta \Gamma_{\alpha\beta\gamma}$ or $e_\beta\Gamma_{\alpha\beta\gamma}i_\gamma$, etc. If, however, the n-matrices contain linear operators, their order cannot be disturbed.

In direct notation the order of n-matrices in a term cannot be disturbed in general. In certain special cases it may be disturbed, provided that the components contain no linear operators. In particular:

1. In the product of two 1-matrices as $\mathbf{e}\cdot\mathbf{i}$ the order may be interchanged, that is, $\mathbf{e}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{e}$.

2. In the product of a 1-matrix and a 2-matrix, their order may be interchanged by taking the transpose of the matrix, that is $\mathbf{A}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{A}_t$.

If a 1-matrix **e** lies between two matrices, as say $\mathbf{A}\cdot(\mathbf{e}\cdot\mathbf{B})$, then a parenthesis must indicate whether the 1-matrix is to be multiplied by the first or by the second matrix, since each case gives a different answer. That is

$$\mathbf{A}\cdot(\mathbf{e}\cdot\mathbf{B}) \neq (\mathbf{A}\cdot\mathbf{e})\cdot\mathbf{B} \qquad 2.4$$

If, however, a 1-matrix is at the beginning or at the end of an expression, the parenthesis may be left out as

$$\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{e}) = \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{e} \qquad 2.5$$

In an expression like $\mathbf{A}\cdot(\mathbf{e}\cdot\mathbf{B})$ the term in parenthesis is a 1-matrix and it can be manipulated as such, that is

$$\mathbf{A}\cdot(\mathbf{e}\cdot\mathbf{B}) = (\mathbf{e}\cdot\mathbf{B})\cdot\mathbf{A}_t = \mathbf{A}\cdot(\mathbf{B}_t\cdot\mathbf{e}) = \mathbf{A}\cdot\mathbf{B}_t\mathbf{e} \tag{2.6}$$

In index notation these precautions in the order of the n-matrices need not be observed in general, since the indices show the correct order of multiplications, irrespective of the order of the base letters. However, *if the components of some of the n-matrices contain " operators " (such as $p = d/dt$ or $1/p$, etc.), then also in index notation the order of n-matrices cannot be disturbed.*

IV. THE ORDER OF MULTIPLICATIONS

(*a*) When three matrices are to be multiplied together as $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$, giving a matrix $\mathbf{M}$, the multiplication may be performed in two different orders:

1. First matrix $\mathbf{A}$ is multiplied with matrix $\mathbf{B}$ giving matrix $\mathbf{A}\cdot\mathbf{B} = \mathbf{G}$, then matrix $\mathbf{G}$ is multiplied with matrix $\mathbf{C}$ in the order $\mathbf{G}\cdot\mathbf{C}$ giving $\mathbf{G}\cdot\mathbf{C} = \mathbf{M}$.
2. First matrix $\mathbf{B}$ is multiplied with matrix $\mathbf{C}$ giving matrix $\mathbf{B}\cdot\mathbf{C} = \mathbf{K}$, then matrix $\mathbf{A}$ is multiplied with matrix $\mathbf{K}$ in the order $\mathbf{A}\cdot\mathbf{K}$ giving $\mathbf{A}\cdot\mathbf{K} = \mathbf{M}$.

That is, the multiplication may be performed as

$$\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C} = (\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} = \mathbf{G}\cdot\mathbf{C} = \mathbf{M} \tag{2.7}$$

or as

$$\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C}) = \mathbf{A}\cdot\mathbf{K} = \mathbf{M} \tag{2.8}$$

However, the order of the matrices cannot be interchanged.

For instance, let the three matrices

$$\mathbf{A} = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline\end{array} \qquad \mathbf{B} = \begin{array}{|c|c|c|}\hline j & k & l \\ \hline m & n & o \\ \hline p & q & r \\ \hline\end{array} \qquad \mathbf{C} = \begin{array}{|c|c|c|}\hline s & t & u \\ \hline v & w & x \\ \hline y & z & a \\ \hline\end{array}$$

be multiplied together in the order $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C} = \mathbf{M}$ ($A_{mn}B_{np}C_{pq} = M_{mq}$).

1. Performing the multiplications in the order $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}$, the first product is

$\mathbf{A}\cdot\mathbf{B} =$

$aj + bm + cp$	$ak + bn + cq$	$al + bo + cr$
$dj + em + fp$	$dk + en + fq$	$dl + eo + fr$
$gj + hm + ip$	$gk + hn + iq$	$gl + ho + ir$

The second product is $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} = \mathbf{M}$

$\mathbf{M} =$

$(aj + bm + cp)s +$ $(ak + bn + cq)v +$ $(al + bo + cr)y$	$(aj + bm + cp)t +$ $(ak + bn + cq)w +$ $(al + bo + cr)z$	$(aj + bm + cp)u +$ $(ak + bn + cq)x +$ $(al + bo + cr)a$
$(dj + em + fp)s +$ $(dk + en + fq)v +$ $(dl + eo + fr)y$	$(dj + em + fp)t +$ $(dk + en + fq)w +$ $(dl + co + fr)z$	$(dj + em + fp)u +$ $(dk + en + fq)x +$ $(dl + eo + fr)a$
$(gj + hm + ip)s +$ $(gk + hn + iq)v +$ $(gl + ho + ir)y$	$(gj + hm + ip)t +$ $(gk + hn + iq)w +$ $(gl + ho + ir)z$	$(gj + hm + ip)u +$ $(gk + hn + iq)x +$ $(gl + ho + ir)a$

2. Performing now the multiplications in the order $\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$ the first product is

$\mathbf{B}\cdot\mathbf{C} =$

$js + kv + ly$	$jt + kw + lz$	$ju + kx + la$
$ms + nv + oy$	$ml + nw + oz$	$mu + no + oa$
$ps + qv + ry$	$pt + q+w\ rz$	$pu + qx + ra$

The second product is $\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C}) = \mathbf{M} =$

$\mathbf{M} =$

$a(js + kv + ly)+$ $b(ms + nv + oy)+$ $c(pa + qv + ry)$	$a(jt + kw + ez)+$ $b(mt + nw + oz)+$ $c(pt + qw + rz)$	$a(ju + kx + la)+$ $b(mu + nx + oa)+$ $c(pu + qx + ra)$
$d(js + kv + ly)+$ $e(ms + nv + oy)+$ $f(ps + qv + ry)$	$d(jt + kw + lz)+$ $e(mt + nw + oz)+$ $f(pt + qw + rz)$	$d(ju + kx + la)+$ $e(mu + nx + oa)+$ $f(pu + qx + ra)$
$g(js + kv + ly)+$ $h(ms + nv + oy)+$ $i(ps + qv + ry)$	$g(jt + kw + lz)+$ $h(mt + nw + oz)+$ $i(pt + qw + rz)$	$g(ju + kx + la)+$ $h(mu + nx + oa)+$ $i(pu + qx + ra)$

Each component of this matrix is the same as that of $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}\cdot$

(*b*) In general, when any number of n-matrices are to be multiplied together as $\mathbf{A}\cdot\mathbf{\Gamma}\cdot\mathbf{B}\cdot\mathbf{C}$, the multiplication may be started with any two neighboring ones and may be continued with any grouping, as for instance $\mathbf{A}\cdot\mathbf{\Gamma}\cdot(\mathbf{B}\cdot\mathbf{C})$ or $\mathbf{A}\cdot(\mathbf{\Gamma}\cdot\mathbf{B})\cdot\mathbf{C}$, etc. *Quite often a particular grouping speeds up the calculations considerably.*

If the components $a, b, c, \cdots j, k, l, \cdots$, etc., of **A**, **B**, and **C** contain "operators," their order in the products of the resultant matrix **M**, such as in ajs, akv, aly, etc., must be kept the same as it follows from the arrow rule.

V. MANIPULATION OF PRODUCTS

(*a*) The *transpose* of the product of two matrices $\mathbf{A} \cdot \mathbf{B}$ is

$$(\mathbf{A} \cdot \mathbf{B})_t = \mathbf{B}_t \cdot \mathbf{A}_t \qquad 2.9$$

A similar formula applies for the *inverse* of the product of two matrices

$$(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} \qquad 2.10$$

In general, the transpose (or the inverse) of a product of matrices is found by taking the transpose (or inverse) of each matrix *in the reverse order*. For instance,

$$(\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C})_t = \mathbf{C}_t \cdot \mathbf{B}_t \cdot \mathbf{A}_t \qquad 2.11$$

$$(\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C})^{-1} = \mathbf{C}^{-1} \cdot \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} \qquad 2.12$$

(*b*) *A product of n-matrices is differentiated by differentiating each n-matrix separately*, just as with scalars. For instance,

$$\frac{\partial \mathbf{A} \cdot \mathbf{e}}{\partial \mathbf{x}} = \frac{\partial \mathbf{A}}{\partial \mathbf{x}} \cdot \mathbf{e} + \mathbf{A} \cdot \frac{\partial \mathbf{e}}{\partial \mathbf{x}} \;\Big|\; \frac{\partial A_{\alpha\beta} e_\beta}{\partial x_\gamma} = \frac{\partial A_{\alpha\beta}}{\partial x_\gamma} e_\beta + A_{\alpha\beta} \frac{\partial e_\beta}{\partial x_\gamma} \qquad 2.13$$

In direct notation the *order* of the n-matrices should not be interchanged.

Similarly the *differential* of products is found by taking the differential of each n-matrix

$$d(\mathbf{A} \cdot \mathbf{e}) = (d\mathbf{A}) \cdot \mathbf{e} + \mathbf{A} \cdot d\mathbf{e} \;\big|\; d(A_{\alpha\beta} e_\beta) = (dA_{\alpha\beta}) e_\beta + A_{\alpha\beta} de_\beta \qquad 2.14$$

(*c*) The *integration* of products will be taken up later.

VI. "SYMMETRIC" AND "SKEW-SYMMETRIC" MATRICES

(*a*) A line drawn from the upper left-hand corner of a matrix to the lower right-hand corner is called the "*main diagonal*" line.

(*b*) A matrix is called "*symmetric*" if the same components occur on both sides of the main diagonal line such as

$$\mathbf{A} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 8 & -2 & 9 \\ b & 8 & 7 & 5 & -6 \\ c & -2 & 5 & 3 & 4 \\ d & 9 & -6 & 4 & 5 \end{array}$$

The transpose of a symmetrical matrix is identical with the original matrix. That is, if **A** is symmetrical, then $\mathbf{A} = \mathbf{A}_t$.

A matrix is called "*skew-symmetric*" ift he same components occur on both sides of the main diagonal line, but with opposite signs,

$$\mathbf{A} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 8 & 9 & -2 \\ b & -8 & 0 & -3 & 0 \\ c & -9 & 3 & 0 & 7 \\ d & 2 & 0 & -7 & 0 \end{array}$$

and if the components along the main diagonal line are zero.

(*c*) *Any general matrix* **A** *may be divided into the sum of two matrices* $\mathbf{B} + \mathbf{C}$ *where* **B** *is a symmetric matrix and* **C** *is a skew-symmetric matrix.* That is

$$\mathbf{A} = \mathbf{B} + \mathbf{C} \qquad \Big| \qquad A_{\alpha\beta} = B_{\alpha\beta} + C_{\alpha\beta}$$

where the symmetrical part is found by

$$\mathbf{B} = \frac{\mathbf{A} + \mathbf{A}_t}{2} \qquad \Big| \qquad B_{\alpha\beta} = \frac{A_{\alpha\beta} + A_{\beta\alpha}}{2} \qquad 2.15$$

and the skew-symmetric part is found by

$$\mathbf{C} = \frac{\mathbf{A} - \mathbf{A}_t}{2} \qquad \Big| \qquad C_{\alpha\beta} = \frac{A_{\alpha\beta} - A_{\beta\alpha}}{2} \qquad 2.16$$

For instance the matrix

$$\mathbf{A} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & K \end{bmatrix} \qquad \mathbf{A}_t = \begin{bmatrix} A & D & G \\ B & E & H \\ C & F & K \end{bmatrix}$$

may be expressed as the sum of two matrices, one symmetric, the other skew-symmetric:

$$\mathbf{A} = \tfrac{1}{2}\begin{array}{|c|c|c|}\hline A+A & B+D & C+G \\ \hline D+B & E+E & F+H \\ \hline G+C & H+F & K+K \\ \hline\end{array} + \tfrac{1}{2}\begin{array}{|c|c|c|}\hline 0 & B-D & C-G \\ \hline D-B & 0 & F-H \\ \hline G-C & H-F & 0 \\ \hline\end{array}$$

VII. MANIPULATION OF MATRIC EQUATIONS

(*a*) *An equation in which every symbol is an n-matrix will be called a "matric equation."* Each side of the equation is composed of the sum of several terms, such as

$P = \mathbf{i}\cdot\mathbf{L}\cdot\mathbf{i}$	$P = L_{\alpha\beta}i_\alpha i_\beta$	2.17
$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{a}\cdot\frac{d\mathbf{i}}{dt} + \mathbf{i}\cdot\mathbf{\Gamma}\cdot\mathbf{i}$	$e_\alpha = R_{\alpha\beta}i_\beta + a_{\alpha\beta}\frac{di_\beta}{dt} + \Gamma_{\alpha\beta\gamma}i_\beta i_\gamma$	2.18
$\mathbf{Z}' + \mathbf{Z}'' = \mathbf{R} + \mathbf{L}\frac{d}{dt}$	$Z'_{\alpha\beta} + Z''_{\alpha\beta} = R_{\alpha\beta} + L_{\alpha\beta}\frac{d}{dt}$	2.19

(*b*) It is important to note that *in a matric equation each term has the same dimension.* That is, each term is 0-matrix, as in equation 2.17, or a 1-matrix as in equation 2.18, or a 2-matrix as in equation 2.19, etc. In index notation *each term has the same free indices.* In equation 2.17 none of the terms has free indices; in equation 2.18 the free index in each term is α; in equation 2.19 they are α and β. Each term may contain, on the other hand, any number of dummy indices.

(*c*) In manipulating a matric equation it must be remembered that *only a 2-matrix (or products of n-matrices forming a 2-matrix) can be transferred to the other side of the equation* by multiplying both sides of the equation by the inverse of the matrix. Products of n-matrices forming a resultant 1-matrix or a 3-matrix cannot be carried from one side of the equation to the other. Also *this transferable matrix must have one free index,* which in direct notation is equivalent to having the matrix at the beginning or at the end of a term and not in the middle. For instance let

$\mathbf{x}\cdot\mathbf{\Gamma}\cdot\mathbf{i} = \mathbf{e}$	$\Gamma_{\alpha\beta\gamma}x_\alpha i_\beta = e_\gamma$

Combining the 3-matrix $\mathbf{\Gamma}$ and the 1-matrix $\mathbf{x}$ into a 2-matrix

$\mathbf{x}\cdot\mathbf{\Gamma} = \mathbf{A}$	$\Gamma_{\alpha\beta\gamma}x_\alpha = A_{\beta\gamma}$

it can be transferred to the right-hand side by multiplying over both sides with the inverse of **A**, that is

$\mathbf{A}\cdot\mathbf{i} = \mathbf{e}$	$A_{\beta\gamma}i_\beta = e_\gamma$
$\mathbf{i} = \mathbf{A}^{-1}\cdot\mathbf{e}$	$i_\beta = (A_{\beta\gamma})^{-1}e_\gamma$
$\mathbf{i} = (\mathbf{x}\cdot\mathbf{\Gamma})^{-1}\cdot\mathbf{e}$	$i_\beta = (\Gamma_{\alpha\beta\gamma}x_\alpha)^{-1}e_\gamma$

(*d*) In the manipulation of matric equations it is often necessary to replace a 1-matrix or a 2-matrix **A** by $\mathbf{A}\cdot\mathbf{I}$ or $\mathbf{I}\cdot\mathbf{A}$, where **I** is the "unit matrix" (Equation 1.5), having unity in its diagonal components and zero elsewhere. (It should be noted that $\mathbf{I}_t = \mathbf{I}$ and $\mathbf{I}^{-1} = \mathbf{I}$. Also multiplication with **I** leaves any *n*-matrix unchanged.) Such cases occur where a common *n*-matrix is to be factored out from a sum of terms. For instance,

$\mathbf{A} - \mathbf{i}\cdot\mathbf{\Gamma}\cdot\mathbf{A} = \mathbf{B}$	$A_{\alpha\beta} - \Gamma_{\gamma\alpha\delta}i_\gamma A_{\delta\beta} = B_{\alpha\beta}$
$(\mathbf{I} - \mathbf{i}\cdot\mathbf{\Gamma})\cdot\mathbf{A} = \mathbf{B}$	$(I_{\alpha\delta} - \Gamma_{\gamma\alpha\delta}i_\gamma)A_{\delta\beta} = B_{\alpha\beta}$
$\mathbf{A} = (\mathbf{I} - \mathbf{i}\cdot\mathbf{\Gamma})^{-1}\cdot\mathbf{B}$	$A_{\delta\beta} = (I_{\alpha\delta} - \Gamma_{\gamma\alpha\delta}\mathrm{i}_\gamma)^{-1}B_{\alpha\beta}$

VIII. MANIPULATION OF INDICES

In manipulating equations in index notation the following should be kept in mind:

1. The free index in every term must be the same, but it may be changed from one letter to another. For instance, in the equation

$$i_\alpha = A_{\alpha\beta}e_\beta + M_{\alpha\beta\gamma}e_\beta e_\gamma$$

the free index α may be changed to ω

$$i_\omega = A_{\omega\beta}e_\beta + M_{\omega\beta\gamma}e_\beta e_\gamma$$

It should be noted that to avoid confusion the new free index should not be the same as any of the dummy indices.

2. The dummy indices may be changed in every term separately. For instance, the equation

$$i_\alpha = A_{\alpha\beta}e_\beta + M_{\alpha\beta\gamma}e_\beta e_\gamma$$

may also be written as

$$i_\alpha = A_{\alpha\delta}e_\delta + M_{\alpha\epsilon\sigma}e_\epsilon e_\sigma$$

3. The same dummy index, say β, may appear in two or more terms at will. In an expression like

$$A_{\alpha\beta}e_\beta + B_{\alpha\gamma}e_\gamma$$

the vector e_β may be factored out by changing the dummy index γ in the second term to β, giving

$$A_{\alpha\beta}e_\beta + B_{\alpha\beta}e_\beta = (A_{\alpha\beta} + B_{\alpha\beta})e_\beta$$

IX. "FORMS"

(*a*) In a matric equation each term is either a 0-matrix, or a 1-matrix, or a 2-matrix, etc. If each term is a 0-matrix (a single number), then each term is called a "*form*." A "form" such as $A_{\alpha\beta}x_\alpha x_\beta$ consists of two types of components:

1. The "variables" x_α or i_α that may occur once or more than once. There may be two or more types of variables as x_α and y_α.
2. The "coefficients" of the variables as $A_{\alpha\beta}$. The coefficients of the form may be the products of several n-matrices.

(*b*) According to the number and type of the variables the following "forms" may be distinguished:

1. $\mathbf{e}\cdot\mathbf{i}$ or $e_\alpha i_\alpha$ is a "*linear*" form.
2. $\mathbf{i}\cdot\mathbf{L}\cdot\mathbf{i}$ or $L_{\alpha\beta}i_\alpha i_\beta$ is a "*quadratic*" form.
3. $A_{\alpha\beta\gamma}i_\alpha i_\beta i_\gamma$ is a "*trilinear*" form.
4. $B_{\alpha\beta\gamma\delta} \ldots . i_\alpha i_\beta i_\gamma i_\delta \ldots .$ is a "*multilinear*" form.

These types of forms contain only one type of variable i_α. With two types of variables the following form often occurs:

5. $\mathbf{x}\cdot\mathbf{D}\cdot\mathbf{y}$ or $D_{\alpha\beta}x_\alpha y_\beta$ is a "*bilinear*" form.

(*c*) The variables i_α in the above forms may be replaced by the "differentials" dx_α, in which case the forms are called "*differential forms*." With i_α or x_α they are called "*algebraic forms*." The most important types of "differential forms" are:

1. The *linear* differential form $e_\alpha dx_\alpha$.
2. The *quadratic* differential form $a_{\alpha\beta}dx_\alpha dx_\beta$.

(*d*) In mechanical or electrical problems, an example of a linear form is

1. Power input $= P = \mathbf{e}\cdot\mathbf{i} = e_\alpha i_\alpha$.

Examples of quadratic forms are:

2. Stored magnetic (or kinetic) energy $= 2T = \mathbf{i}\cdot\mathbf{a}\cdot\mathbf{i} = a_{\alpha\beta}i_\alpha i_\beta$.
3. Dissipation function $= 2F = \mathbf{i}\cdot\mathbf{r}\cdot\mathbf{i} = r_{\alpha\beta}i_\alpha i_\beta$.
4. Stored electrostatic (elastic) energy $= 2V = \mathbf{x}\cdot\mathbf{s}\cdot\mathbf{x} = s_{\alpha\beta}x_\alpha x_\beta$.

When i_α is expressed as dx_α/dt, the first three of the above four forms may be considered as " differential forms " instead of " algebraic forms."

(*e*) If each term in a matric equation is a 1-matrix, each term may be considered as a " *set of forms.*" For instance:

1. $\mathbf{Z} \cdot \mathbf{i}$ or $Z_{\alpha\beta} i_\beta$ is a set of linear forms.
2. $\Gamma_{\alpha\beta\gamma} x_\alpha y_\beta$ is a set of bilinear forms.

(*f*) *The significance of the concept of " form " is that, whereas the variety of matric equations is great, the variety of their component parts, the " forms," is small.* An extensive mathematical literature has been built up on the theory of forms, dealing with their characteristics, their reduction to simplified forms, etc. All textbooks on modern algebra or higher algebra chiefly deal with the theory of " forms." Their theory leads often to reduction in the *numerical* calculations necessary.

(*g*) The following theorem on quadratic forms may be considered as an example of simplification. It was shown in Section VII that any matrix may be considered as the sum of a symmetrical and a skew-symmetrical matrix. The theorem states that:

> In any quadratic form $\mathbf{i} \cdot \mathbf{G} \cdot \mathbf{i}$ the matrix $\mathbf{G}$ may always be replaced by its symmetrical part $(\mathbf{G} + \mathbf{G}_t)/2$.

That is, *the coefficient of a quadratic form*, algebraic or differential, *is always a symmetrical matrix*, since the asymmetrical part reduces the form to zero. That is

$$\mathbf{i} \cdot \left(\frac{\mathbf{G} - \mathbf{G}_t}{2}\right) \cdot \mathbf{i} = 0 \qquad \Bigg| \qquad \left(\frac{G_{\alpha\beta} - G_{\beta\alpha}}{2}\right) i_\alpha i_\beta = 0$$

For instance if:

$$\mathbf{G} = \begin{array}{c|cc} & \mathbf{a} & \mathbf{b} \\ \hline \mathbf{a} & A & B \\ \mathbf{b} & C & D \end{array} \qquad \mathbf{G}_t = \begin{array}{c|cc} & \mathbf{a} & \mathbf{b} \\ \hline \mathbf{a} & A & C \\ \mathbf{b} & B & D \end{array} \qquad \mathbf{i} = \begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \hline i_a & i_b \end{array}$$

$$\frac{\mathbf{G} - \mathbf{G}_t}{2} = \begin{array}{c|cc} & \mathbf{a} & \mathbf{b} \\ \hline \mathbf{a} & 0 & (B - C)/2 \\ \mathbf{b} & -(B - C)/2 & 0 \end{array}$$

$$(1/2)(\mathbf{G} - \mathbf{G}_t) \cdot \mathbf{i} = \begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \hline i_b(B - C)/2 & -i_a(B - C)/2 \end{array}$$

$$(1/2)\mathbf{i} \cdot (\mathbf{G} - \mathbf{G}_t) \cdot \mathbf{i} = (1/2)[i_b(B - C)i_a - i_a(B - C)i_b] = 0$$

Hence

$$\mathbf{i}\cdot\mathbf{G}\cdot\mathbf{i} = \mathbf{i}\cdot\frac{\mathbf{G}+\mathbf{G}_t}{2}\cdot\mathbf{i} \tag{2.20}$$

X. THE "FIRST GENERALIZATION POSTULATE"

(*a*) The interesting fact should be noted that a *single coil* is characterized by the quantities

$$e,\ i,\ R,\ L,\ C,\ \text{and}\ Z$$

while a *set of coils* is characterized by the n-matrices

$$\mathbf{e},\ \mathbf{i},\ \mathbf{R},\ \mathbf{L},\ \mathbf{C},\ \text{and}\ \mathbf{Z}$$

That is, a set of coils is characterized by the same type and number of symbols as a single coil, with the difference that *single numbers are replaced by n-matrices of various dimensions.*

Hence it should be noted that an n-matrix represents not just any haphazard collection of numbers. *The components of each n-matrix correspond to some definite physical (or geometrical) concept, and in each problem only as many n-matrices can be introduced as there are physical (or geometrical) concepts involved.* Their number can be increased or decreased only by following strict rules that correspond mathematically to the physics of the problem at hand.

(*b*) Later on it will be found that in general complex systems are not only characterized by the same number of symbols as simple systems, but the *whole method of reasoning* to be used in their performance analysis follows quite closely the analysis of a simple system, with the difference that each quantity is replaced by an n-matrix in the analysis.

In other words, *before analyzing any complex system with several variables, it will be found advantageous first to analyze a similar system having only one (or more) degrees of freedom. Then the same reasoning can be repeated in substantially the same manner, replacing each quantity by an appropriate n-matrix, however.* It will then be found also that the *final equation* of the complex system with n degrees of freedom will look substantially the same as that of a simple system with one (or more) degrees of freedom, each quantity being replaced, however, by an n-matrix.

This labor-saving device will be called "*the First Generalization Postulate,*" and it can be stated as follows:

"*The method of analysis and the final equations describing the performance of a complex physical system (with n degrees of freedom) may be obtained by following step by step those of the simplest but most*

general unit of the system, provided each quantity is replaced by an appropriate n-matrix." The simplest unit of the system may have one or more degrees of freedom.

It seems logical that the mere *collection* of system elements should not introduce any physical phenomenon that has not already been contained in the simplest element, hence *new symbols, representing new physical concepts, should not arise in combining several elements.* It is not always obvious, though, what the simplest system element is, and not just *any* formulation of the method of analysis and of the equation for the simplest element is capable of generalization. It will be found usually that only the shortest (but still more general) and at the same time the most *elegant* scalar formulation of the physical phenomenon is in a form to be replacable by an n-matric expression. The scalar formulation may involve, of course, one or more equations.

It should be remembered that the simplest unit of a system may contain two or more variables, also that a system may be composed of two or more radically different types of elements so that each of these elements may require a different equation. Also a system containing *one* type of element may be divided into *several* groups according to the different behavior of the various elements.

(*c*) Since the manipulation of n-matrices differs slightly from that of ordinary quantities, in the parallel analysis the ordinary equations should have the following forms:

1. Instead of dividing by a number as $1/Z$ multiply by Z^{-1}, corresponding to the inverse of $\mathbf{Z}$ which is written as $\mathbf{Z}^{-1}$.

2. Instead of squaring a number as Z^2, use it in the form ZZ, corresponding to $\mathbf{Z} \cdot \mathbf{Z}$.

3. The quantities should be kept in the proper order as the arrangement is built up by the analysis if the n-matrices are to be expressed in direct notation. If the index notation is used, the order is immaterial in the absence of operators.

These rules for the manipulation of quantities are about the same as those of " operators " like $p = d/dt$.

(*d*) If the analysis of a problem is carried through in terms of n-matrices, it will eliminate the necessity of carrying along for page after page a set of ordinary equations. *It is emphasized, however, that there are certain types of operations and a minimum number of manipulations that are not eliminated by the use of n-matrices.* Such necessary operations are, for instance:

1. In the analysis of a set of linear equations a certain amount of determinant calculations cannot be avoided.

2. *In the solution of a set of differential equations, the finding of the*

roots of an nth degree algebraic equation cannot be avoided by the use of any imaginable symbolism.

In particular it is emphasized that, once the *final* equations are arrived at by either method, the same amount of numerical additions and multiplications is needed to get the numerical answer. However, it *often* happens that the final answer arrived at in terms of n-matrices is in a far more organized form than an answer expressed in terms of ordinary quantities and hence it requires less numerical labor; or owing to its organized form the numerical calculation may be performed by a computer on a calculating machine. Also it often happens *that the use of n-matrices suggests a new method of attack which is unsuspected when the usual reasoning is followed and so the final answer comes out in a new form requiring far less numerical work.* Numerous examples of such cases will be given later. That is, *the use of n-matrices in general effects great savings*:

1. *In the analytical work.*
2. *In the numerical calculations.*

It should be remarked that *it is not always apparent just what quantities should be grouped together to form an n-matrix.* Often several choices are open, but usually only one selection leads to the simplest formulation of a problem. More often the organization into n-matrices appears to be hopeless at first sight. One of the purposes of this treatise is to set up basic principles for organizing certain electrical engineering concepts in terms of n-way matrices.

XI. STATIONARY NETWORKS

A very simple example of the First Generalization Postulate is the analysis of a stationary network. The relation between the impressed voltages e and $\mathbf{e}$, the currents i and $\mathbf{i}$, and the impedances Z and $\mathbf{Z}$ of a *single coil* and of a *network* with n degrees of freedom is given by Ohm's law for both cases as

$$e = Zi \qquad\qquad \mathbf{e} = \mathbf{Z} \cdot \mathbf{i} \qquad 2.21$$

If the voltages are known and the currents are unknown, these equations may be solved by multiplying both sides by the inverse of Z or $\mathbf{Z}$

$$Z^{-1}e = Z^{-1}Zi \qquad\qquad \mathbf{Z}^{-1} \cdot \mathbf{e} = \mathbf{Z}^{-1} \cdot \mathbf{Z} \cdot \mathbf{i}$$

$$Z^{-1}e = i \qquad\qquad \mathbf{Z}^{-1} \cdot \mathbf{e} = \mathbf{i}$$

That is, *the currents* $\mathbf{i}$ *are found by finding first the inverse of* $\mathbf{Z}$ *and then*

multiplying it by **e**. If the inverse of the impedance is defined as the admittance and denoted by a new symbol, then

$$i = Z^{-1}e \qquad \Bigg| \qquad \mathbf{i} = \mathbf{Z}^{-1}\cdot\mathbf{e}$$

$$i = Ye \qquad \Bigg| \qquad \mathbf{i} = \mathbf{Y}\cdot\mathbf{e} \qquad 2.22$$

For instance, the inverse of matrix **Z** given in Section II is found to be:

$$\mathbf{Y} = \mathbf{Z}^{-1} = \begin{array}{c|cccc|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \hline \mathbf{a} & Yaa & Yab & Yac & Yad \\ \mathbf{b} & Yab & Ybb & Ybc & Ybd \\ \mathbf{c} & Yac & Ybc & Ycc & Ycd \\ \mathbf{d} & Yad & Ybd & Ycd & Ydd \\ \hline \end{array}$$

This matrix, the "*admittance matrix*" represents the self- and mutual admittances of the various coils of Fig. 2.1 assuming that, when the admittance of one coil is measured, *all the other coils are short-circuited.* This matrix is also symmetrical if **Z** is symmetrical.

The current flowing in the various circuits of Fig. 2.1 due to the impressed voltage **e** given in Section II is then

$$\mathbf{i} = \mathbf{Y}\cdot\mathbf{e} = \begin{array}{c|c|} \hline \mathbf{a} & Y_{aa}e_a + Y_{ab}e_b + Y_{ac}e_c + Y_{ad}e_d \\ \mathbf{b} & Y_{ab}e_a + Y_{bb}e_b + Y_{bc}e_c + Y_{bd}e_d \\ \mathbf{c} & Y_{ac}e_a + Y_{bc}e_b + Y_{cc}e_c + Y_{cd}e_d \\ \mathbf{d} & Y_{ac}e_a + Y_{bd}e_b + Y_{cd}e_c + Y_{dd}e_d \\ \hline \end{array}$$

XII. TRAVELING WAVES ON MULTICONDUCTORS *

(*a*) As an example of applying the First Generalization Postulate to a set of *differential equations*, consider the motion of electromagnetic waves along a transmission system.

For a *single conductor*, let the resistance of a unit length be R, its inductance L, its leakage conductance G, and its capacity C. Then along an infinitesimal distance dx the voltage drop is

$$-de = -\frac{\partial e}{\partial x}dx = Ridx + L\frac{di}{dt}dx = \left(R + L\frac{d}{dt}\right)idx$$

and the current change is

$$-di = -\frac{\partial i}{\partial x}dx = Gedx + C\frac{de}{dt}dx = \left(G + C\frac{d}{dt}\right)edx$$

* This section may be left out at first reading.

If $R + L\, d/dt = Z$, also $G + C\, d/dt = Y$, and if dx is cancelled, then the equations become

$$-\frac{\partial e}{\partial x} = Zi \quad \text{and} \quad -\frac{\partial i}{\partial x} = Ye \qquad 2.23$$

(*b*) Now let a transmission system with *n parallel conductors* be considered with electromagnetic and electrostatic coupling between them. The resistances and the self- and mutual inductances may be arranged in the form of a matrix

$$\mathbf{Z} = \begin{array}{c|cccc|} & \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline \mathbf{a} & Z_{aa} & Z_{ab} & \cdots & Z_{an} \\ \mathbf{b} & Z_{ba} & Z_{bb} & \cdots & Z_{bn} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{n} & Z_{na} & Z_{nb} & \cdots & Z_{nn} \\ \hline \end{array}$$

where $Z_{aa} = R_{aa} + L_{aa}\, d/dt$. The leakage conductances and the self- and mutual capacities also may be arranged in the form of a matrix

$$\mathbf{Y} = \begin{array}{c|cccc|} & \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline \mathbf{a} & Y_{aa} & Y_{ab} & \cdots & Y_{an} \\ \mathbf{b} & Y_{ba} & Y_{bb} & \cdots & Y_{bn} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{n} & Y_{na} & Y_{nb} & \cdots & Y_{nn} \\ \hline \end{array}$$

where $Y_{aa} = G_{aa} + C_{aa}\, d/dt$. (This matrix is not the inverse of **Z**, being independent of it.)

Let the potentials of the various conductors be arranged in a row to form a 1-matrix

$$\mathbf{e} = \begin{array}{|cccc|} \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline e_a & e_b & \cdots & e_n \\ \hline \end{array}$$

Similarly the currents form the 1-matrix

$$\mathbf{i} = \begin{array}{|cccc|} \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline i_a & i_b & \cdots & i_n \\ \hline \end{array}$$

Since the conductors are parallel, the distance along each conductor may be considered as a single quantity x.

Altogether five n-matrices are introduced, in particular two 2-matrices **Z** and **Y**, two 1-matrices **e** and **i**, and one 0-matrix x.

(*c*) The differential equations containing only voltages will be developed for a single conductor and for n conductors in parallel columns. The equations for n conductors will be given in both direct and index notations.

The voltage drops are

$$-\frac{\partial e}{\partial x} = Zi \qquad \Bigg| \qquad -\frac{\partial \mathbf{e}}{\partial x} = \mathbf{Z}\cdot\mathbf{i} \qquad \Bigg| \qquad -\frac{\partial e_\alpha}{\partial x} = Z_{\alpha\beta} i_\beta$$

The current changes are

$$-\frac{\partial i}{\partial x} = Ye \qquad \Bigg| \qquad -\frac{\partial \mathbf{i}}{\partial x} = \mathbf{Y}\cdot\mathbf{e} \qquad \Bigg| \qquad -\frac{\partial i_\beta}{\partial x} = Y_{\beta\gamma} e_\gamma$$

Differentiating the first equation with respect to x

$$-\frac{\partial^2 e}{\partial x^2} = Z\frac{\partial i}{\partial x} \qquad \Bigg| \qquad -\frac{\partial^2 \mathbf{e}}{\partial x^2} = \mathbf{Z}\cdot\frac{\partial \mathbf{i}}{\partial x} \qquad \Bigg| \qquad -\frac{\partial^2 e_\alpha}{\partial x^2} = Z_{\alpha\beta}\frac{\partial i_\beta}{\partial x}$$

Substituting the second equation into the last equation, when the signs are changed, results in

$$\frac{\partial^2 e}{\partial x^2} = ZYe \qquad \Bigg| \qquad \frac{\partial^2 \mathbf{e}}{\partial x^2} = \mathbf{Z}\cdot\mathbf{Y}\cdot\mathbf{e} \qquad \Bigg| \qquad \frac{\partial^2 e_\alpha}{\partial x^2} = Z_{\alpha\beta} Y_{\beta\gamma} e_\gamma \qquad 2.24$$

The last equation gives the differential equations for the propagation of the voltage waves **e** which are functions of x and t. It is emphasized that the analysis can be carried through clear to the complete solution of the differential equations in terms of n-matrices. However, *the use of n-matrices does not eliminate the necessity of finding the roots of an nth* (*or 2nth*) *degree algebraic equation.* The use of n-matrices eliminates only the carrying along of n ordinary equations page after page.

(*d*) In order to show the set of n ordinary equations that are represented by the single matric equation 2.24 let the indicated operations be performed.

$$\mathbf{Z}\cdot\mathbf{Y} = \begin{array}{c|c|c|c|c|} & \mathbf{a} & \mathbf{b} & & \mathbf{n} \\ \hline \mathbf{a} & Z_{aa}Y_{aa} + Z_{an}Y_{na} & Z_{aa}Y_{ab} + Z_{an}Y_{nb} & \cdots & Z_{aa}Y_{an} + Z_{an}Y_{nn} \\ \hline \mathbf{b} & Z_{ba}Y_{aa} + Z_{bn}Y_{na} & Z_{ba}Y_{ab} + Z_{bn}Y_{nb} & \cdots & Z_{ba}Y_{an} + Z_{bn}Y_{nn} \\ \hline & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \hline \mathbf{n} & Z_{na}Y_{aa} + Z_{nn}Y_{na} & Z_{na}Y_{ab} + Z_{nn}Y_{nb} & \cdots & Z_{na}Y_{an} + Z_{nn}Y_{nn} \\ \hline \end{array}$$

$$\mathbf{Z}\cdot\mathbf{Y}\cdot\mathbf{e} = \begin{array}{c|c|}
\mathbf{a} & (Z_{aa}Y_{aa} + Z_{an}Y_{na})e_a + (Z_{aa}Y_{ab} + Z_{an}Y_{nb})e_b + (Z_{aa}Y_{an} + Z_{an}Y_{nn})e_n \\
\mathbf{b} & (Z_{ba}Y_{aa} + Z_{bn}Y_{na})e_a + (Z_{ba}Y_{ab} + Z_{bn}Y_{nb})e_b + (Z_{ba}Y_{an} + Z_{bn}Y_{nn})e_n \\
\vdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbf{n} & (Z_{na}Y_{aa} + Z_{nn}Y_{na})e_a + (Z_{na}Y_{ab} + Z_{nn}Y_{nb})e_b + (Z_{na}Y_{an} + Z_{nn}Y_{nn})e_n
\end{array}$$

$$\frac{\partial^2\mathbf{e}}{\partial x^2} = \begin{array}{|c|c|c|c|}
\mathbf{a} & \mathbf{b} & \ldots & \mathbf{n} \\
\hline
\dfrac{\partial^2 e_a}{\partial x^2} & \dfrac{\partial^2 e_b}{\partial x^2} & \cdots & \dfrac{\partial^2 e_n}{\partial x^2}
\end{array}$$

Equating corresponding components of the last two 1-matrices according to equation 2.24, the set of n ordinary equations representing the single matric equation 2.24 are:

$$\left.\begin{aligned}
\frac{\partial^2 e_a}{\partial x^2} &= (Z_{aa}Y_{aa} + \cdots Z_{an}Y_{na})e_a + (Z_{aa}Y_{ab} + \cdots Z_{an}Y_{nb})e_b \\
&\quad + \cdots (Z_{zz}Y_{an} + \cdots Z_{an}Y_{nn})e_n \\
\frac{\partial^2 e_b}{\partial x^2} &= (Z_{ba}Y_{aa} + \cdots Z_{bn}Y_{na})e_a + (Z_{ba}Y_{ab} + \cdots Z_{bn}Y_{nb})e_b \\
&\quad + \cdots (Z_{ba}Y_{an} + \cdots Z_{bn}Y_{nn})e_n \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\frac{\partial^2 e_n}{\partial x^2} &= (Z_{na}Y_{aa} + \cdots Z_{nn}Y_{na})e_a + (Z_{na}Y_{ab} + \cdots Z_{nn}Y_{nb})e_b \\
&\quad + \cdots (Z_{na}Y_{an} + \cdots Z_{nn}Y_{nn})e_n
\end{aligned}\right\} \quad 2.25$$

The differential equations analogous to equation 2.24 in terms of currents are:

$$\frac{\partial^2 i}{\partial x^2} = YZi \qquad \Bigg| \qquad \frac{\partial^2 \mathbf{i}}{\partial x^2} = \mathbf{Y}\cdot\mathbf{Z}\cdot\mathbf{i} \qquad \Bigg| \qquad \frac{\partial^2 i_\alpha}{\partial x^2} = Y_{\alpha\beta}Z_{\beta\gamma}i_\gamma \quad 2.26$$

The matrix $\mathbf{Y}\cdot\mathbf{Z}$ is different from $\mathbf{Z}\cdot\mathbf{Y}$.

XIII. POWER SERIES DEVELOPMENT *

(*a*) To show the use of the First Generalization Postulate in a problem where 3-matrices and n-matrices of higher dimensions occur, let the development of several functions of *several* variables in terms of power series be considered. Such cases occur, for instance, when a *non-linear* relation exists between quantities, such as between currents and voltages in vacuum tubes, or between m.m.f.'s and fluxes in

* This section may be left out at the first reading.

saturated magnetic circuits, etc. A power series development of the variables is necessary when a set of equations cannot be solved otherwise.

First *one* function of *one* variable will be developed in terms of single quantities, then the same steps will be repeated in terms of n-matrices for *several* functions of *several* variables.

(*b*) Any curve in a plane $y = f(x)$ can be expressed as a power series

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \qquad 2.27$$

where the coefficients A, B, C, $D \cdots$ are known or unknown quantities (provided certain conditions stated in textbooks are satisfied).

(*c*) A curve in a three-dimensional space is represented by the intersection of two surfaces

$$\begin{aligned} y_a &= f_a(x_a, x_b) \\ y_b &= f_b(x_a, x_b) \end{aligned} \qquad 2.28$$

Each of the dependent variables y_a and y_b may be expressed as a power series in terms of the independent variables x_a and x_b such as

$$\begin{aligned} y_a = A_a &+ (B_{aa}x_a + B_{ab}x_b) + \\ &+ (C_{aaa}x_a^2 + C_{aab}x_ax_b + C_{aba}x_bx_a + C_{abb}x_b^2) + \\ &+ (D_{aaaa}x_a^3 + D_{aaab}x_a^2x_b + D_{aaba}x_a^2x_b + D_{aabb}x_ax_b^2 + \\ &+ D_{abaa}x_bx_a^2 + D_{abab}x_ax_b^2 + D_{abba}x_ax_b^2 + D_{abbb}x_b^3) + \\ &+ (E_{aaaaa}x_a^4 + E_{aaaab}x_a^3x_b + \cdots) \\ &+ \cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \end{aligned} \qquad 2.29$$

The coefficients of the variables as D_{abaa} are known or unknown quantities. A similar equation can be written for y_b

$$\begin{aligned} y_b = A_b &+ (B_{ba}x_a + B_{bb}x_b) \\ &+ (C_{baa}x_a^2 + C_{bab}x_ax_b + C_{bba}x_bx_a + C_{bbb}x_b^2) \\ &+ (D_{baaa}x_a^3 + D_{baab}x_a^2x_b + D_{baba}x_a^2x_b + D_{babb}x_ax_b^2 \\ &+ D_{bbaa}x_bx_a^2 + D_{bbab}x_ax_b^2 + D_{bbba}x_ax_b^2 + D_{bbbb}x_b^3) \\ &+ (E_{baaaa}x_a^4 + E_{baaab}x_a^3x_b + \cdots) \\ &+ \cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \end{aligned} \qquad 2.30$$

If instead of two functions and two variables there are n functions and n variables (all real)

$$\begin{aligned} y_a &= f_a(x_a,\ x_b,\ x_c \cdots x_n) \\ y_b &= f_b(x_a,\ x_b,\ x_c \cdots x_n) \\ y_c &= f_c(x_a,\ x_b,\ x_c \cdots x_n) \\ &\cdots\cdots\cdots\cdots \\ y_n &= f_n(x_a,\ x_b,\ x_c \cdots x_n) \end{aligned} \tag{2.31}$$

there are n such series equations as the foregoing, each parenthesis containing n, n^2, $n^3 \cdots$ terms instead of 2, 2^2, $2^3 \cdots$ terms.

(*d*) In order to represent the n ordinary equations as *one* matric equation, let the following n-matrices be defined:

1. All the dependent variables are arranged in a row forming a 1-matrix

$$y_\alpha = \begin{array}{c} \begin{array}{ccccc} a & b & c & \cdots & n \end{array} \\ \begin{array}{|c|c|c|c|c|} \hline y_a & y_b & y_c & \cdots & y_n \\ \hline \end{array} \end{array}$$

2. All the independent variables are arranged in a 1-matrix

$$x_\alpha = \begin{array}{c} \begin{array}{ccccc} a & b & c & \cdots & n \end{array} \\ \begin{array}{|c|c|c|c|c|} \hline x_a & x_b & x_c & \cdots & x_n \\ \hline \end{array} \end{array}$$

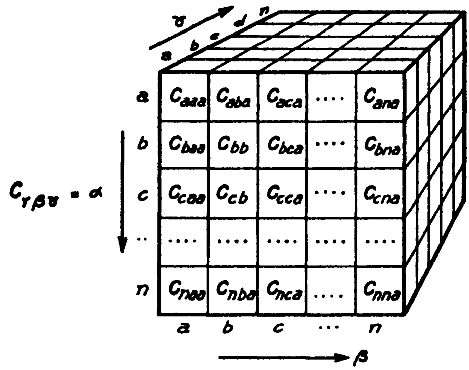
3. All the A coefficients are arranged in a 1-matrix

$$A_\alpha = \begin{array}{c} \begin{array}{ccccc} a & b & c & \cdots & n \end{array} \\ \begin{array}{|c|c|c|c|c|} \hline A_a & A_b & A_c & \cdots & A_n \\ \hline \end{array} \end{array}$$

4. All the B coefficients of x_α are arranged in a square forming a 2-matrix

$$B_{\alpha\beta} = \begin{array}{c|c|c|c|c|c|} \alpha \backslash \beta & a & b & c & \cdots & n \\ \hline a & B_{aa} & B_{ab} & B_{ac} & \cdots & B_{an} \\ \hline b & B_{ba} & B_{bb} & B_{bc} & \cdots & B_{bn} \\ \hline c & B_{ca} & B_{cb} & B_{cc} & \cdots & B_{cn} \\ \hline \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline n & B_{na} & B_{nb} & B_{nc} & \cdots & B_{nn} \\ \hline \end{array}$$

5. All the C coefficients of $x_\alpha x_\beta$ are arranged in a cube forming a 3-matrix.



6. All the D coefficients of $x_\alpha x_\beta x_\gamma$ are arranged into n cubes, forming a 4-matrix $D_{\alpha\beta\gamma\delta}$. The E coefficients form a 5-matrix, $E_{\alpha\beta\gamma\delta\epsilon}$, and so on.

(*e*) *In terms of these n-matrices the n power-series equations are written as one matric equation*

$$y_\alpha = A_\alpha + B_{\alpha\beta}x_\beta + C_{\alpha\beta\gamma}x_\beta x_\gamma + D_{\alpha\beta\gamma\delta}x_\beta x_\gamma x_\delta + E_{\alpha\beta\gamma\delta\epsilon}x_\beta x_\gamma x_\delta x_\epsilon + \cdots \qquad 2.32$$

This equation has the same form as the single equation 2.27 except:

1. Each quantity is replaced by an n-matrix.

2. The nth power of a quantity, say x^4, is replaced by n products $x_\beta x_\gamma x_\delta x_\epsilon$.

It should be noted that in the equation:

1. Each term is a 1-matrix, that is, in each term one free index exists. All the other indices in each term are dummy indices.

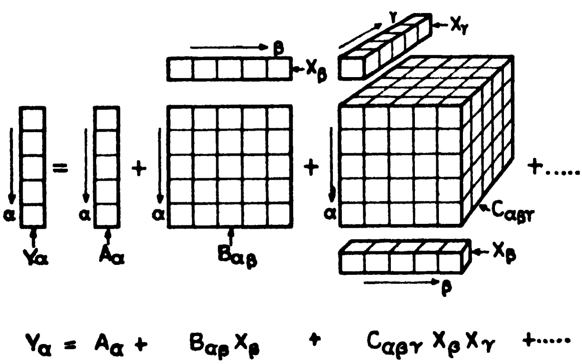


FIG. 2.2.—A Power Series in Terms of n-Way Matrices

2. Each free index on each side of the equation is denoted by the same letter α.

3. In each term a n-matrix is multiplied by the 1-matrix x_α several times. For instance, the 3-matrix $C_{\alpha\beta\gamma}$ is multiplied first by the 1-matrix x_γ giving the 2-matrix $C_{\alpha\beta\gamma}x_\gamma = F_{\alpha\beta}$,

then the 2-matrix $F_{\alpha\beta}$ is multiplied again by the 1-matrix x_β as $F_{\alpha\beta}x_\beta = (C_{\alpha\beta\gamma}x_\gamma)x_\beta$, giving the 1-matrix G_α. Each term in the equation is such a 1-matrix as shown in Fig. 2.2.

XIV. THE INVERSE SERIES *

(*a*) In order to show an example in which 3-matrices are *manipulated*, consider three terms of the above series with y replaced by e and x by i

$$e_\alpha = B_{\alpha\beta}i_\beta + C_{\alpha\beta\gamma}i_\beta i_\gamma + \cdots \qquad 2.33$$

Let it be assumed that the components of $B_{\alpha\beta}$ and $C_{\alpha\beta\gamma}$ are known, also the components of e_α (which represent, say, some impressed voltages on a non-linear system). *The problem is to solve this series for the unknown* i_α, that is, to express i_α as a function of $B_{\alpha\beta}$, $C_{\alpha\beta\gamma}$, and e_α.

The unknown i_β may be expressed as a function of e_α in a power series (called the " inverse " series)

$$i_\beta = K_{\beta\epsilon}e_\epsilon + M_{\beta\epsilon\sigma}e_\epsilon e_\sigma + \cdots \qquad 2.34$$

where the coefficients $K_{\beta\epsilon}$ and $M_{\beta\epsilon\sigma}$ are unknown functions of the previous coefficients $B_{\alpha\beta}$ and $C_{\alpha\beta\gamma}$. The problem is then to find K and M as a function of B and C.

(*b*) *Following the First Generalization Postulate, the analysis will be performed first in terms of single quantities.* That is, first the following problem will be solved: given the power series

$$e = Bi + Ci^2 + \cdots \qquad 2.35$$

find the unknown i, that is, in the inverse series

$$i = Ke + Me^2 + \cdots \qquad 2.36$$

express the unknown K and M as a function of the known B and C.

The steps are the following:

1. Substituting the second equation into the first

$$e = B(Ke + Me^2) + C(Ke + Me^2)^2 + \cdots \qquad 2.37$$

Since all expressions in which e occurs in higher than the second power are going to be neglected, the above equation reduces to

$$e = BKe + BMe^2 + CK^2e^2 + \cdots \qquad 2.38$$

$$e = BKe + (BM + CK^2)e^2 + \cdots \qquad 2.39$$

* This section may be left out at the first reading.

2. Equating corresponding coefficients of e and e^2 on each side of the equation

$$1 = BK \tag{2.40}$$

$$0 = (BM + CK^2) \tag{2.41}$$

3. Solving these two equations for the unknown K and M

$$K = B^{-1} \tag{2.42}$$

$$M = - C(B^{-1})^3 = - CK^3 \tag{2.43}$$

4. Hence the value of i in terms of e, B, and C is

$$i = B^{-1}e - C(B^{-1})^3e^2 \tag{2.44}$$

or

$$i = Ke - CK^3e^2 \tag{2.45}$$

where $K = B^{-1}$.

(*c*) *These same steps will now be repeated by replacing each quantity with an n-matrix.*

1. Substituting equation 2.34 into equation 2.33.

$$\begin{aligned} e_\alpha = {} & B_{\alpha\beta}(K_{\beta\epsilon}e_\epsilon + M_{\beta\epsilon\sigma}e_\epsilon e_\sigma) \\ & + C_{\alpha\beta\gamma}(K_{\beta\epsilon}e_\epsilon + M_{\beta\epsilon\sigma}e_\epsilon e_\sigma)(K_{\gamma\omega}e_\omega + M_{\gamma\omega\pi}e_\omega e_\pi) \end{aligned} \tag{2.46}$$

It should be noted that in equation 2.34 the free index is denoted first by β then by γ for the purpose of substitution. Similarly in the latter case the dummy indices are changed to

$$i_\gamma = K_{\gamma\omega}e_\omega + M_{\gamma\omega\pi}e_\omega e_\pi \tag{2.47}$$

to avoid confusion in substituting i_β twice in succession (see Section IX).

Neglecting powers of e_ϵ higher than 2, the above equation is

$$e_\alpha = B_{\alpha\beta}K_{\beta\epsilon}e_\epsilon + B_{\alpha\beta}M_{\beta\epsilon\sigma}e_\epsilon e_\sigma + C_{\alpha\beta\gamma}K_{\beta\epsilon}K_{\gamma\omega}e_\epsilon e_\omega \tag{2.48}$$

Factoring $e_\epsilon e_\sigma$

$$e_\alpha = B_{\alpha\beta}K_{\beta\epsilon}e_\epsilon + (B_{\alpha\beta}M_{\beta\epsilon\sigma} + C_{\alpha\beta\gamma}K_{\beta\epsilon}K_{\gamma\sigma})e_\epsilon e_\sigma \tag{2.49}$$

2. Equating corresponding coefficients of e_ϵ and $e_\epsilon e_\sigma$ on each side of the equation (writing $e_\epsilon I_{\epsilon\alpha}$ for e_α where $I_{\epsilon\alpha}$ is the unit matrix):

$$I_{\epsilon\alpha} = B_{\alpha\beta}K_{\beta\epsilon} \tag{2.50}$$

$$0 = B_{\alpha\delta}M_{\delta\epsilon\sigma} + C_{\alpha\beta\gamma}K_{\beta\epsilon}K_{\gamma\sigma} \tag{2.51}$$

3. Solving these two equations for the unknowns $K_{\alpha\beta}$ and $M_{\alpha\beta\gamma}$

$$K_{\beta\epsilon} = I_{\epsilon\alpha}(B_{\alpha\beta})^{-1} = (B_{\epsilon\beta})^{-1} \qquad 2.52$$

$$M_{\delta\epsilon\sigma} = -\,(C_{\alpha\beta\gamma}K_{\beta\epsilon}K_{\gamma\sigma})(B_{\alpha\delta})^{-1} = -\,C_{\alpha\beta\gamma}K_{\delta\alpha}K_{\beta\epsilon}K_{\gamma\sigma} \qquad 2.53$$

These matric equations are the same as the corresponding ordinary equations 2.42 and 2.43. That is, *the matrix $K_{\alpha\beta}$ is found by taking the inverse of the matrix $B_{\alpha\beta}$ and the 3-matrix $M_{\alpha\beta\gamma}$ is found by multiplying the 3-matrix $C_{\alpha\beta\gamma}$ by the matrix $K_{\alpha\beta}$ three times in succession* in the order indicated by the indices, then taking its negative.

Since $(B_{\alpha\beta})^{-1} = K_{\beta\alpha}$, that is, *since in taking the inverse of a matrix the order of the indices changes*, the three $K_{\alpha\beta}$ matrices in the last expression have their free indices in different position. That is, the free index of $K_{\delta\alpha}$ is its *first* index, while the free indices of $K_{\beta\epsilon}$ and $K_{\gamma\sigma}$ are their *second* indices.

4. Hence the value of i_α in terms of $B_{\alpha\beta}$ and $C_{\alpha\beta\gamma}$ are

$$i_\alpha = K_{\alpha\beta}e_\beta - C_{\gamma\delta\epsilon}K_{\alpha\gamma}K_{\delta\tau}K_{\epsilon\sigma}e_\tau e_\sigma \qquad 2.54$$

where $K_{\alpha\beta} = (B_{\beta\alpha})^{-1}$.

It should be noted that without the concepts of n-matrices it is an extremely laborious procedure to find the inverse of a set of power series equations. *Because of the absence of such rules as equation* 2.53 each time the inverse of a set of equations is to be found, the whole analytical procedure has to be gone through all over again. If the finding of the inverse of a power series is just one step in some analytical development, then the analysis is rarely undertaken in ordinary symbolism, since after the first few steps the mechanical difficulties involved in handling the numerous terms become insurmountable, not even mentioning the mental labor necessary to keep the physics of the problem and the analysis clear.

XV. ELIMINATION OF VARIABLES

(*a*) To show an example where the First Generalization Postulate is used for the manipulation of *several* matric equations, consider *three* linear equations:

$$\begin{array}{l|l}
e_1 = Z_1 i_1 + Z_2 i_2 + Z_3 i_3 & \mathbf{e}_1 = \mathbf{Z}_1\cdot\mathbf{i}_1 + \mathbf{Z}_2\cdot\mathbf{i}_2 + \mathbf{Z}_3\cdot\mathbf{i}_3 \\
e_2 = Z_4 i_1 + Z_5 i_2 + Z_6 i_3 & \mathbf{e}_2 = \mathbf{Z}_4\cdot\mathbf{i}_1 + \mathbf{Z}_5\cdot\mathbf{i}_2 + \mathbf{Z}_6\cdot\mathbf{i}_3 \\
e_3 = Z_7 i_1 + Z_8 i_2 + Z_9 i_3 & \mathbf{e}_3 = \mathbf{Z}_7\cdot\mathbf{i}_1 + \mathbf{Z}_8\cdot\mathbf{i}_2 + \mathbf{Z}_9\cdot\mathbf{i}_3
\end{array}\left.\vphantom{\begin{array}{l}a\\b\\c\end{array}}\right\} \quad 2.55$$

The left-hand column represents three linear ordinary equations with three unknowns, i_1, i_2, and i_3. The right-hand column repre-

sents three linear matric equations, in which the three 1-matrices $\mathbf{e}$ and the nine 2-matrices $\mathbf{Z}$ are known quantities, while the three 1-matrices $\mathbf{i}$ are unknown. Each matric equation may represent *any number* of ordinary equations.

(*b*) Let it be assumed that in the problem at hand the knowledge of the values of $\mathbf{i}_3$ is not needed; hence *let* $\mathbf{i}_3$ *be eliminated, leaving only two matric equations with* $\mathbf{i}_1$ *and* $\mathbf{i}_2$ *as unknowns.* The elimination of $\mathbf{i}_3$ will be performed in parallel columns for the ordinary and for the matric equations.

1. From the third equation let $\mathbf{i}_3$ be expressed in terms of $\mathbf{i}_1$ and $\mathbf{i}_2$. Placing the terms containing $\mathbf{i}_2$ on the left-hand side of the equation

$$\begin{array}{l|l} Z_9 i_3 = e_3 - Z_7 i_1 - Z_8 i_2 & \mathbf{Z}_9\cdot\mathbf{i}_3 = \mathbf{e}_3 - \mathbf{Z}_7\cdot\mathbf{i}_1 - \mathbf{Z}_8\cdot\mathbf{i}_2 \end{array}$$

Multiplying both sides of the equation by $\mathbf{Z}_9^{-1}$

$$\begin{array}{l|l} i_3 = Z_9^{-1}(e_3 - Z_7 i_1 - Z_8 i_2) & \mathbf{i}_3 = \mathbf{Z}_9^{-1}\cdot(\mathbf{e}_3 - \mathbf{Z}_7\cdot\mathbf{i}_1 - \mathbf{Z}_8\cdot\mathbf{i}_2) \end{array}$$

2. Let this value of $\mathbf{i}_3$ be substituted into the other two equations

$$\begin{array}{l|l} e_1 = Z_1 i_1 + Z_2 i_2 & \mathbf{e}_1 = \mathbf{Z}_1\cdot\mathbf{i}_1 + \mathbf{Z}_2\cdot\mathbf{i}_2 + \\ \quad + Z_3 Z_9^{-1}(e_3 - Z_7 i_1 - Z_8 i_2) & \quad + \mathbf{Z}_3\cdot{}_9^{-1}\cdot(\mathbf{e}_3 - \mathbf{Z}_7\cdot\mathbf{i}_1 - \mathbf{Z}_8\cdot\mathbf{i}_2) \\ e_2 = Z_4 i_1 + Z_5 i_2 + & \mathbf{e}_2 = \mathbf{Z}_4\cdot\mathbf{i}_1 + \mathbf{Z}_5\cdot\mathbf{i}_2 + \\ \quad + Z_6 Z_9^{-1}(e_3 - Z_7 i_1 - Z_8 i_2) & \quad + \mathbf{Z}_6\cdot\mathbf{Z}_9^{-1}\cdot(\mathbf{e}_3 - \mathbf{Z}_7\cdot\mathbf{i}_1 - \mathbf{Z}_8\cdot\mathbf{i}_2) \end{array}$$

Factoring $\mathbf{i}_1$ and $\mathbf{i}_2$ in both equations:

$$\begin{array}{l|l} e_1 - Z_3 Z_9^{-1} e_3 = & \mathbf{e}_1 - \mathbf{Z}_3\cdot\mathbf{Z}_9^{-1}\cdot\mathbf{e}_3 = \\ \quad = (Z_1 - Z_3 Z_9^{-1} Z_7) i_1 + & \quad = (\mathbf{Z}_1 - \mathbf{Z}_3\cdot\mathbf{Z}_9^{-1}\cdot\mathbf{Z}_7)\cdot\mathbf{i}_1 + \\ \quad + (Z_2 - Z_3 Z_9^{-1} Z_8) i_2 & \quad + (\mathbf{Z}_2 - \mathbf{Z}_3\cdot\mathbf{Z}_9^{-1}\cdot\mathbf{Z}_8)\cdot\mathbf{i}_2 \\ e_2 - Z_6 Z_9^{-1} e_3 = & \mathbf{e}_2 - \mathbf{Z}_6\cdot\mathbf{Z}_9^{-1}\cdot\mathbf{e}_3 = \\ \quad = (Z_4 - Z_6 Z_9^{-1} Z_7) i_1 + & \quad = (\mathbf{Z}_4 - \mathbf{Z}_6\cdot\mathbf{Z}_9^{-1}\cdot\mathbf{Z}_7)\cdot\mathbf{i}_1 + \\ \quad + (Z_5 - Z_6 Z_9^{-1} Z_8) i_2 & \quad + (\mathbf{Z}_5 - \mathbf{Z}_6\cdot\mathbf{Z}_9^{-1}\cdot\mathbf{Z}_8)\cdot\mathbf{i}_2 \end{array} \qquad 2.56$$

In these two matric equations there are only two unknowns $\mathbf{i}_1$ and $\mathbf{i}_2$.

The two equations may be written shortly as

$$\begin{array}{l|l} e'_1 = Z'_1 i_1 + Z'_2 i_2 & \mathbf{e}'_1 = \mathbf{Z}'_1\cdot\mathbf{i}_1 + \mathbf{Z}'_2\cdot\mathbf{i}_2 \\ e'_2 = Z'_3 i_1 + Z'_4 i_2 & \mathbf{e}'_2 = \mathbf{Z}'_3\cdot\mathbf{i}_1 + \mathbf{Z}'_4\cdot\mathbf{i}_2 \end{array} \Bigg\} \quad 2.57$$

The new coefficients $\mathbf{Z}'$ of the unknowns are matrices and are found

from the original matrices by performing the indicated operations, as for instance

$$Z_1' = Z_1 - Z_3 Z_9^{-1} Z_7 \qquad \Big| \qquad \mathbf{Z}_1' = \mathbf{Z}_1 - \mathbf{Z}_3 \cdot \mathbf{Z}_9^{-1} \cdot \mathbf{Z}_7$$

(*c*) Of course the manipulation of a set of matric equations involves only incidentally the finding of the unknown quantities. More often the manipulation involves, say, the development of certain criteria between the known quantities in order that the system should perform in some desired manner. Or the manipulation may involve a search for a new equation that includes say less concepts, or more details, or covers more systems, etc. The variety of the manipulations of matric equations is at least as wide as that of ordinary equations. In fact, *matric equations are manipulated for more reasons than ordinary equations are,* since the organization of quantities into n-way matrices introduces additional concepts, that are non-existent in the absence of such an organization.

CHAPTER III

THE SECOND GENERALIZATION POSTULATE

I. THE CREATION OF NEW MATHEMATICAL ENTITIES

(*a*) In the previous chapter the First Generalization Postulate was introduced showing that, if a *set* of ordinary equations is represented as *one* matric equation, the latter has the same form for any number of degrees of freedom as for one single degree of freedom. (In case of *several* matric equations, they have the same form as the similar number of ordinary equations that represent the simplest analogous system.) This postulate leads to great savings in *thought* and *labor* in setting up the equations and in manipulating them.

However, as soon as a set of equations is set up, not haphazardly, but in a systematic manner in terms of n-way matrices, the resulting equation immediately calls into existence three interrelated concepts that have hitherto been hidden from view in the unorganized manner of attack of engineering problems. These concepts occupy the central foreground of modern physical and mathematical analysis. The three concepts are:

1. Transformation.
2. Invariance.
3. Group.

With the systematic introduction of these concepts begins the study of what is known as " tensor analysis " or " absolute differential calculus." The ideas developed hitherto are useful in the understanding of tensor analysis but they themselves are not yet tensor analysis.

(*b*) It will be shown that the organization of a set of quantities into a single n-matrix and its representation by a single central letter $A_{\alpha\beta\gamma}$ signifies far mòre than just the introduction of a *shorthand* notation and a labor-saving device. *The organization of a set of quantities and the simultaneous introduction of the concepts of " transformation," " invariance," and " group " represent the creation of an entirely new mathematical entity $A_{\alpha\beta\gamma}$, that is endowed with properties that its building blocks, the "n-matrices " or their " components," do not possess. The creation of new entities from a mere collection of n-way matrices by means*

of "organization," and the endowment of these new entities with new properties, constitute the underlying purpose of tensor analysis.

This creation is of the same nature as building out of a conglomeration of atoms a new entity, a molecule, which is endowed by the *mere process of organization* with new characteristics, new properties, that its component parts, the atoms, do not possess; it is of the same nature as the organization of a community of people into a state, having functions that none of its constituent members has. The creation of these new mathematical entities and the endowment of these entities with new properties are equivalent to discovering in the corresponding physical (or geometrical) phenomena new physical (or geometrical) entities obeying new physical laws.

(*c*) *It cannot be emphasized sufficiently that an n-way matrix itself is not a new mathematical entity.* That is, the fact that the various currents in a system are represented by one symbol **i**, the voltages by **e**, and the impedances by **z** does not represent a creation of new entities, neither does the fact that the current **i** is imagined to be represented by a point in an n-dimensional space. *An n-way matrix is just a shorthand notation, a labor-saving device,* but it has *as yet* no additional properties that its components themselves do not possess. For instance, *both* have *rules of manipulations* associated with them; that is, both *n-matrices and their components* can be added, multiplied, differentiated, etc. The situation of using n-matrices is analogous to using a steam shovel where previously many laborers with pick and shovel were employed. The single steam shovel does *the same work* but more systematically and *faster* (and with less labor trouble) that laborers can do with a large number of picks and shovels.

In order to endow an n-way matrix with hitherto non-existent characteristics, and thereby to create a new mathematical entity, *it is absolutely necessary to introduce a new content into a matric equation that the corresponding ordinary equations do not possess.* This new content is introduced with the aid of the three interrelated concepts of "transformation," "invariance," and "group."

(*d*) It is also emphasized that it is not necessary to use n-matrices in order to introduce these new concepts and contents into the analysis. In fact, *no textbook or publication on tensor analysis employs n-way matrices as a stepping stone to introduce this new method of reasoning.* The textbooks simply use expressions such as a "set of 2^3 quantities," without arranging them into a cube with 2 rows, columns, and layers, merely writing the eight quantities in a row side by side. Of course, the fixed indices take care of their correct manipulation.

However, the experience in working out numerous engineering

problems with this new method of reasoning has led to the systematic use of rows, squares, and cubes in the actual calculations, which is also employed in this volume. But their use has little to do with tensor analysis; it only *facilitates* its presentation and application.

II. THE "SECOND GENERALIZATION POSTULATE"

(*a*) When a current i flows in a *single* stationary coil (or mesh) Ohm's law gives its equation of voltage as

$$e = zi \tag{3.1}$$

The concepts occurring in the equation are three in number, namely e, z, and i.

When *any number* of stationary coils with any type of mutual impedances between them are interconnected into a network with n meshes, the First Generalization Postulate states that the equation of voltage of the *set of coils* has the same form as that of the *single coil* except that each quantity e, z, and i is replaced by some appropriate n-matrix of various dimensions $\mathbf{e}$, $\mathbf{z}$, and $\mathbf{i}$, giving the resultant equation of voltage of the whole system with n degrees of freedom as

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i} \qquad \Big| \qquad e_\alpha = z_{\alpha\beta} i^\beta \tag{3.2}$$

Suppose now that the interconnection of the coils is destroyed and the individual coils are interconnected into a mesh network in a different manner, without, however, destroying or changing the individual self- and mutual impedances. The question is now: What is the equation of voltage of the new system in terms of n-matrices?

A fundamental assumption of tensor analysis states:

1. *The new system has the same number and types of n-matrices as the old system* (namely $\mathbf{e}'$, $\mathbf{z}'$, and $\mathbf{i}'$) *but they now have different components.*

2. *The equation of the new system in terms of n-matrices is exactly the same as the equation of the old system*, namely, it is

$$\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}' \qquad \Big| \qquad e_{\alpha'} = z_{\alpha'\beta'} i^{\beta'} \tag{3.3}$$

3. *The n-matrices of the new system may be established from those of the old system by a routine transformation.*

These statements, or their equivalent, will be called the "*Second Generalization Postulate.*"

That is, in changing over to a new interconnection, no new types of n-matrices are introduced and the arrangement of the existing ones

is not changed. *The only difference is that the existing n-matrices* $\mathbf{e}'$ *and* $\mathbf{z}'$ *have different components from those that they formerly had* and that the variables $\mathbf{i}$ are changed to new variables $\mathbf{i}'$.

The step in going from one type of interconnection to another type will be called a " transformation " or (to employ a phrase that is much used, but is not quite descriptive of this step) a " transformation of the reference frame." It may also be called a " *change of variables* " since the set of variables $\mathbf{i}$ changes to another set of variables $\mathbf{i}'$.

If the new interconnection is again changed to another one, that is, if a new transformation is introduced, the set of equations describing the third arrangement of coils are still

$$\mathbf{e}'' = \mathbf{z}'' \cdot \mathbf{i}'' \qquad\qquad e_{\alpha''} = z_{\alpha''\beta''} i^{\beta''}$$

where now $\mathbf{e}''$ and $\mathbf{z}''$ have again different components and $\mathbf{i}''$ represents still different sets of variables. (The use of the upper index in i^β will be explained later.)

(*b*) As another example of the use of the Second Generalization Postulate let the equation of voltage of a *single moving coil* be:

$$e = Ri + \frac{d\phi}{dt} + \psi v$$

where Ri is the resistance drop in the coil, $d\phi/dt$ is the voltage induced in the coil as the result of the time variation of the flux lines ϕ linking it (assuming the coil stationary), and where ψv is the voltage generated in it owing to its motion through the flux density ψ with an instantaneous velocity v (assuming the flux lines constant in time). The concepts occurring in the ordinary equation are seven in number, namely t, v, e, i, ϕ, ψ, and R.

Next, instead of one coil let *several coils* with mutual inductances between them move all with the instantaneous velocity v. Let these moving coils be, say, those of an alternator. The First Generalization Postulate states that their equation contains seven n-matrices, t, v, $\mathbf{e}$, $\mathbf{i}$, $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, $\mathbf{R}$, and it has the same form as the corresponding ordinary equation, namely

$$\mathbf{e} = \mathbf{R} \cdot \mathbf{i} + \frac{d\boldsymbol{\phi}}{dt} + \boldsymbol{\psi} v \qquad\qquad e_\alpha = R_{\alpha\beta} i^\beta + \frac{d\phi_\alpha}{dt} + \psi_\alpha v$$

Now, if these same coils are interconnected in any *other* manner with the aid of brushes, or slip-rings, or taps, etc., forming a new machine, say a shunt-polyphase commutator-motor, then *the Second Generalization Postulate states that the equation of the new machine still*

contains the same seven n-matrices and they are arranged in the same manner

$$\mathbf{e}' = \mathbf{R}' \cdot \mathbf{i}' + \frac{d\boldsymbol{\phi}'}{dt} + \boldsymbol{\psi}'v' \qquad \bigg| \qquad e_{\alpha'} = R_{\alpha'\beta'} i^{\beta'} + \frac{d\phi_{\alpha'}}{dt} + \psi_{\alpha'} v'$$

except that now the n-matrices $\mathbf{e}'$, $\mathbf{i}'$, $\mathbf{R}'$, $\boldsymbol{\phi}'$, *and* $\boldsymbol{\psi}'$ *have different components.* But the *form* of the equation has not changed.

For a third type of interconnection the equation still keeps its form except that the primes are replaced by double primes. *There exists a large variety of interconnections (transformations) of n coils for which the equations keep their form.*

(*c*) As a third example, let the differential equation for the propagation of electromagnetic waves along a *single conductor* be

$$\frac{\partial^2 e}{\partial x^2} = Z\,Ye$$

containing the four concepts x, e, Z, and Y.

If the waves travel along *several parallel* conductors, the First Generalization Postulate introduces the four n-matrices x, $\mathbf{e}$, $\mathbf{Z}$, $\mathbf{Y}$ and gives their differential equation as

$$\frac{\partial^2 \mathbf{e}}{\partial x^2} = \mathbf{Z} \cdot \mathbf{Y} \cdot \mathbf{e} \qquad \bigg| \qquad \frac{\partial^2 e_\alpha}{\partial x^2} = Z_{\alpha\beta}\, Y^{\beta\gamma} e_\gamma$$

When the conductors are joined together or are terminated, etc., in various manners, the differential equations of propagation are, by the Second Generalization Postulate,

$$\frac{\partial^2 \mathbf{e}'}{\partial x^2} = \mathbf{Z}' \cdot \mathbf{Y}' \cdot \mathbf{e}' \qquad \bigg| \qquad \frac{\partial^2 e_{\alpha'}}{\partial x^2} = Z_{\alpha'\beta'}\, Y^{\beta'\gamma'} e_{\gamma'}$$

These examples can be continued indefinitely.

(*d*) In all these transformations it is tacitly assumed that in changing over from one system to another *nothing is disturbed except the interconnection* (consisting of infinitely short conductors having neither resistance, nor inductance, etc.). That is, *during the transformation no new physical phenomenon is introduced* such as motion where none existed before the interconnection or an electrostatic field where none existed previously.

Just because the number of physical concepts and their interrelations remains unchanged, it is possible to keep the number of mathematical symbols and their interrelations also unchanged. It is one of the goals of tensor analysis in analyzing any physical problem

to establish only as many symbols as there are actually existing natural phenomena corresponding to them, and to interrelate these symbols in a parallel manner as the corresponding physical manifestations are interrelated. That is, *in the equations of tensor analysis*:

1. *Each symbol (base letter) corresponds to a physical or geometrical entity.*
2. *Each operation on the symbols corresponds to an analogous physical or geometrical relation existing between the entities.*

III. THE CONCEPT OF "GEOMETRIC OBJECT"

(*a*) The Second Generalization Postulate states that a single symbol **A** stands not for a *single n-matrix*, but for a *large number of n-matrices*, each having the same dimensions and the same number of axes, but different components.

From now on, each symbol or base letter will stand for an infinite number of n-matrices that will be considered to form a new mathematical entity, called the "geometric object." The particular n-matrix under consideration will be represented by the variable indices (having primes or double primes, etc.) attached to the base letter. That is, in each reference frame with each geometric object is associated an n-matrix which gives the components of *the* geometric object in that particular reference frame. As the reference frames vary, the *components* of the geometric object (identified by the indices) also vary, but the geometric object itself (represented by the base letter) remains unchanged.

In physical problems a "geometric object" stands for some *physical object*, as the velocity vector of a moving body or stresses in a deformed body. The expression "geometric object" used for $A_{\alpha\beta\gamma}$ could just as well be replaced by "physical object" or "mathematical object." The geometrical nomenclature is retained in tensor literature since these concepts were first developed in conjunction with geometrical problems.

Hence the *components* of the velocity vector v^α of a point measured along a particular reference frame are $v^{\alpha'}$, along another frame its components are $v^{\alpha''}$, along a third one they are $v^{\alpha'''}$, etc. Although all these components are different, the velocity v of the point itself is unchanged.

The representation of a "geometric object" along a particular reference frame includes not only the components that are arranged in a row or a square or a cube, but also the fixed indices themselves written alongside. That is, each time the components of a geometric object are given, it is absolutely necessary to state the reference frames along which the

components are expressed. If the identifying indices along the components are omitted, only an "n-matrix" is defined by the given components but not a "geometric object." For instance *the expression "matrix" refers to a set of quantities arranged in a rectangle, ignoring the fixed indices written alongside the rectangle.*

(*b*) With the introduction of the new entity—the "geometric object"—in place of an "n-matrix," a new terminology and notation will be introduced. The reason for each of these changes will be stated as the development of new concepts continues. In particular:

1. In index notation an n-matrix is distinguished from a geometric object by enclosing the indices of the former in parenthesis as $Z_{(\alpha)(\beta)}$. That is, $Z_{\alpha\beta}$ is a geometric object having components in an infinite number of reference frames; $Z_{(\alpha)(\beta)}$ is an n-matrix having components in a single reference frame only. In direct notation no such differentiation exists since *in simpler problems their manipulation is analogous.*

2. *An equation in which each symbol stands for a geometric object instead of an n-matrix will be called an "invariant equation"* instead of a "matric equation." That is, an invariant equation as $e_\alpha = z_{\alpha\beta}i^\beta$ is valid for an infinite number of physical systems, while a matric equation as $e_{(\alpha)} = z_{(\alpha)(\beta)}i_{(\beta)}$ is valid for only one particular physical system having n axes.

3. An n-dimensional geometric object will be called a "*geometric object of valence n.*"

4. The expression "*current vector*" will be used for **i**, "*voltage vector*" for **e**, "*impedance tensor*" for **z**, and "*admittance tensor*" for **y**.

5. The current vector i^α and the admittance tensor $y^{\alpha\beta}$ will have upper instead of lower indices.

IV. MATHEMATICAL REPRESENTATION OF A GEOMETRIC OBJECT

(*a*) Strictly speaking, to represent a single geometric object (or physical object) mathematically it is necessary to show all its components along all the possible reference frames that may exist. That is, *a geometric object is fully represented mathematically only by a large, usually an infinite, number of n-matrices.*

In practice this representation is accomplished in the following manner:

1. Out of its infinite number of n-matrices only *one* is fully given by showing all its components.

2. The particular reference frame along which these components are known is also given.

3. All other possible reference frames along which the geometric object may possess components are defined.

4. A *routine procedure*, a "formula," is given by which the components of any one of the other infinite n-matrices may be calculated *in a routine manner*, whenever they are needed.

The *ability* to find any one of the infinite number of n-matrices belonging to a geometric object whenever it is wanted is equivalent to completely defining the geometric object itself.

(*b*) Summarized, a "geometric object" is defined if:

1. A particular *n-matrix* is given along one reference frame.
2. All the *axes* of this particular reference frame are given.
3. All the possible *reference frames* of the geometric object are defined.
4. The *formula* for finding the various n-matrices along the various reference frames is also given.

V. THE TRANSFORMATION TENSOR

(*a*) When an n-matrix representing the components of a geometric object along some reference frame is given, the particular axes are shown by placing fixed indices to each row, column, layer, etc., of the n-matrix.

Each of the other reference frames is defined by a 2-matrix $\mathbf{C} = C^{\alpha}_{\alpha'}$ *called a "transformation matrix"* (to be studied in detail in Chapter IV), that shows how the new reference frame (or system) differs from the original reference frame (or system). Each new reference frame has its own transformation matrix $C^{\alpha}_{\alpha'}$ relating it to the old reference frame, hence with every geometric object a whole group of transformation matrices is associated. *The totality of all transformation matrices forms one entity, the "transformation tensor"* $C^{\alpha}_{\alpha'}$.

The formulas by which the components of the geometric object along all the other reference frames are found are called "*transformation formulas*" or "*equations of transformations*" or "*laws of transformation.*" Each geometric object $A_{\alpha\beta\gamma}$ has its own law of transformation that involves only $A_{\alpha\beta\gamma}$ and the transformation tensor $C^{\alpha}_{\alpha'}$.

Hence the concept of "geometric object" involves the following additional concepts:

1. An n-matrix (or a set of k^n quantities).
2. A group of transformation matrices.
3. A law of transformation.

(*b*) In terms of these new concepts, the Second Generalization Postulate may also be stated as follows:

If the matric equation of the physical phenomenon taking place in a particular system (or reference frame) with any number of degrees of freedom is known, then this same equation is valid for an infinite variety

of similar systems (or reference frames) in which the same physical phenomenon takes place, if each n-way matrix is replaced by a geometric object. The components of each geometric object along any of the new systems are found from those of the original system by routine "formulas of transformation" with the aid of a "transformation tensor" $C^{\alpha}_{\alpha'}$.

(*c*) Hence by the Second Generalization Postulate the analysis of any new system consists of the following steps (if the invariant equation of *one* system already has been established):

1. Find the transformation matrix **C** of the new system, showing how it differs from the old system.

2. Find the new components of the geometric objects for the new system by using the transformation formula of each geometric object.

When the new system or reference frame differs in some *essential* manner from the old system (for instance, the junctions vary in time instead of being stationary or the reference frame is no longer rectilinear), then the equation of the new system has a different form. Later on other generalization postulates will be developed to take care of this fundamental dissimilarity of the old and the new systems or reference frames.

(*d*) In order to avoid lengthy and roundabout expressions of stating each time that a given set of k^n quantities (n-matrix) represents "*the components of the geometric object* **A** *along the given reference frame*" it will often be stated in its place that the given n-matrix represents "*the geometric object* **A**." (If the fixed indices alongside the n-matrices are assumed to represent *unit vectors* along the axes, the n-matrix *plus* the unit vectors do represent the geometric object.)

VI. THE PURPOSE OF THE GENERALIZATION POSTULATES

(*a*) Summarizing the statements of the previous section:

1. The First Generalization Postulate changes an *ordinary* equation valid from one (or a few) degrees of freedom to a *matric* equation valid for n degrees of freedom; that is, it generalizes a phenomenon from one (or a few) axes to any number of axes.

2. The Second Generalization Postulate changes a *matric* equation valid for *one particular* reference frame to an *invariant* equation valid for an infinite variety of other reference frames *of the same type.*

3. The remaining generalization postulates will change an *invariant* equation valid for one type of reference frame to *tensor* equations valid for several *other types* of reference frames (say both rectilinear and curvilinear axes, or stationary and moving axes, and so on).

(*b*) These postulates have been formulated in these pages in order

to reduce to a few *routine* standardized steps the mental labor needed to formulate the large variety of engineering problems. Generally speaking, the analysis of engineering problems in terms of ordinary equations requires an entirely new procedure, a new physical picture, etc., for each particular system to be analyzed, with very little reference to the analysis of other systems. For instance, each rotating electrical machine has a different theory (as a glance into any textbook will show), so that engineers specializing, say, in induction-motor analysis know little about the analysis of the salient-pole alternator, and vice versa. It is the purpose of this treatise to point out that the various types of rotating electrical machines or stationary networks or transmission systems, etc., are fundamentally identical; *their equations differ only because the reference frames assumed are different in each case. Hence in learning the analysis of one particular machine or system the engineer learns at the same time the analysis of a large variety of analogous machines or systems.*

(*c*) It will also be shown in this treatise that stationary networks, rotating machines, transmission lines, etc., are not isolated types of structures whose theories and equations (even in the language of tensor analysis) are independent of each other. *All engineering structures may be looked upon as links of various dimensions in a single chain, which obeys one single law,* which, however, manifests itself (mathematically) in different forms, for instance as the dynamical equation of motion of Lagrange on one hand and the field equations of Maxwell on the other.

The attempt to preserve this unity of all equations of performance of engineering and non-engineering systems will manifest itself throughout this treatise. Networks, the subject matter of this volume, are considered as the simplest possible links, namely, a collection of zero-dimensional (junctions) and one-dimensional (coils) links in a chain of multi-dimensional links. (Rotating machines will be considered as a collection of two-dimensional links, etc.)

(*d*) It is emphasized, however, that it is not the sole purpose of tensor analysis to establish standardized steps to set up the equations for another set of reference frames or systems if they are known for one particular system. A still more important function of tensor analysis is to set up an equation in a form in which *every symbol should correspond to an actually existing physical phenomenon in nature* and not be just a figment of human imagination.

Still another important function of tensor analysis is to set up equations in a form or to transform them to a reference frame in which they can be solved in the simplest possible manner, with exact or

approximate methods. In other words, a set of equations is to be transferred to a new reference frame or is to be brought to a new form not because it corresponds to a physical system, but because in that frame the algebraic, differential, or integral equations can be more easily visualized, or solved.

(*e*) It should be noted that *the " Second Generalization Postulate " endows a matric equation with a new content that a set of ordinary equations does not possess. That is, a matric equation is endowed with the additional property of being valid for a large variety of systems or reference frames, instead of just for one particular system or reference frame. On the other hand, a set of ordinary equations is valid only for one particular system or reference frame, and it cannot be endowed with this new interpretation.*

For instance, the ordinary equations of performance of, say, a series polyphase-commutator motor cannot by any stretch of the imagination be endowed with the additional property of representing at the same time the equations of performance of, say, a salient-pole synchronous motor with amortisseur windings. But, if their performance is expressed first in terms of n-way matrices and then the transformation tensor **C** changing the connection of one machine into the other is established (thereby changing the n-way matrices to geometric objects), then the same equation represents the performance of both machines and the ordinary equation of each machine can be established from that of the other by a *routine* calculation, without the aid of any physical analysis or other supplementary help.

The key to the creation of these new entities (that is, to the endowment of n-way matrices with new properties) *is the concept of " transformation tensor "* **C**, *whose study forms the main interest of this volume. As the development continues throughout the treatise, the geometric objects will be endowed with additional properties and the invariant equations will be endowed with additional " contents."*

VII. NETWORKS

The simplest type of engineering structure consists of a collection of *one-dimensional members* (pieces of wires, bracings, pipes, etc.) *joined together* at fixed points, on which forces (voltages, weights, etc.) are superimposed along the members and at the junctions. A *systematic* study of such structures, called " *networks*," is the purpose of this volume.

Only special types of networks will be considered, namely, electromagnetic networks. It will also be assumed that *the superimposed*

electromagnetic quantities propagate instantaneously throughout the whole network; that is, the networks under consideration will have *lumped* design constants, instead of distributed constants.

Although the study of networks is undertaken in the language of the electrical engineer, it is emphasized that the *method of attack* has little to do with *electrical* engineering. It can be applied with appropriate changes of expressions (like force for voltage, etc.) to the study of *mechanical* networks also. The method of reasoning in its broad outlines has been borrowed partly from a branch of geometry called " topology " or " analysis situs " and partly from another branch called " differential geometry." The mathematical tool used in the reasoning is " tensor algebra " and " tensor analysis."

The study of networks will be so formulated that it should serve as a logical stepping stone to the study of more complex engineering structures, where the component members are *more than one-dimensional*, their joints are *not fixed*, the propagation is *not instantaneous*, etc.

VIII. THE BUILDING BLOCKS OF NETWORKS

(*a*) Let several lumped coils, with electromagnetic couplings between some of them, be interconnected in any combination to form a network such as Fig. 3.1. The presence of leads at the junction points of the coils indicates that *the " network " is not an isolated structure* but in the general case it forms part of a larger network from which it has been detached. That is, *the leads themselves may be connected to other coils, which, however, do not appear as part of the network under consideration.*

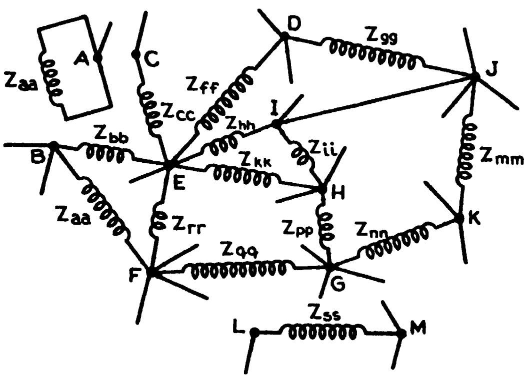


FIG. 3.1.—A "Network" Detached from a Larger System

The following component parts of a network will be distinguished:

1. The " *coils* " Z_{aa}, Z_{bb}, that are joined together to form a network. There are 15 coils in Fig. 3.1.

2. The two ends of a coil, where it is joined to other coils, called "*junctions*." In connecting n coils into networks always the $2n$

junctions are joined together, thereby decreasing their number. When two or more junctions are connected with an impedanceless wire such as I and J, they will be considered to form only one junction. There are altogether 12 junctions on Fig. 3.1. (The junctions are also called " vertices," or " nodes," or " branch-points.")

These two concepts, " coils " and " junctions," are the building blocks of lumped networks. With the coils certain quantities (design constants), Z_{aa}, Y^{aa}, etc., are associated, while the junctions are formed by conductors having neither resistance nor inductance.

In this volume the limitation is put upon the physical nature of the lumped " coils " of a network that their characteristics are not influenced by the superimposed electromagnetic quantities. Similar limitation is put upon the form of the mathematical expressions Z_{aa}, Y^{aa}. They may be real or complex numbers R, $R - jX$, or functions of time $R \cos \omega t$, or *linear operators* $L d/dt$ or $\int$, etc., but are not functions of electromagnetic quantities.

The limitation is put upon the physical nature of the *junctions* between the coils that *the interconnections momentarily do not vary in time,* the junctions are fixed, *at least at the instant under consideration.*

The interconnections are assumed momentarily stationary; the conductors forming the coils, however, may have an instantaneous velocity, as in a rotating machine with stationary reference axes.

(*b*) In a given network it is important to note how many *independent component-networks,* or " *sub-networks,*" can be distinguished, having *no physical connections* with one another. However, electrical or magnetic coupling may exist between the sub-networks. In Fig. 3.1 the number of sub-networks is three.

The number of coils is denoted by α_1, the number of junctions by α_0, and the number of independent sub-networks by R_0.

IX. THE ANALYTICAL UNITS OF A NETWORK

The two physical units of networks, namely, the *coils* and *junctions,* do not by themselves suffice for the analytical study of networks. *Certain combinations of these units* are necessary for their performance analysis. These new units are:

1. Any closed circuit traced through the network, called a " *mesh,*" such as BEF or $BEHGF$.

In finding the minimum number of meshes of a network, each coil must be traced out at least once. The number of meshes in Fig. 3.1 is six.

2. *Any two junctions* located on the *same* independent sub-network, called a "*junction-pair,*" such as B–E, F–H, B–J, L–M, etc.

In finding the minimum number of junction-pairs, each junction must be included at least once. The number of junction-pairs in Fig. 3.1 is nine.

It will be assumed that the expressions " mesh " and " junction-pair " will also include the *direction* in which they are traced out. That is, the mesh BEF is the negative of mesh FEB; similarly the junction-pair E–K is the negative of the junction-pair K–E. Hence *with every mesh and junction-pair is associated the concept of "orientation."*

The number of meshes of a network is denoted by μ and the number of junction-pairs by ρ_1.

(*a*) The five concepts hitherto introduced, namely α_0, α_1, R_0, μ, and ρ_1, are not independent of one another. The following two relations should be remembered:*

1. *The number of junction-pairs of a network is equal to the number of junctions minus the number of sub-networks.* That is,

$$\boxed{\rho_1 = \alpha_0 - R_0} \qquad 3.1$$

2. *The number of coils forming the network is equal to the sum of the number of meshes and the number of junction-pairs.* That is,

$$\boxed{\alpha_1 = \mu + \rho_1} \qquad 3.2$$

In the network of Fig. 3.1

$$\rho_1 = \alpha_0 - R_0 \quad \text{gives} \quad 9 = 12 - 3$$

and

$$\alpha_1 = \mu + \rho_1 \quad \text{gives} \quad 15 = 6 + 9.$$

Before any network can be analyzed, either its number of *meshes* or its number of *junction-pairs* or *both* must be known, depending on what are assumed as variables. In complex networks the quickest procedure is to find first the number of junction-pairs (number of junctions minus number of sub-networks) then to find the number of meshes (number of coils minus number of junction-pairs).

* For proof of these relations see Veblen: Analysis Situs, Am. Math. Soc. 1931, pp. 15 and 18.

X. ANOTHER PHYSICAL INTERPRETATION

Often it is found advantageous to replace the concept of "*mesh*" and "*junction-pair*" by another set of concepts that are analogous to them and give the same answer but *represent a different physical picture.* These analogous sets are:

1. The "*branch*" representing those parts of a network *in which the same current flows.* A branch may consist of one coil, or several coils in series, or it may consist of a conductor connecting two junctions and changing them into one junction. A *branch* quantity may replace a corresponding *mesh* quantity in all analysis, as will be shown later.

2. The "*open mesh*" representing *any* circuit through the coils that starts at one junction of a junction-pair and ends at the other junction. The open-mesh circuit includes also the leads at the junction-pair. *In all analysis the open mesh may replace the corresponding junction-pair.* In Fig. 3.1 an open mesh of the junction-pair E-F consists of coils Z_{kk}, Z_{pp}, and Z_{qq}, or coils Z_{ff}, Z_{gg}, Z_{mm}, Z_{nn}, and Z_{qq}, or it consists of the single coil Z_{rr}, etc.

Summarized, the following network concepts have now been introduced:

1. The network itself and its sub-networks.
2. Coils and junctions.
3. Meshes and junction-pairs.
4. Branches and open meshes.

XI. THE SUPERIMPOSED ELECTROMAGNETIC QUANTITIES

(*a*) The question arises: Why is it necessary to introduce the concepts of "mesh" and "junction-pair"?

Now, as long as the network is unexcited there is no need to introduce these concepts. They become necessary only when electromagnetic quantities, say voltages or currents (or both), are *superimposed* upon the network.

Two types of superimposed quantities will be assumed to exist on a network:

1. *Impressed* quantities that originate outside the network.

2. *Response* quantities representing the reaction of the network to the impressed quantitites.

Both impressed and response quantities may be either *voltages* or *currents* or both.

(*b*) Let it be assumed first that in a network with n coils and

k meshes a single voltage e_a is impressed in series with a coil (Fig. 3.2). As a response, in each of the n coils a different current i^a, $i^b \cdots$ flows.

Now these n currents are not independent of one another. It is sufficient to determine only k of them by setting up k equations and solving them for the k unknowns; that is, *it is sufficient to determine first only as many of the response currents* **i** *as there are meshes in the network* as shown in Fig. 3.2. The others are easily determined from these, without solving any equations.

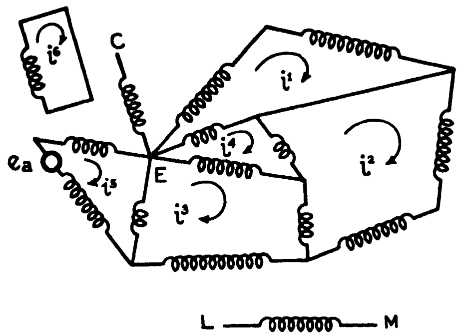


FIG. 3.2.—Impressed Coil-voltage e_a and Response Mesh-currents **i**

(*c*) Instead of the voltage let a current I^a be impressed *across* a coil of the same network (Fig. 3.3). (That is, let I^a flow into one of the junctions and I^a flow out of its other junction.) As a response, in all the coils a difference of potential E_a, $E_b \cdots$ appears.

Again, these n differences of potentials are not independent of one another. It is sufficient to determine first only $n - k$ of them by setting up $n - k$ equations and solving them for the $n - k$ unknowns. That is, *it is sufficient to determine only as many of the response differences of potential* **E** *as there are junction-pairs in the network,* as shown in Fig. 3.3. The others are easily determined from these.

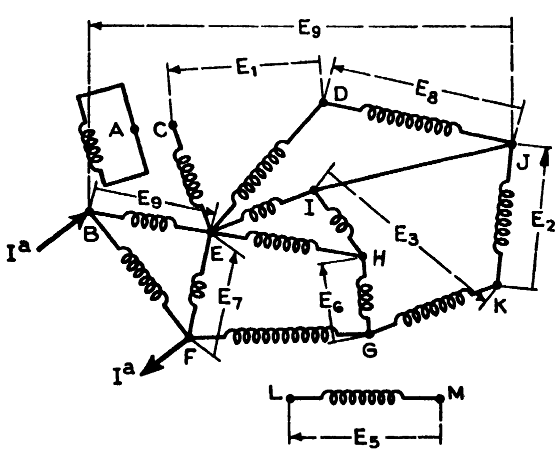


FIG. 3.3.—Impressed Junction-current I^a and Response Voltages **E**

(*d*) If, instead of a single voltage e_a, n different voltages **e** are impressed in series with the n coils, still it is sufficient to determine only k response currents **i** by setting up k equations. It makes no difference where these k currents flow, as long as they are independent of one another.

Similarly, if, instead of a single current I^a, n different currents **I** are impressed across the n coils, still it is sufficient to determine only $n - k$ response differences of potential **E** by setting up $n - k$ equations. Again it makes no difference where these $n - k$ potentials are assumed, as long as they are independent of one another.

Hence, when *voltages* **e** are impressed in series with the coils, as many equations are set up to find the response *currents* **i** as there are meshes. Similarly, when *currents* **I** are impressed in shunt with the coils, as many equations are set up to find the response *differences of potentials* **E** as there are junction-pairs. When both voltages **e** and currents **I** are impressed (Fig. 3.4), then (in general) as many equations are set up to find the response quantities **i** and **E** as there are coils.

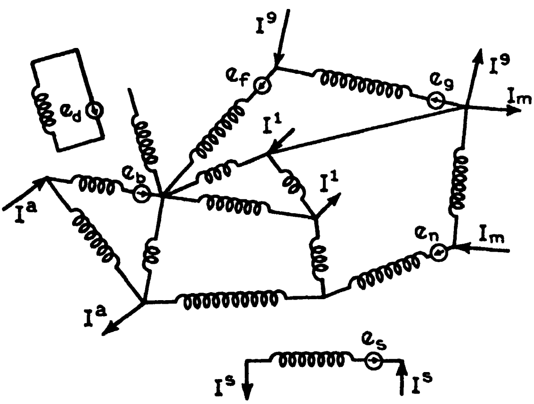


FIG. 3.4.—Impressed Coil-voltages **e** and Junction-currents **I**

(*e*) To summarize: for purposes of analysis it is assumed that:

1. The *impressed* quantities **e** and **I** appear on the *physical* units of the network, namely **e** on the coils and **I** on its junctions.
2. The *response* quantities **i** and **E** appear on the *analytical* unit of the network, namely **i** around the meshes and **E** across the junction-pairs.

Since there are n actually impressed voltages **e** and only k response currents **i**, in setting up the k equations *the n impressed voltages* **e** *are replaced by k voltages around the meshes.* Similarly since there are n actually impressed currents **I** and only $n - k$ response voltages **E**, in setting up the $n - k$ equations *the n impressed currents* **I** *are replaced by $n - k$ currents across the junction-pairs.*

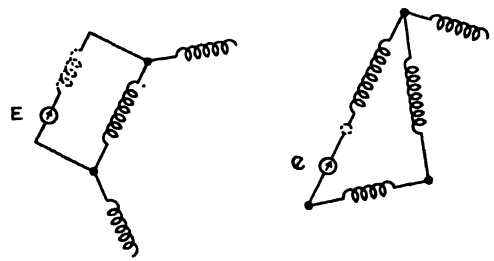
(*f*) The *impressed* quantities **e** and **I** may be looked upon as *discontinuities* introduced in the *response* quantities, namely, in the potentials **E** along an open mesh and in the current **i** around a closed mesh respectively.

From a thermodynamical point of view it is more logical to use "*withdrawn currents*" instead of "*impressed currents*" by changing their sign, that is it would be more appropriate to assume that the

components of I represent currents flowing into outside loads. However, impressed currents will be used in most places to follow the terminology of impressed voltages without a change of signs.

XII. ARBITRARINESS OF THE NOMENCLATURE

(*a*) In actual problems the " impressed " currents I may flow into loads outside the system (that are not shown on the network diagram); nevertheless, for analytical purposes they are considered as impressed currents. Similarly, the "response" differences of potentials may be actual impressed voltages, but they are assumed as response voltages.

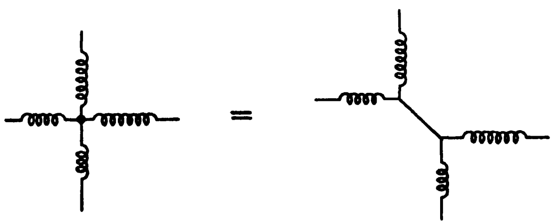


(*a*) Changing E to e (*b*) Changing e to E

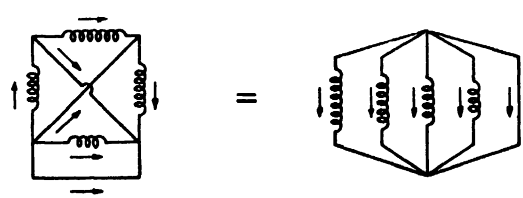
FIG. 3.5

That is, in manipulating the equations, *impressed* and *response* quantities will be made interchangeable by a change of sign, but in organizing the method of attack the distinction between impressed and response quantities will be maintained.

(*b*) The determination of what should be called a "junction" and what a " coil " often is quite arbitrary. For instance, one coil may be divided into two coils in series by introducing a junction.



(*a*) Changing a Junction into a Branch



(*b*) Changing the Number of Branches by "Stretching"

FIG. 3.6

Also *whether an impressed voltage is to be considered a coil or a junction-pair voltage is quite arbitrary.* A junction-pair voltage may be considered as a coil voltage by assuming that an impedance of zero value is in series with it (Fig. 3.5*a*). Similarly, a coil voltage may be assumed as a junction-pair

voltage by assuming the two terminals of the voltage as two junctions forming a junction-pair, Fig. 3.5*b*.

(*c*) It is also possible to change any junction into a branch with zero impedance, and vice versa, by simply stretching the point out into a line, and vice versa, as shown in Fig. 3.6*a*. That is, the number of branches in which currents flow can be varied arbitrarily by adding or subtracting impedanceless branches.

Two networks transformable into one another by *stretching* are shown in Fig. 3.6*b*. In the left-hand network there are seven branches; in the right-hand network, five branches.

XIII. THE TWO TYPES OF VARIABLES

In *setting up* the equation of performance of any physical system the question of what quantities are *known* and what are *unknown* in a given problem is temporarily unimportant. The important point is to decide what quantities may be assumed to be *variable* and what are *fixed*. Only when the equations have been once established, does the role of unknown quantities begin.

In setting up the equations of performance of networks, either of two different quantities may be assumed as variables (that, however, are not necessarily unknown), namely either of the two *response* quantities:

1. *The currents* **i** (i^n) *flowing around the meshes.*

2. *The "differences of potential"* **E** (E_a) *appearing across the junction-pairs.*

Maxwell in his " Electricity and Magnetism " gave both methods of attack, but *only the first method is used generally by engineers.* The second method is used only incidentally, and there appears to be but one modern textbook, by Herzog-Feldmann, which uses the second method consistently.

The reasons for neglecting the second method appears to be numerous:

1. Maxwell sets up the equations in terms of the " *absolute potentials* " appearing at each junction and thereby introduces one more variable and one more equation than are actually needed.

2. Maxwell then assumes the potential at *one* of the junctions as a reference potential " the ground potential " and calculates the differences of potential appearing between this particular junction and the other junctions. (In communication networks this reference junction is also called the " datum " point.)

3. The method of reasoning followed in setting up the equations *differs radically* from that followed when mesh currents are assumed as variables.

The resultant equations are quite cumbersome; they lack the simplicity and flexibility of the mesh method of attack and can be used in practical problems only with difficulty.

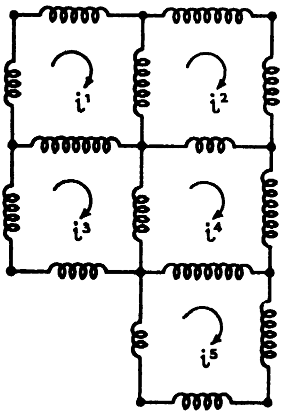
In this volume all the above limitations are avoided by the introduction of the concept of "junction-pairs." With its aid it is possible to assume the "differences of potential" appearing between any two points of the network as the (known or unknown) *variables,* thereby allowing great flexibility in the analysis and in its application, and *introducing complete parallelism with the mesh method of analysis.* It will be found that often the concept of "*junction-pair*" is of greater utility to the engineer than the concept of "*mesh,*" since the voltages and currents across a junction-pair are *actual, physical quantities,* whereas the voltages and currents in the meshes are *hypothetical quantities.*

XIV. THE THREE TYPES OF NETWORK PERFORMANCE

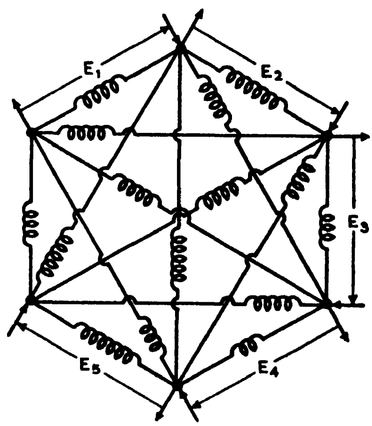
(*a*) There are three ways of looking at any network:

1. A network may be considered as a collection of *meshes.* In this case the variables are the currents **i** flowing in the meshes.

2. A network may be considered as a collection of *junction-pairs.* In this case the variables are the differences of potential **E** appearing across the junction pairs.



(*a*) Typical Mesh Network



(*b*) Typical Junction Network

FIG. 3.7.—Networks with 15 Coils

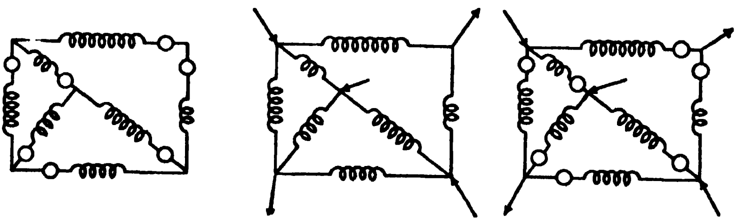
Whether the mesh currents or the differences of potential should be considered as variables depends usually on whether the network has more meshes or more junction-pairs. A typical mesh network of 15 coils is shown in Fig. 3.7*a* containing 5 meshes and 10 junction-pairs

(15 = 5 + 10). A typical junction-network of 15 coils is shown in Fig. 3.7*b* containing 5 junction-pairs and 10 meshes (15 = 5 + 10).

3. *When a network operates under the most general operating conditions*, instead of considering it as a collection of meshes or as a collection of junction-pairs to obtain the simplest analysis, *it has to be considered as a collection of both meshes and junction-pairs.* In this case the variables are the currents **i** flowing in the meshes and the differences of potential **E** appearing across the junction pairs. The maximum number of variables that may be assumed is the same as the sum of the number of meshes and junction-pairs, that is, as the number of coils.

Owing to the orthogonal properties assumed by the meshes and the junction-pairs of such a network, the latter will be called an "*orthogonal network.*"

One and the same network may be looked upon as a mesh, or a junction, or an orthogonal network, depending on the assumed nature of the impressed voltages or currents, as shown in Fig. 3.8.



(*a*) Mesh Network (*b*) Junction Network (*c*) Orthogonal Network

FIG. 3.8.—Types of Impressed Quantities

(*b*) In index notation the following *variable* indices will be used to differentiate between these three types of networks:

1. The meshes will be denoted by $m, k, n \cdots$.
2. The junction-pairs will be denoted by $u, v, w \cdots$.
3. Both meshes and junction-pairs will be denoted by $\alpha, \beta, \gamma \cdots$.

The three types of network will have analogous methods of attack.

XV. SUMMARY OF THE "EQUATIONS OF PERFORMANCE"

(*a*) Each type of network has a different "equation of performance." In particular:

1. In analyzing a network as a collection of *meshes*, the equation to be set up first is the equation of voltage

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i} \qquad \Big| \qquad e_m = z_{mn} i^n \qquad 3.3$$

where the variable is $\mathbf{i}$. The number of ordinary equations is the same as the number of meshes.

2. In analyzing a network as a collection of *junction-pairs* the equation to be set up first is the equation of current

$$\mathbf{I} = \mathbf{Y} \cdot \mathbf{E} \qquad \Big| \qquad I^u = Y^{uv} E_v \qquad 3.4$$

where the variable is $\mathbf{E}$. The number of ordinary equations is the same as the number of junction-pairs.

3. In analyzing a network as a collection of *both meshes and junction-pairs*, the equation to be set up first is either the equation of voltage

$$\mathbf{E} + \mathbf{e} = \mathbf{z} \cdot (\mathbf{i} + \mathbf{I}) \qquad \Big| \qquad E_\alpha + e_\alpha = z_{\alpha\beta}(i^\beta + I^\beta) \qquad 3.5$$

or the equation of current

$$\mathbf{i} + \mathbf{I} = \mathbf{Y} \cdot (\mathbf{E} + \mathbf{e}) \qquad \Big| \qquad i^\alpha + I^\alpha = Y^{\alpha\beta}(E_\beta + e_\beta) \qquad 3.6$$

where the variables are both $\mathbf{i}$ and $\mathbf{E}$. The number of ordinary equations in the two cases are not necessarily the same. In the most general case their number is the same, namely. the sum of the number of meshes and junction-pairs.

(*b*) *It is emphasized that the points of view of considering a given network as a collection of meshes only, or as a collection of junction-pairs only, are special cases and are due to the special assumptions as to the nature of the impressed quantities that are known or of the response quantities that are to be found. Every network actually is a collection of both meshes and junction-pairs having as many degrees of freedom as there are coils, and the leads at the junctions* (shown in Fig. 3.1) *originally belong to the network.*

It is also emphasized that the equation of performance of a network is not simply a generalized Ohm's law, $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$, *containing three concepts, but the more general equation of voltage* $\mathbf{E} + \mathbf{e} = \mathbf{z} \cdot (\mathbf{i} + \mathbf{I})$ *containing five concepts that reduces to Ohm's law as a special case. The general equation of voltage cannot be replaced by such a simplified form as* $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$, *either mathematically or physically, except in special cases. In a general network four distinct electrical concepts appear*, $\mathbf{e}$, $\mathbf{I}$, $\mathbf{E}$, $\mathbf{i}$, and their number cannot be reduced.

(*c*) Of course, in most problems *the number of variables and the number of unknowns are not necessarily the same.* Also the number of equations available is usually not the same as the number of variables. This difference in the number of equations, the number of variables, and the number of unknowns is responsible for the large variety of manipulations possible with a set of equations.

(*d*) *The systematic study of networks will be undertaken in the above order.* That is, first *mesh* networks will be studied representing one extreme case of performance, then *junction* networks representing the other extreme case, and finally *orthogonal* networks in which both mesh and junction voltages and currents exist.

Also it will be assumed first that in all three cases the number of unknowns is the same as the number of equations that are established, leading to the simplest type of manipulations. Afterward any number of unknowns will be assumed, leading to more complicated manipulations.

XVI. TYPES OF NETWORKS

There are networks which *by the nature of their engineering application* are to be considered either as mesh or as junction networks. In particular:

1. *Multiwinding transformers, transmission lines, and rotating electrical machines are primarily "mesh" networks.* Their design constants, that are calculated from their structure, are the self- and mutual impedances $\mathbf{z}$, and their equation of performance is set up as $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$.

2. *Multielectrode vacuum tubes are primarily "junction" networks.* Their design constants are the self- and mutual admittances $\mathbf{Y}$, and their equation of performance is $\mathbf{I} = \mathbf{Y} \cdot \mathbf{E}$.

3. *Communication, distribution, and general static networks may be considered either as "mesh" or as "junction" networks or as "orthogonal" networks* since most of their elements consist of a lumped impedance Z, whose admittance is $Y = 1/Z$.

It will be found that the consideration of a communication, distribution, or any general network as a "*junction*" network offers far more variety of manipulations than the mesh point of view. Also the concepts offered by the junction point of view correspond more closely to the problems that arise in engineering practice.

Also it may be stated that in general:

1. Purely *magnetic* networks, or electrical networks interlinked with magnetic networks, are primarily *mesh* networks.

2. Purely *electrostatic* networks, or electrical networks interconnected with them, are primarily *junction* networks.

3. Electrical networks interlinked with both magnetic and electrostatic networks are primarily orthogonal networks.

It will be also found that in problems of synthesis (where the performance is given and the network is to be constructed) *all networks have to be considered as "orthogonal" networks in order to pass easily from one network to another.*

XVII. THE CONCEPT OF "PRIMITIVE SYSTEM"

The Second Generalization Postulate states that, if the components of the geometric objects, say e_α, $Z_{\alpha\beta}$, $\Gamma_{\alpha\beta\gamma}$, etc., of *one* particular system are known, then for *any other* system of the same type (infinite in number) the new components of the geometric objects can be established in a routine manner with the aid of a transformation tensor **C**. The equations of performance of all systems of the same type are the same in terms of geometric objects.

It should be noted that it is entirely *immaterial* what that particular system is whose geometric objects and invariant equations are assumed to be known. On the other hand, *before the postulate can be applied it is absolutely necessary that the equation of performance of at least one particular system should be known already in terms of geometric objects and also the components of the geometric objects for this particular system. Then this system can be used as a starting point for the calculation of all the other systems.*

Now, when the performance of a large number of systems is to be analyzed, it seems the logical procedure to select out of the large variety of systems *one particular system*, for which:

1. It is comparatively easy to set up the components of the various geometric objects e_α, $Z_{\alpha\beta}$, $\Gamma_{\alpha\beta\gamma}$, etc., that are needed for certain performance calculations.

2. It is comparatively easy to set up the various *transformation tensors* $C^{\alpha}_{\alpha'}$ representing the difference between this system and *any of the other* systems.

3. It is comparatively easy to calculate the new components of the various geometric objects $e_{\alpha'}$, $Z_{\alpha'\beta'}$, $\Gamma_{\alpha'\beta'\gamma'}$, etc., of all the other systems that also are to be analyzed.

This particular system, selected out of the large variety of systems, to be used as a starting point will be called in this treatise the "primitive system."

Stationary static networks, vacuum tubes, rotating machines, transmission lines, etc., each will have its own primitive system. In general *there will be as many different types of primitive systems as there are different types of invariant equations.* Or in other words there are as many types of primitive systems as there are different types of:

1. Fundamental systems to be analyzed.
2. Points of view to be employed.

Also, as more and more details of the systems are to be considered, *the primitive system itself may assume increasingly more complex forms* parallel with the actual system.

It should be most emphatically understood that one is at liberty to select any one of the large variety of systems as the starting point, that is as the primitive system, and derive from its performance the performance of all the other systems. The selection of some particular system as the primitive system is a question of personal judgment whether that system satisfies one or more of the above-mentioned considerations or some other particular conditions.

Later on, many occasions will arise to derive the performance of a new system from that of some other system instead of the primitive system because of the greater simplicity of calculations afforded by such a selection or because of other considerations. Hence *the use of the primitive system as a starting point in the analysis is only a convenience, not an absolute necessity.*

But whatever system is used as the starting point to find the performance of one or more new systems, *in all cases the calculations are of a routine nature and require the same steps for each new system,* namely:

1. A transformation tensor **C** is established.
2. The new components of the various geometric objects are calculated with the aid of "transformation formulas."
3. The equation of performance already established for the generalized system remains unchanged.
4. The invariant equation is *manipulated* for various reasons.
5. The equation is *solved* for the unknowns, if there are any.

This last step, of course, sometimes may encounter insurmountable difficulties depending on the type of equations that have been established. Quite often it happens that a set of equations that appears to be insoluble yields an easy solution as soon as a new, more appropriate reference frame is introduced.

XVIII. THE PRIMITIVE "MESH" NETWORK

(*a*) It is shown in Section X that, in setting up the equations of performance of *lumped networks with stationary interconnections,* three different points of view may be employed for each network:

1. The *mesh currents* **i** are assumed as *variables.*
2. The *differences of potential* **E** appearing across junction-pairs are assumed as variables.
3. Both *mesh currents* **i** and *differences of potential* **E** are assumed as variables.

The first point of view assumes the network of *n* coils as a *collection of k meshes,* the second point of view as a *collection of n − k junction-*

pairs, and the general point of view as a *collection of both k meshes and n − k junction-pairs*. The first two points of view are *special cases* of the last point of view.

It will be shown that orthogonal networks with n coils may be analyzed either as mesh networks with n (or less) meshes or as junction networks with n (or less) junction-pairs; hence *it will be sufficient to set up two types of primitive networks, namely a primitive " mesh " network and a primitive " junction " network*, representing the two extreme cases of coil arrangements into networks.

(*b*) The question now arises: What is the simplest network that consists of a collection of n coils and k meshes?

The answer is quite obvious in this case. A set of n individual coils each short-circuited upon itself (shown in Fig. 3.9) is the simplest

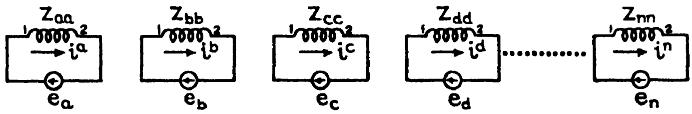


FIG. 3.9.—The "*Primitive*" Mesh Network

possible collection of n coils in n meshes. Hence *a set of n individual coils without any interconnection between them will be assumed as the "primitive mesh network."* Some or all of the coils may have asymmetrical mutual impedances between them.

It should be noted that, in establishing the primitive network with n coils and n meshes as a starting point for the analysis of a network with n coils and k meshes, it is tacitly assumed that the number of meshes k (and the number of junction-pairs $n - k$) of a network is not a determining factor for its analysis. *The only determining factor is the number of coils n*, which is kept unchanged for both actual and primitive networks. That is, when a physical network is given it is possible to change arbitrarily its number of meshes and its number of junction-pairs *together* by assuming junctions to exist between a generator and a coil, etc. (The number of coils n may also be changed though by assuming coils with zero impedance, or dividing one coil into two coils in series, etc.)

(*c*) Once the primitive mesh network has been established, the next step is to set up the various *geometric objects* that play a part in its physical analysis, then to set up its *equation of performance* in terms of these geometric objects.

In accomplishing these steps the First Generalization Postulate is used as a guiding principle by analyzing first the *simplest unit* of the primitive network as follows:

1. The simplest unit of the primitive mesh network is assumed to be *one mesh.*

2. The concepts necessary for its performance analysis are e, i, and z. *The limitations of Section VI(a) are made about the form of e, i, and z.* They may be transient or steady-state, time functions, or they may be linear operators, etc.

3. Its equation of voltage is $e = zi$.

Now repeating the same steps for the primitive network:

4. The primitive network is established as a collection of n simple units.

5. The three geometric objects of the primitive mesh network with n meshes are (see also Section IV, Chapter I)

$$\mathbf{e} = \begin{array}{c} \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \cdots & \mathbf{n} \end{array} \\ \begin{array}{|c|c|c|c|c|} \hline e_a & e_b & e_c & \cdots & e_n \\ \hline \end{array} \end{array} \qquad 3.7$$

$$\mathbf{i} = \begin{array}{c} \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \cdots & \mathbf{n} \end{array} \\ \begin{array}{|c|c|c|c|c|} \hline i_a & i_b & i_c & \cdots & i_n \\ \hline \end{array} \end{array} \qquad 3.8$$

$$\mathbf{z} = \begin{array}{c|c|c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \cdots & \mathbf{n} \\ \hline \mathbf{a} & Z_{aa} & Z_{ab} & Z_{ac} & \cdots & Z_{an} \\ \hline \mathbf{b} & Z_{ba} & Z_{bb} & Z_{bc} & \cdots & Z_{bn} \\ \hline \mathbf{c} & Z_{ca} & Z_{cb} & Z_{cc} & \cdots & Z_{cn} \\ \hline \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \mathbf{n} & Z_{na} & Z_{nb} & Z_{nc} & \cdots & Z_{nn} \\ \hline \end{array} \qquad 3.9$$

The impedance tensor $\mathbf{z}$ in general is not symmetrical and the two mutual impedances between two elements Z_{ab} and Z_{ba} may be different in the two directions. For instance, in a rotating machine Z_{ac} is zero while Z_{ca} is not zero, so that the form of the impedance tensor $\mathbf{z}$ of its primitive network with four coils is:

$$\mathbf{z} = \begin{array}{c|c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \hline \mathbf{a} & Z_{aa} & Z_{ab} & 0 & 0 \\ \hline \mathbf{b} & Z_{ba} & Z_{bb} & Z_{bc} & Z_{bd} \\ \hline \mathbf{c} & Z_{ca} & Z_{cb} & Z_{cc} & Z_{cd} \\ \hline \mathbf{d} & 0 & 0 & Z_{dc} & Z_{dd} \\ \hline \end{array} \qquad 3.10$$

Again it is emphasized that *the components of the various geometric objects may assume a large variety of forms. They may be constants, or*

time functions, or linear operators, etc. Even in case of coils with self- and mutual inductances only, the components Z_{aa}, Z_{ab} do not necessarily have to be the standard self- and mutual reactances measured in ohms. *They may be any hypothetical design constants,* as will be shown in the study of multiwinding transformers.

6. The equation of voltage of the primitive mesh network in terms of geometric objects is:

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i} \qquad \Big| \qquad e_m = z_{mn} i^n \tag{3.11}$$

The single invariant equation is equivalent to the set of n ordinary equations of voltage

$$\left.\begin{aligned} e_a &= Z_{aa}i^a + Z_{ab}i^b + Z_{ac}i^c + \cdots Z_{an}i^n \\ e_b &= Z_{ba}i^a + Z_{bb}i^b + Z_{bc}i^c + \cdots Z_{bn}i^n \\ e_c &= Z_{ca}i^a + Z_{cb}i^b + Z_{cc}i^c + \cdots Z_{cn}i^n \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ e_n &= Z_{na}i_a + Z_{nb}i_b + Z_{nc}i_c + \cdots Z_{nn}i_n \end{aligned}\right\} \tag{3.12}$$

(*d*) In most problems the *single* invariant equation $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ is subdivided into *several* invariant equations in various manners. Such subdivisions will be studied in later chapters.

When the equation is treated as one unit without subdivision, its manipulation and solution follow closely those of the analogous ordinary equation as shown in Section XI, Chapter II. That is, the currents flowing in the coils of the primitive mesh are found by $\mathbf{i} = \mathbf{z}^{-1} \cdot \mathbf{e}$.

A more general form of the primitive mesh network and its equation of performance will be given in Chapter XVI.

(*e*) In the present case the establishment and the analysis of the primitive system are quite simple and almost obvious. However, as the system to be analyzed or the point of view employed increases in complexity, both the establishment and the method of analysis of the primitive system become less obvious and more involved. But, ***no matter how complex the primitive systems are, the above six steps or their equivalent have to be followed in all cases for their analysis.***

XIX. THE PRIMITIVE "JUNCTION" NETWORK

(*a*) When a network with n coils is to be analyzed by assuming the *differences of potential* appearing across $n - k$ junction-pairs as the variables, again the first step is to establish a primitive "junction" network and to set up its equation of performance in terms of geometric objects. The mesh network of Fig. 3.9 cannot be used as the primitive network for the *new point of view* since no junction-pairs exist,

the number of meshes being the same as the number of coils. Hence *a new primitive network has to be established for the new point of view.*

(*b*) The question now is: What is the simplest network that consists of a collection of n coils in $n - k$ junction-pairs?

The primitive junction-network consists of n open-circuited coils as shown in Fig. 3.10. It contains n independent sub-networks having $2n$ junctions, hence having $2n - n = n$ junction-pairs. *The two ends of each coil form a junction-pair* through which the current enters and departs. Two junctions on two coils cannot form a junction-pair since the latter would lie on two independent sub-networks.

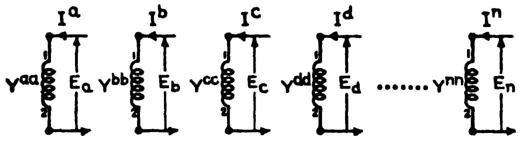


FIG. 3.10.—The "*Primitive*" Junction Network

(*c*) To find the equation of performance of the primitive junction-network the First Generalization Postulate is used as in the previous section. That is:

1. The simplest unit of the primitive network is *one open-circuited coil* shown in Fig. 3.10.

2. The concepts necessary for its performance calculation are E, I, and Y, where I is the current impressed on the coil, E is the *difference of potential* appearing across the two junctions due to I flowing through the coil, and Y is the admittance of the coil.

3. The equation of performance of the simple unit is now not an equation of voltage, but an equation of current:

$$I = YE$$

where I and Y are assumed to have fixed values, and E, the difference of potential appearing on the junction-pair, is assumed as variable. If E is unknown then it is found as $E = Y^{-1}I$

Repeating the same steps for the primitive network:

4. The primitive junction-network for n coils is established as in Fig. 3.10, where n may be any number.

5. The three geometric objects necessary for its performance analysis are:

$$\mathbf{E} = \begin{array}{ccccc} a & b & c & \dots & n \\ \hline E_a & E_b & E_c & \dots & E_n \end{array} \qquad 3.13$$

$$\mathbf{I} = \begin{array}{ccccc} a & b & c & & n \\ \hline I^a & I^b & I^c & \dots & I^n \end{array} \qquad 3.14$$

$$\mathbf{Y} = \begin{array}{c|ccccc|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \cdots & \mathbf{n} \\ \hline \mathbf{a} & Y^{aa} & Y^{ab} & Y^{ac} & \cdots & Y^{an} \\ \mathbf{b} & Y^{ba} & Y^{bb} & Y^{bc} & \cdots & Y^{bn} \\ \mathbf{c} & Y^{ca} & Y^{cb} & Y^{cc} & \cdots & Y^{cn} \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{n} & Y^{na} & Y^{nb} & Y^{nc} & \cdots & Y^{nn} \\ \hline \end{array} \qquad 3.15$$

The components of $\mathbf{E}$ are the *differences of potential* appearing across the junction-pairs. The components of $\mathbf{I}$ are the currents flowing into and out of the junction-pairs. The components of $\mathbf{Y}$ represent the *self- and mutual admittances* of the individual coils. The matrix may be asymmetrical, and some of its components may be zero.

6. The equation of current of the primitive junction network is

$$\boxed{\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}} \quad \Big| \quad \boxed{I^u = Y^{uv}E_v} \qquad 3.16$$

where the components of $\mathbf{E}$ may be positive or negative depending whether it is an *impressed* junction voltage or not.

The single invariant equation represents the following set of n ordinary equations

$$\left.\begin{aligned} I^a &= Y^{aa}E_a + Y^{ab}E_b + Y^{ac}E_c + \cdots Y^{an}E_n \\ I^b &= Y^{ba}E_a + Y^{bb}E_b + Y^{bc}E_c + \cdots Y^{bn}E_n \\ I^c &= Y^{ca}E_a + Y^{cb}E_b + Y^{cc}E_c + \cdots Y^{cn}E_n \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ I^n &= Y^{na}E_a + Y^{nb}E_b + Y^{nc}E_c + \cdots Y^{nn}E_n \end{aligned}\right\} \qquad 3.17$$

(*d*) Before this invariant equation is manipulated, in most problems it is divided into several component equations in various manners. Such subdivisions are treated in later chapters.

When the equation is not subdivided but is to be solved as one unit, the unknown junction potentials $\mathbf{E}$ are found by

$$\boxed{\mathbf{E} = \mathbf{Y}^{-1}\cdot\mathbf{I}} \quad \Big| \quad \boxed{E_v = (Y^{uv})^{-1}I^u} \qquad 3.18$$

The replacement of the single invariant equation by several equations and the variety of manipulations appear when, say, ***not all*** of the junction-pairs have voltages or currents impressed on them.

Later on, a more general form of the primitive junction network and its equation of performance will be given.

CHAPTER IV

THE TRANSFORMATION TENSOR

I. STEPS IN THE ANALYSIS

(*a*) The analysis of a whole group of physical systems in terms of geometric objects may be assumed to consist of three steps:

1. The equation of performance (an equation of voltage or an equation of current, etc.) valid for each member of the group is first *established.*

2. The equation is *manipulated* in various manners for various reasons. *For purposes of manipulation the single invariant equation of performance is usually subdivided into several invariant equations.*

3. The unknown quantities, if there are any, are *solved for.*

In the first few chapters the equations of performance of the three types of stationary networks (mesh, junction, and orthogonal networks) will be set up, manipulated, and solved for the variables. However, the invariant equation will not be subdivided, but will be manipulated first as one unit.

(*b*) For most practical problems the single invariant equation of performance has to be subdivided into several invariant equations. Needs for the subdivision may arise in a variety of manners:

1. Some of the unknowns may not be needed, hence they may be eliminated.

2. Some of the variables or impedances, etc., may be known; some may be unknown.

3. The physical system itself divides *functionally* into several parts. For instance, at some terminals the currents may remain constant under all loads, or the differences of potentials may remain constant or may obey some predetermined law. At some parts the impedances may vary, while the currents in another part are varied in some desired manner, etc., etc.

4. The impedances or the impressed voltages or currrents in certain parts or in the whole system may be *replaced* by new quantities, while the performance of the system still satisfies certain requirements or criteria, etc.

The study of the *subdivision* of the invariant equation of performance will be undertaken in later chapters.

(c) *In this chapter will be investigated the setting up of the equations of voltage* $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$ of a mesh network having n coils and n meshes, if that of the primitive mesh network, $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$, is known. Afterward any mesh network with n coils and *less than* n meshes will be studied.

II. ALL-MESH NETWORKS

(a) In analyzing any network as a collection of *meshes*, it will be first assumed that *the network to be analyzed has as many meshes as it has coils*; this will be called an " all-mesh " network. The analysis of mesh networks with n coils but with *less than n meshes* (of far greater practical importance) is a *special case* of the analysis of those with n meshes; hence the study of the latter will be taken up first.

There are a large number of ways in which n coils may be arranged to form n meshes. For instance, seven different ways of arranging five coils into five meshes are shown in Fig. 4.1, including their common primitive network. All the apparently different networks of Fig. 4.1 have the following common property:

With a given voltage $\mathbf{e}$ *impressed in series with each coil, the currents* $\mathbf{i}$ *flowing in each coil are the same no matter how the n coils are interconnected into n meshes.*

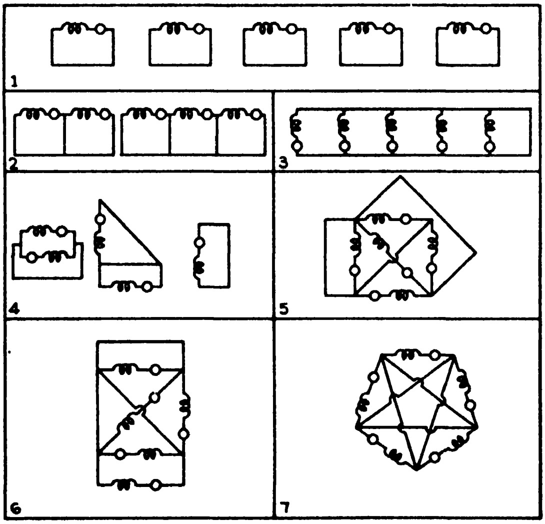


FIG. 4.1.—Seven Different Interconnections of Five Coils into Five Meshes

The currents in each coil remain the same since *in each network each coil is short-circuited upon itself.* The various interconnections consist of taking the primitive network with five independent meshes and interconnecting its junctions in various manner by conductors having no impedances. None of the networks contains any junction-pairs.

(*b*) Although the currents in the coils remain unchanged, the currents flowing in the various junction wires are different in each type of connection. The components of each geometric object in *the equations of voltage* $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ of each network are the same if they are expressed in terms of the currents flowing through the coils, but *the components assume a different value for each network* if they are expressed in terms of currents flowing also in the impedanceless junction wires. In fact, it is found that even for the same network several sets of n equations can be set up, each set being different from the others and from those of other networks.

(*c*) One of the purposes of tensor analysis is to *systematize the setting up* of this large variety of sets of equations. *The systematization is performed by considering the analysis of all the possible networks at the same time* and, (1) pointing out those characteristics that are *identical* for all the networks and denoting them by separate symbols, (2) pointing out those characteristics in which all networks *differ* from each other and denoting them also by separate symbols.

In order that the complexity of the network should not hinder the understanding of the new concepts and method of reasoning to be introduced, the simplest possible network will be analyzed first, namely, one containing only two coils.

III. INTERCONNECTION OF COILS

(*a*) Let two coils Z_{aa} and Z_{bb} without any interconnection between them be given as shown in Fig. 4.2. Let *any* voltages be impressed in series with them, and let currents flow in them. (One specific value of the impedances, currents, and voltages is also shown in Table 4.1.) The network is equivalent to a primitive mesh network with two meshes, and its equation of voltage is easily set up as

$$\begin{aligned} e_a &= Z_{aa}i^a + Z_{ab}i^b \\ e_b &= Z_{ba}i^a + Z_{bb}i^b \end{aligned} \qquad 4.1$$

In terms of the three geometric objects $\mathbf{e}$, $\mathbf{i}$, and $\mathbf{z}$, the above set of ordinary equations can be written as:

$$\boxed{\mathbf{e} = \mathbf{z}\cdot\mathbf{i}} \qquad (4.2)$$

Where

$$\mathbf{e} = \begin{array}{cc} a & b \\ \hline e_a & e_b \end{array} \qquad (4.3)$$

$$\mathbf{i} = \begin{array}{cc} a & b \\ \hline i^a & i^b \end{array} \qquad (4.4)$$

$$\mathbf{z} = \begin{array}{c|cc} & a & b \\ \hline a & Z_{aa} & Z_{ab} \\ b & Z_{ba} & Z_{bb} \end{array} \qquad (4.5)$$

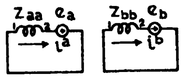


FIG. 4.2.—The Primitive Network

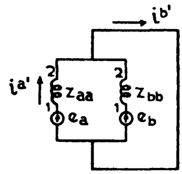


FIG. 4.3.—The New Network

TABLE 4.1

$Z_{aa} = 2$ $\qquad Z_{bb} = 3$

$e_a = 1$ $\qquad e_b = 2$

$i^a = 1/2$ $\qquad i^b = 2/3$

$$\mathbf{e} = \begin{array}{cc} a & b \\ \hline 1 & 2 \end{array}$$

$$\mathbf{i} = \begin{array}{cc} a & b \\ \hline 1/2 & 2/3 \end{array}$$

$$\mathbf{z} = \begin{array}{c|cc} & a & b \\ \hline a & 2 & 0 \\ b & 0 & 3 \end{array}$$

$$P = \mathbf{e}\cdot\mathbf{i} = 11/6$$

The currents are found by $\mathbf{i} = \mathbf{z}^{-1}\cdot\mathbf{e} = \mathbf{y}\cdot\mathbf{e} =$

$$\mathbf{i} = \begin{array}{|cc|} \hline Z_{bb}/D & -Z_{ab}/D \\ -Z_{ba}/D & Z_{aa}/D \\ \hline \end{array} \cdot \begin{array}{|c|} \hline e_a \\ e_b \\ \hline \end{array} = \begin{array}{c|c|} a & Z_{bb}e_a/D - Z_{ab}e_b/D \\ b & -Z_{ba}e_a/D + Z_{aa}e_b/D \\ \hline \end{array} \qquad (4.6)$$

(where $D = Z_{aa}Z_{bb} - Z_{ab}Z_{ba}$) representing the currents i^a and i^b flowing in each isolated coil.

(*b*) Let it be assumed now that the two coils are interconnected as shown in Fig. 4.3, forming again two meshes, without changing the voltages impressed in series with the individual coils or the self- and mutual impedances of the coils. Since *after* interconnection each coil is still short-circuited upon itself as it was *before* interconnection, *the current through each coil also remains unchanged.*

However, now *the added conductors introduce new current-paths, and the equation of voltage can be expressed in terms of other currents*

beside those flowing in the coils. The new currents are selected either *arbitrarily* or by some requirement of the problem at hand. The number of new currents to be selected is the same as the number of meshes, that is, as the number of the old currents. Hence the performance of the system of Fig. 4.3 can be expressed now, for instance, in terms of:

1. The current flowing through the added impedanceless branch.
2. The current flowing through coil Z_{aa}.

(If the performance of the new system is expressed in terms of the two currents flowing through the two coils, the previous equation of voltage remains unchanged.)

The two branches in which the two new currents flow will be denoted by *primed* letters as **a**$'$ and **b**$'$, and the currents flowing through them will be denoted as $i^{a'}$ and $i^{b'}$.

(*c*) The purpose of the analysis is to set up the voltage equation of the new interconnected system, having exactly the same form in terms of geometric objects as it had before the interconnection. That is, the purpose of the analysis is to find the new components of the three geometric objects $\mathbf{e}'$, $\mathbf{i}'$, and $\mathbf{z}'$ existing in the voltage equation of the new system $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$.

In any general problem the components of $\mathbf{z}'$ and $\mathbf{e}'$ are usually known, and the value of $\mathbf{i}'$ is unknown. Irrespective of what quantities are assumed to be known or unknown, *in setting up the equations of a mesh network the currents* $\mathbf{i}'$ *are always assumed as variables.*

(*d*) As a general procedure all variable indices belonging to the new system will have a prime attached to them as shown in Table 4.2 In direct notation the geometric objects themselves will have a prime attached, since their variable indices are absent.

TABLE 4.2

	Before Interconnection	After Interconnection
Fixed indices....................	$a, b, c \cdots$	$a', b', c' \cdots$
Variable indices.................	$k, m, n \cdots$	k', m', n'
Geometric objects (index notation)	e_m, i^m, z_{mn}	$e_{m'}, i^{m'}, z_{m'n'}$
Geometric objects (direct notation)	$\mathbf{e}, \mathbf{i}, \mathbf{z}$	$\mathbf{e}', \mathbf{i}', \mathbf{z}'$

In index notation *the base letter, say* z, *remains unchanged in any system; only the indices attached to it vary* as the system is changed to another system from z_{mn} to $z_{m'n'}$.

Also the current vector will have an upper index as i^m instead of a lower index. The reason for this notation will be shown presently. The components of i^m will be called "*old currents*" and the components of $i^{m'}$ "*new currents*," while the base letter i will be called the "*current vector*."

IV. RELATION BETWEEN OLD AND NEW CURRENTS

(*a*) *The first step in analyzing the new network of Fig. 4.3 is to consider what currents should be the new variables.* Because of the existence of two closed circuits (meshes), two currents have to be assumed as variables. *The currents assumed must be independent of one another,* that is, none of them may be expressible in terms of the others.

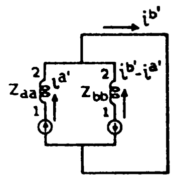


FIG. 4.4. — The New Currents in the Coils

Let the two currents be $i^{a'}$ and $i^{b'}$ flowing in coil Z_{aa} and in the impedanceless branch as shown in Fig. 4.4. That is, let the performance of the new system be expressed in terms of $i^{a'}$ and $i^{b'}$, while that of the old system is expressed in terms of i^a and i^b.

(*b*) Once the new currents (variables) have been selected, *the next step is to express the currents flowing in each coil in terms of the new variables $i^{a'}$ and $i^{b'}$.* For this step *Kirchhoff's First Law* is used, stating that: "*The sum of the currents entering a junction is zero.*"

Hence the current i^x in coil Z_{bb} of Fig. 4.3 is found by summing up the currents entering a junction. That is

$$i^{a'} - i^{b'} + i^x = 0$$

from which

$$i^x = i^{b'} - i^{a'} \tag{4.7}$$

This value of the current flowing through coil Z_{bb} is shown in Fig. 4.4.

(*c*) *The next step is to set up relations between the old currents i^a, i^b and the new currents $i^{a'}$, $i^{b'}$* (that is, between the old components i of the current vector and the new components i′).

In the new system both sets of currents can be identified:

1. The *old set of currents* i^a and i^b are the currents flowing in the individual coils.

2. The *new set of currents* $i^{a'}$ and $i^{b'}$ have been arbitrarily assumed flowing anywhere in the network.

As a consequence, *the relation between the old and the new set of currents can be set up* by a simple inspection of the new network *by considering the old and the new currents flowing through each coil* as follows:

1. Considering the currents flowing through coil Z_{aa}

$$\text{Old current } = i^a$$

$$\text{New current } = i^{a'}$$

2. Considering the currents flowing through coil Z_{bb}

$$\text{Old current } = i^b$$

$$\text{New current } = i^{b'} - i^{a'}.$$

Since through each coil identical currents flow before and after the interconnection, the following relations can be set up between the old currents i^a, i^b and the new currents $i^{a'}$, $i^{b'}$:

$$\begin{aligned} i^a &= i^{a'} &&= \quad i^{a'} \\ i^b &= i^{b'} - i^{a'} &&= - i^{a'} + i^{b'} \end{aligned} \qquad 4.8$$

These equations are valid no matter what the value of the currents is as long as the circuit is not changed. Hence *the above equations are also identities.*

(*d*) It is emphasized that in general problems it is not possible to set up a relation between the old and the new variables by a simple inspection of the new system only. *Both old and new systems must be inspected, in general.* However, in some cases of stationary networks it is possible to select the old system in such a simple form that it is imagined to be incorporated in the new system. In the present example the individual coils of the new system form the old system. That, however, is not always the case.

V. THE TRANSFORMATION TENSOR

(*a*) *The step of setting up a relation between the currents (or velocities) of the old and the new system is the central feature of finding the performance of the new system. Once this relation is established, then the remaining work of setting up the equations of performance of the new system from that of the known old system* (or of finding any other property of the new system) *is purely automatic.*

The set of linear equations 4.8 can be expressed in terms of geometric objects analogously to the set of linear equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$, as

$$\boxed{\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'} \qquad \boxed{i^m = C^m_{m'}\, i^{m'}} \qquad 4.9$$

where

$$\mathbf{i} = \begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \hline i^a & i^b \end{array} \qquad \begin{array}{c|cc} m & a & b \\ \hline i^m = & i^a & i^b \end{array} \qquad 4.10$$

$$\mathbf{i}' = \begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \hline i^{a'} & i^{b'} \end{array} \qquad \begin{array}{c|cc} m & a' & b' \\ \hline i^{m'} = & i^{a'} & i^{b'} \end{array} \qquad 4.11$$

The coefficients of the new variables form a matrix called the "transformation matrix," (or rather the "components of the transformation tensor along the *two* given reference frames")

$$\mathbf{C} = \begin{array}{c|cc} & \mathbf{a}' & \mathbf{b}' \\ \hline \mathbf{a} & 1 & 0 \\ \mathbf{b} & -1 & 1 \end{array} \qquad C^m_{m'} = \begin{array}{c|cc} m \backslash m & a' & b' \\ \hline a & 1 & 0 \\ b & -1 & 1 \end{array} \qquad 4.12$$

This two-dimensional matrix forms the backbone of tensor analysis. It shows the relation between the old variables and the new variables (currents). The reason for the use of upper and lower indices will be shown presently.

(*b*) The process of setting up the transformation matrix $C^m_{m'}$ for a new system consists then of three steps:

1. Decide what should be called the new currents $i^{m'}$ of the new system.

2. Set up a linear relation between the old currents i^m and the new currents $i^{m'}$ flowing in each coil.

That is, put on the left-hand side of the equations the old currents and on the right-hand side some linear combination of the new currents.

3. From the coefficients of the new currents form a matrix, which is the required "transformation matrix" $C^m_{m'}$.

(*c*) It should be noted that the positions of the indices of the transformation tensor $C^m_{m'}$ differ from those of the impedance tensor z_{mn}, inasmuch as in z_{mn} both indices m and n refer to the same set of old fixed indices a, b, c, whereas in $C^m_{m'}$ the upper index m refers to the old indices a, b, c and the lower index m' refers to the new indices a', b', c'. That is, *in z_{mn} or in $z_{m'n'}$ the indices written in front of the matrix and those written above it are identical, while in the matrix $C^m_{m'}$ the two sets of indices are different.*

(*d*) *The determinant of the transformation tensor of an all-mesh*

network C *is never zero*, that is, the inverse of matrix 4.12 can always be calculated. If the determinant of C is zero, it is a sign that the new variables are incorrectly assumed and are not independent of each other (that is, one of them can be expressed in terms of the others).

The *inverse transformation tensor* is found by calculating the inverse of matrix 4.12. It is denoted by interchanging its upper and lower indices. That is,

$$\mathbf{C}^{-1} = \begin{array}{c|c|c|} & \mathbf{a} & \mathbf{b} \\ \hline \mathbf{a}' & 1 & 0 \\ \hline \mathbf{b}' & 1 & 1 \\ \hline \end{array} \qquad C_m^{m'} = \begin{array}{c|c|c|} m' \backslash m & a & b \\ \hline a' & 1 & 0 \\ \hline b' & 1 & 1 \\ \hline \end{array} \qquad 4.13$$

VI. THE "INVARIANCE" OF "FORMS"

(*a*) The immediate problem under investigation is the following: There is given a network, the "primitive network" of Fig. 4.2. With that network the following concepts are associated:

1. Geometric objects—such as

$$\mathbf{e}, \mathbf{z}, P \qquad \Big| \qquad e_m, z_{mn}, P$$

and the variable $\mathbf{i} = i^m$ (which also is a geometric object).

2. Invariant equations such as

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i} \qquad \Big| \qquad e_m = z_{mn} i^n \qquad 4.14$$

$$P = \mathbf{e} \cdot \mathbf{i} \qquad \Big| \qquad P = e_m i^m \qquad 4.15$$

All these geometric objects and equations are known.

There is given now another network of Fig. 4.3. With that network are also associated exactly the same concepts as with the given network, namely:

1. Geometric objects, such as

$$\mathbf{e}', \mathbf{z}', P' \qquad \Big| \qquad e_{m'}, z_{m'n'}, P'$$

and the variable $\mathbf{i}' = i^{m'}$.

2. Invariant equations, such as

$$\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}' \qquad \Big| \qquad e_{m'} = z_{m'n'} i^{n'} \qquad 4.16$$

$$P' = \mathbf{e}' \cdot \mathbf{i}' \qquad \Big| \qquad P' = e_{m'} i^{m'} \qquad 4.17$$

However, none of the new components of these geometric objects has as yet been determined (hence none of the new equations can be

established), with the exception of the single relation set up between the old and the new variables

$$\mathbf{i} = \mathbf{C} \cdot \mathbf{i}' \qquad \Bigg| \qquad i^m = C^m_{m'} i^{m'} \qquad 4.18$$

$$\mathbf{i}' = \mathbf{C}^{-1} \cdot \mathbf{i} \qquad \Bigg| \qquad i^{m'} = C^{m'}_{m} i^m \qquad 4.19$$

defining the components of the transformation tensor which, however, is insufficient to determine the new components of the geometric objects, and thereby the new equations. It is still necessary to determine the " formula of transformation " of one other geometric object.

(*b*) In order to establish the formula of transformation of a geometric object *it is necessary to find at least one physical quantity which is the same for both systems, that is, which does not change as the reference frame changes.* The mathematical representation of the " *invariance* " of this physical quantity serves as the second relation needed for finding the transformation formulas.

This second relation is established by the recognition that, when the coils of the primitive network are interconnected, the total instantaneous power input into the whole system remains " invariant," unchanged. That is

$$\boxed{P = P'} \qquad 4.20$$

or, in tensor parlance, *the power input P is an invariant under the transformation $C^m_{m'}$*, since the current in each coil is unchanged.

It is emphasized that this second relation is not an assumption (as the first relation $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ is) but it is found from *physical considerations* as a corollary of the existence of the first relation.

(*c*) Hence in addition to the relation

$$\mathbf{i} = \mathbf{C} \cdot \mathbf{i}' \qquad \Bigg| \qquad i^m = C^m_{m'} i^{m'} \qquad 4.21$$

which determines the components of the transformation tensor, the following relation can also be set up between the old and the new components of geometric objects, representing the invariance of the power input

$$\boxed{\mathbf{e} \cdot \mathbf{i} = \mathbf{e}' \cdot \mathbf{i}'} \qquad \Bigg| \qquad \boxed{e_m i^m = e_{m'} i^{m'}} \qquad 4.22$$

which will determine the transformation formula of one other geometric object, the voltage vector $\mathbf{e}$.

It should be noted that *the power input* $\mathbf{e} \cdot \mathbf{i}$ is a " *linear form* " (Section X, Chapter II).

(*d*) To summarize: before the equations of a new system can be determined from those of a known system, two steps have to be taken:

1. *A relation has to be set up between the old and the new variables* i^m *and* $i^{m'}$. This step determines the transformation matrix $C^m_{m'}$. Each time a new system is given this step must be performed.

2. *A "form" has to be assumed "invariant" under the transformation.* This assumption has to be taken for only one new system and need not be taken for any other new system.

VII. THE "TRANSFORMATION FORMULA" OF THE VOLTAGE VECTOR e_m

(*a*) From the previous considerations two relations, equations 4.21 and 4.22 are now available to determine the values of the new geometric objects $\mathbf{e}'$ and $\mathbf{z}'$. Both these relations are valid for *any* value of the variables $\mathbf{i}$ and $\mathbf{i}'$ or of the impressed voltages $\mathbf{e}$ and $\mathbf{e}'$, or of the interconnections $\mathbf{C}$; hence they are "identities" rather than equations. That is, the available identities are:

$$\mathbf{i} \equiv \mathbf{C} \cdot \mathbf{i}' \qquad\Big|\qquad i^m \equiv C^m_{m'} i^{m'} \tag{4.23}$$

$$\mathbf{e} \cdot \mathbf{i} \equiv \mathbf{e}' \cdot \mathbf{i}' \qquad\Big|\qquad e_m i^m \equiv e_{m'} i^{m'} \tag{4.24}$$

(*b*) Now substituting $\mathbf{i}$ from the first identity into the second one

$$\mathbf{e} \cdot \mathbf{C} \cdot \mathbf{i}' \equiv \mathbf{e}' \cdot \mathbf{i}' \qquad\Big|\qquad e_m C^m_{m'} i^{m'} \equiv e_{m'} i^{m'} \tag{4.25}$$

Since these identities are valid for any value of the variables $\mathbf{i}'$, the vector $\mathbf{i}'$ can be cancelled on both sides of the identities, leaving

$$\mathbf{e} \cdot \mathbf{C} \equiv \mathbf{e}' \qquad\Big|\qquad e_m C^m_{m'} \equiv e_{m'} \tag{4.26}$$

Hence the new components $\mathbf{e}'$ *of the voltage vector are found from the old components* $\mathbf{e}$ *by the "transformation formula"*

$$\boxed{\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}} \qquad\Big|\qquad \boxed{e_{m'} = C^m_{m'} e_m} \tag{4.27}$$

that is, by premultiplying $\mathbf{e}$ by the transpose of the transformation tensor.

(*c*) Calculating $\mathbf{e}'$ by multiplying the transpose of matrix 4.12 with the vector 4.3, its components are

$$\mathbf{e}' = \begin{array}{c|cc} & \mathbf{a} & \mathbf{b} \\ \hline a' & 1 & -1 \\ b' & 0 & 1 \end{array} \cdot \begin{array}{c|c} \mathbf{a} & e_a \\ \mathbf{b} & e_b \end{array} = \begin{array}{cc} \mathbf{a}' & \mathbf{b}' \\ \hline e_a - e_b & e_b \end{array} \tag{4.28}$$

That is, in setting up the equation of voltage $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$ for the new system, *the new voltage components are the sum of the voltages impressed around the two meshes*, namely, $e_a - e_b$ when the left-hand mesh is considered, and e_b when the right-hand mesh is considered.

(*d*) The inverse relation follows from equation 4.27 by multiplying both sides of the equation on the left-hand side by $\mathbf{C}_t^{-1}$ (or by $C_k^{m'}$)

$$\mathbf{C}_t^{-1} \cdot \mathbf{e}' = \mathbf{C}_t^{-1} \cdot \mathbf{C}_t \cdot \mathbf{e} \quad \Big| \quad C_k^{m'} e_{m'} = C_k^{m'} C_{m'}^{m} e_m \qquad 4.29$$

But

$$\mathbf{C}_t^{-1} \cdot \mathbf{C}_t = \mathbf{I} \quad \Big| \quad C_k^{m'} C_{m'}^{m} = \delta_k^m \qquad 4.30$$

where $\mathbf{I} = \delta_k^m$ is the " unit tensor." Since multiplying any geometric object by $\mathbf{I}$ leaves it unchanged, $\mathbf{I}$ can be left out, giving the *inverse transformation formula* (changing the free index k to m)

$$\boxed{\mathbf{e} = \mathbf{C}_t^{-1} \cdot \mathbf{e}'} \quad \Big| \quad \boxed{e_m = C_m^{m'} e_{m'}} \qquad 4.31$$

Comparing this formula with that of $\mathbf{i}$

$$\mathbf{i} = \mathbf{C} \cdot \mathbf{i}' \quad \Big| \quad i^m = C_{m'}^{m} i^{m'} \qquad 4.32$$

it can be seen that $\mathbf{e}$ *has a different transformation formula from* $\mathbf{i}$.

VIII. THE TRANSFORMATION FORMULA OF z_{mn}

(*a*) When the transformation formula of *one* geometric object belonging to an equation of the new system has been determined (by recognizing an invariant " form ") the transformation formulas of *all the other* geometric objects occurring in the equation can be determined *automatically* from the fundamental assumption of tensor analysis, namely, from the Second Generalization Postulate, that *in terms of geometric objects the form of all equations describing the old system or the new system is the same.*

That is, if the equation of voltage of the old system is

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i} \quad \Big| \quad e_m = z_{mn} i^n \qquad 4.33$$

then the equation of voltage of the new system has exactly the same form, namely

$$\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}' \quad \Big| \quad e_{m'} = z_{m'n'} i^{n'} \qquad 4.34$$

(*b*) From the requirement that the form of the voltage equation should not change, the transformation formula of $\mathbf{z}$ can be derived as follows:

Substitute the transformation formulas of **e** and **i** (equations 4.31 and 4.32) into the old equation of voltage 4.33.

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i} \qquad \bigg| \qquad e_m = z_{mn} i^n \tag{4.35}$$

$$\mathbf{C}_t^{-1} \cdot \mathbf{e}' = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}' \qquad \bigg| \qquad C_m^{k'} e_{k'} = z_{mn} C_{n'}^n i^{n'} \tag{4.36}$$

Multiplying both sides of the equation on the left-hand side by $\mathbf{C}_t = C_{m'}^m$

$$\mathbf{C}_t \cdot \mathbf{C}_t^{-1} \cdot \mathbf{e}' = \mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}' \qquad \bigg| \qquad C_{m'}^m C_m^{k'} e_{k'} = C_{m'}^m z_{mn} C_{n'}^n i^{n'}$$

Since $\mathbf{C}_t \cdot \mathbf{C}_t^{-1}$ is the unit tensor $\mathbf{I} = \delta_{m'}^{k'}$ it can be dropped, leaving

$$\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}' \qquad \bigg| \qquad e_{m'} = z_{mn} C_{m'}^m C_{n'}^n i^{n'} \tag{4.37}$$

Since the new equation must have the form

$$\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}' \qquad \bigg| \qquad e_{m'} = z_{m'n'} i^{n'} \tag{4.38}$$

from the comparison of the last two equations follows that

$$\boxed{\mathbf{z}' = \mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}} \qquad \bigg| \qquad \boxed{z_{m'n'} = z_{mn} C_{m'}^m C_{n'}^n} \tag{4.39}$$

In other words, the components of $\mathbf{z}'$ for the new system are found from its components in the old system $\mathbf{z}$ by multiplying the latter with the transformation tensor $C_{m'}^m$ *twice in succession.*

(*c*) Performing the multiplication in two steps as indicated, the first step is

$$\mathbf{z} \cdot \mathbf{C} = \begin{bmatrix} Z_{aa} & Z_{ab} \\ Z_{ab} & Z_{bb} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{array}{c|cc} & a' & b' \\ \hline a & Z_{aa} - Z_{ab} & Z_{ab} \\ b & Z_{ab} - Z_{bb} & Z_{bb} \end{array} \tag{4.40}$$

The second step is $\mathbf{C}_t \cdot (\mathbf{z} \cdot \mathbf{C}) =$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} Z_{aa} - Z_{ab} & Z_{ab} \\ Z_{ab} - Z_{bb} & Z_{bb} \end{bmatrix} = \begin{bmatrix} Z_{aa} - 2Z_{ab} + Z_{bb} & Z_{ab} - Z_{bb} \\ Z_{ab} - Z_{bb} & Z_{bb} \end{bmatrix}$$

Hence the new components of the impedance tensor for the new network are

$$\mathbf{z}' = \begin{array}{c|cc} & a' & b' \\ \hline a' & Z_{aa} - 2Z_{ab} + Z_{bb} & Z_{ab} - Z_{bb} \\ b' & Z_{ab} - Z_{bb} & Z_{bb} \end{array} \tag{4.41}$$

IX. THE NEW EQUATION OF VOLTAGE

(*a*) The impressed voltage vector of the new system has been given in equation 4.28 as

$$\mathbf{e}' = \begin{array}{cc} \mathbf{a}' & \mathbf{b}' \\ \hline e_a - e_b & e_b \end{array} \qquad 4.42$$

Hence the equation of voltage of the new system $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$ is equivalent in terms of ordinary quantities to

$$\begin{aligned} e_a - e_b &= (Z_{aa} - 2Z_{ab} + Z_{bb})i^{a'} + (Z_{ab} - Z_{bb})i^{b'} \\ e_b &= (Z_{ab} - Z_{bb})i^{a'} + Z_{bb}i^{b'} \end{aligned} \qquad 4.43$$

It may be written in a more recognizable form as

$$\begin{aligned} e_a - e_b &= Z_{aa}i^{a'} - Z_{bb}(i^{b'} - i^{a'}) - Z_{ab}i^{a'} + Z_{ab}(i^{b'} - i^{a'}) \\ e_b &= Z_{bb}(i^{b'} - i^{a'}) + Z_{ab}i^{a'} \end{aligned} \qquad 4.44$$

giving the voltage equations of the two meshes.

(*b*) In the voltage equation $\mathbf{i}'$ is still unknown. Its value is found by the formula

$$\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}' = \mathbf{y}' \cdot \mathbf{e}' \qquad \Big| \; i^{n'} = (z_{m'n'})^{-1}e_{m'} = y^{n'm'}e_{m'} \qquad 4.45$$

by finding the inverse of $\mathbf{z}'$ and multiplying it by $\mathbf{e}'$.

The inverse of $\mathbf{z}'$ is $\mathbf{y}'$

$$\mathbf{y}' = \frac{1}{\text{Det.}} \times \begin{array}{c|c|c} & \mathbf{a}' & \mathbf{b}' \\ \hline \mathbf{a}' & Z_{bb} & Z_{bb} - Z_{ab} \\ \hline \mathbf{b}' & Z_{bb} - Z_{ab} & Z_{aa} - 2Z_{ab} + Z_{bb} \end{array} \qquad 4.46$$

where $\text{Det.} = D = (Z_{aa} - 2Z_{ab} + Z_{bb})Z_{bb} - (Z_{ab} - Z_{bb})^2$. Hence

$$\mathbf{i}' = \mathbf{y}' \cdot \mathbf{e}' = \overrightarrow{\begin{array}{|c|c|} \hline Z_{bb}/D & (Z_{bb} - Z_{ab})/D \\ \hline (Z_{bb} - Z_{ab})/D & (Z_{aa} - 2Z_{ab} + Z_{bb})/D \\ \hline \end{array}} \cdot \begin{array}{|c|} \hline e_a - e_b \\ \hline e_b \\ \hline \end{array}\downarrow =$$

$$= \mathbf{i}' = \begin{array}{c|c|} & \\ \hline \mathbf{a}' & \dfrac{Z_{bb}}{D}(e_a - e_b) + \dfrac{(Z_{bb} - Z_{ab})}{D}e_b \\ \hline \mathbf{b}' & \dfrac{(Z_{bb} - Z_{ab})}{D}(e_a - e_b) + \dfrac{(Z_{aa} - 2Z_{ab} + Z_{bb})}{D}e_b \\ \hline \end{array} = \begin{array}{c|c|} & \\ \hline \mathbf{a}' & i^{a'} \\ \hline \mathbf{b}' & i^{b'} \\ \hline \end{array} \qquad 4.47$$

giving the new currents $i^{a'}$ and $i^{b'}$ flowing in the system in terms of the impressed coil voltages e_a, e_b and the individual coil-impedances.

(*c*) If the *numerical* values given in Table 4.1 are substituted into the new *n*-matrices the resultant numerical values are given in Table 4.3.

It should be noted that the transformation formulas of **e** and **z** give only the new voltage equation $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$. The equation has to be further manipulated in order to find the unknown currents.

TABLE 4.3

$$\mathbf{z}' = \begin{array}{c|cc} & \mathbf{a}' & \mathbf{b}' \\ \hline \mathbf{a}' & 5 & -3 \\ \mathbf{b}' & -3 & 3 \end{array}$$

$$\mathbf{e}' = \begin{array}{cc} \mathbf{a}' & \mathbf{b}' \\ \hline -1 & 2 \end{array}$$

$$\mathbf{i}' = \begin{array}{cc} \mathbf{a}' & \mathbf{b}' \\ \hline 1/2 & 7/6 \end{array}$$

$$P' = \mathbf{e}' \cdot \mathbf{i}' = 11/6$$

X. CURRENTS AND VOLTAGES IN INDIVIDUAL COILS

(*a*) When the new currents $\mathbf{i}'$ have been determined, often it is necessary to know the *currents* flowing in and the *difference of potentials* appearing across each individual coil.

Since the old system (the primitive network) has been so selected that each of its axes consists of individual coils, therefore the components of **i** and **e** represent the currents and voltages appearing in the individual coils. Hence a relation has to be set up between **i** and **e** and the known new currents $\mathbf{i}'$. Since **e** is now *across* a coil, instead of *in series* with it, it will be denoted by $\mathbf{e}_c$. Both **e** and $\mathbf{e}_c$ have the same numerical value in an all-mesh network.

(*b*) The relation between **i** and $\mathbf{i}'$ is given by the equation of transformation $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$. Hence the currents flowing in the individual coils are found by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$. *This relation already has been set up in establishing the transformation matrix.*

The relation between $\mathbf{e}_c$ and $\mathbf{i}'$ can be set up by replacing **i** in $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ by $\mathbf{C} \cdot \mathbf{i}'$ and **e** by $\mathbf{e}_c$, giving $\mathbf{e}_c = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$, *where* $\mathbf{z} \cdot \mathbf{C}$ *already has been calculated in finding* $\mathbf{z}'$ *by* $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$.

Hence *the currents* **i** *in the individual coils are found from the known* $\mathbf{i}'$ by

$$\boxed{\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'} \qquad \Bigg| \qquad \boxed{i^m = C^m_{m'} i^{m'}} \qquad 4.48$$

and the voltage drops $\mathbf{e}_c$ *across the individual coils are found by* $\mathbf{e}_c = \mathbf{z} \cdot \mathbf{i}$ or by

$$\boxed{\mathbf{e}_c = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'} \qquad \Bigg| \qquad \boxed{e_m = Z_{mn} C^n_{n'} i^{n'}} \qquad 4.49$$

It is emphasized that, if the old currents **i** and voltages **e** represent not those of the primitive network but some other reference network

(a situation which occurs quite often, as will be shown later), then the above formulas do not give the currents and voltages in the individual coils but in some other collection of coils. Hence *the last two formulas should be used only if the old network is the primitive network.*

(*c*) For the network under consideration the *currents* in the individual coils, i^a and i^b, are found by equation 4.8, that is by $i^a = i^{a'}$ and $i^b = i^{b'} - i^{a'}$.

The differences of potential appearing across the individual coils are by $(\mathbf{z} \cdot \mathbf{C}) \cdot \mathbf{i}'$, where $\mathbf{z} \cdot \mathbf{C}$ is given in equation 4.40.

$$(\mathbf{z}\cdot\mathbf{C})\cdot\mathbf{i}' = \begin{array}{c|c|c|} & \mathbf{a}' & \mathbf{b}' \\ \hline \mathbf{a} & Z_{aa} - Z_{ab} & Z_{ab} \\ \hline \mathbf{b} & Z_{ab} - Z_{bb} & Z_{bb} \\ \hline \end{array} \cdot \begin{array}{c|c|} \hline \mathbf{a}' & i^{a'} \\ \hline \mathbf{b}' & i^{b'} \\ \hline \end{array} = \begin{array}{c|c|} \hline \mathbf{a} & (Z_{aa} - Z_{ab})i^{a'} + Z_{ab}i^{b'} \\ \hline \mathbf{b} & (Z_{ab} - Z_{bb})i^{a'} + Z_{bb}i^{b'} \\ \hline \end{array} = \mathbf{e}_c \qquad 4.50$$

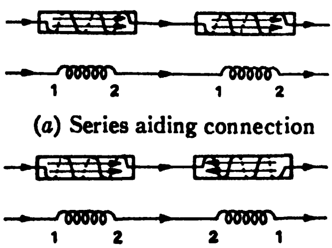
each component giving the voltages induced in the respective coil in terms of the new currents found.

XI. SIGN CONVENTIONS

(*a*) Two coils with mutual inductance between them may be connected in series in two different manners:

1. The flux coming from the second coil links the first in the same direction as its own flux, Fig. 4.5*a*. The connection is called "*series aiding.*"

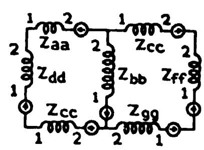
2. The flux coming from the second coil links the first in a direction



(*a*) Series aiding connection

(*b*) Series opposing connection

FIG. 4.5



Z_{aa} — Z_{dd} series aiding
Z_{aa} — Z_{bb} series opposing

FIG. 4.6

opposite to its own flux, Fig. 4.5*b*. The connection is called "*series opposing.*"

In going around a closed circuit *in any direction*, two coils are considered *series aiding* if connected in the order 1–2, 1–2 or 2–1, 2–1. (Such, in Fig. 4.6, are coils Z_{aa} and Z_{dd} or coils Z_{bb} and Z_{ce}, etc.) Two

coils are considered *series opposing* if they are connected in the order 1–2, 2–1 or 2–1, 1–2. Such are coils Z_{aa} and Z_{bb} or coils Z_{cc} and Z_{gg} in Fig. 4.6.

(*b*) In the " primitive mesh network " Fig. 4.7 it will be assumed that the positive currents flow in each coil from 1 to 2 and that the positive voltages are impressed in each closed circuit also from 1 to 2.

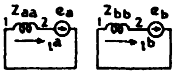


FIG. 4.7.—Sign Conventions in the Primitive Network

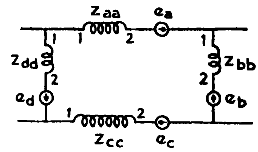


FIG. 4.8.—Voltages Around a Mesh

In the new network in each individual coil the new currents may flow in any directions; also in each closed circuit the positive voltages may be impressed in any direction.

(*c*) Considering any closed mesh (Fig. 4.8) of the new system, along each individual coil may exist an impressed voltage (battery, generator, etc.) This individual impressed voltage in the new system is considered as one *old* component of the voltage vector **e**, having the same subscript as the coil alongside it. *The new components of the voltage vector* **e**′ *are the total voltages existing around each closed mesh.* Hence each component of **e**′ may contain several impressed voltages.

(*d*) Of course any other sign convention may be introduced.

XII. SUMMARY OF STEPS IN SETTING UP THE EQUATIONS

(*a*) In order to set up the equation of voltage of any mesh network the following steps are made:

A. The first step is to set up the " *primitive mesh network* " and the components of the three geometric objects **i**, **z**, and **e** belonging to it.

1. The primitive network may be separately drawn if desired; it consists of all the coils of the given network short-circuited upon themselves, each short circuit including any impressed voltage that may exist alongside the coil.

If the given network contains a branch in which no impedance but only an impressed voltage exists, then in simpler problems a coil with zero impedance may be assumed to exist in that branch and this coil is also included in the primitive network. If neither an impedance nor a voltage exists in a branch (being a short circuit), then this branch is not included in the primitive network.

(In general, a branch with an impressed voltage and without impedance should be considered as a junction-pair instead of a mesh. Such a method of analysis is given in Chapter XVI.)

2. The impedance tensor $\mathbf{z}$ of the primitive network is set up, having as many rows and columns as there are coils. The main diagonal components contain all the self-impedances, the others all the mutual impedances, of the various coils. Its matrix is symmetrical in many stationary network analyses.

3. The impressed voltage vector $\mathbf{e}$ of the primitive network has as many components as there are coils and it contains the voltages impressed in the given network, each impressed voltage being assumed to belong to some coil. Usually most of the components of $\mathbf{e}$ are zero.

4. The currents $\mathbf{i}$ flowing in the individual meshes are assumed as the variables.

Later on it will be shown that the " primitive " network may be replaced by *any other* convenient network as a starting point.

B. The next step is to set up the *transformation matrix* $\mathbf{C}$ changing the primitive network to the actual network.

1. Assume as many new but independent *currents* $\mathbf{i}'$ anywhere in the new system as there are closed circuits (meshes) in it. Their order is immaterial.

2. *Write along each individual coil the currents flowing through it,* expressed in terms of the assumed new currents, using Kirchhoff's First Law.

3. Equate the old currents and the new currents flowing in each individual coil; that is, set up the relations

$$\boxed{\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'} \qquad \Big| \qquad \boxed{i^m = C^m_{m'} i^{m'}} \qquad 4.51$$

There are as many equations as there are coils. The left-hand side contains the old currents, the right-hand side the new currents.

4. *The coefficients of the new currents* $\mathbf{i}'$ *form the transformation matrix* $\mathbf{C}$.

C. The next step consists of finding the new components of $\mathbf{z}'$ and $\mathbf{e}'$ and the equation of performance $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$ of the new network.

1. The new components of the impedance tensor are found in two steps by the transformation formula

$$\boxed{\mathbf{z}' = \mathbf{C}_t\cdot\mathbf{z}\cdot\mathbf{C}} \qquad \Big| \qquad \boxed{z_{m'n'} = z_{mn} C^m_{m'} C^n_{n'}} \qquad 4.52$$

by first finding $\mathbf{z} \cdot \mathbf{C}$, then $\mathbf{C}_t \cdot (\mathbf{z} \cdot \mathbf{C})$, or first finding $\mathbf{C}_t \cdot \mathbf{z}$, then $(\mathbf{C}_t \cdot \mathbf{z}) \cdot \mathbf{C}$.

2. The mesh voltages are found by the transformation formula

$$\boxed{\mathbf{e}' = \mathbf{C}_t\cdot\mathbf{e}} \qquad\Bigg|\qquad \boxed{e_{m'} = C^m_{m'}e_m} \tag{4.53}$$

3. The equation of voltage of the new network is

$$\boxed{\mathbf{e}' = \mathbf{z}'\cdot\mathbf{i}'} \qquad\Bigg|\qquad \boxed{e_{m'} = z_{m'n'}i^{n'}} \tag{4.54}$$

D. Once the equation of voltage of a system has been established, it may be subjected to numerous types of manipulations. *Usually it is subdivided into several invariant equations* for purposes of manipulations. Such subdivisions are treated in later chapters.

When the equation of voltage is treated as one unit without subdivision, the unknown currents are found in two steps:

1. The inverse of the impedance tensor $\mathbf{z}'$ is calculated, giving the admittance tensor

$$\boxed{\mathbf{y}' = \mathbf{z}'^{-1}} \qquad\Bigg|\qquad \boxed{y^{n'm'} = (z_{m'n'})^{-1}} \tag{4.55}$$

2. The unknown currents are found by

$$\boxed{\mathbf{i}' = \mathbf{y}'\cdot\mathbf{e}'} \qquad\Bigg|\qquad \boxed{i^{n'} = y^{n'm'}e_{m'}} \tag{4.56}$$

giving the values of the arbitrarily selected new currents in terms of the applied coil voltages and the coil impedances.

Once the unknown currents $\mathbf{i}'$ have been calculated, then:

3. The currents flowing in the individual coils are found by the relation:

$$\boxed{\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'} \qquad\Bigg|\qquad \boxed{i^m = C^m_{m'}i^{m'}} \tag{4.57}$$

which already has been established in step B3.

4. The voltages induced in the individual coils are found by $\mathbf{e}_c = \mathbf{z}\cdot\mathbf{i}$ or by

$$\boxed{\mathbf{e}_c = \mathbf{z}\cdot\mathbf{C}\cdot\mathbf{i}'} \qquad\Bigg|\qquad \boxed{e_m = z_{mn}C^n_{n'}i^{n'}} \tag{4.58}$$

where $\mathbf{z}\cdot\mathbf{C}$ has already been calculated as a first step in finding $\mathbf{C}_t\cdot\mathbf{z}\cdot\mathbf{C}$ in step C1.

(*b*) Of course, in a simple network without mutual impedances it is easier to set up the equations of voltage for the various meshes by simply inspecting the diagram of the network and applying Kirchhoff's laws than to follow through the above steps. However, as the network becomes more complex or a series of circuits is to be analyzed or new reference frames are to be introduced, then the above steps reduce the amount of mental and physical labor in proportion to the complexity

of the network. The rigorous application of these steps is needed if the equations are to be subdivided for further systematic manipulations. Their use also suggests new methods of attacks, labor-saving devices, etc.

One of the advantages of this method of attack is that *no attention has to be paid to the existence of mutual inductances between the various windings, nor to their direction of linkage, and so on.* The magnitudes and signs of the mutual voltages are *automatically* taken care of by the use of the transformation tensor **C**.

XIII. SUMMARY OF STEPS IN THE REASONING

(*a*) Let n coils be interconnected into n-mesh networks in a large variety of manners. In setting up the n equations of voltage of each network by Kirchhoff's laws, it is found that:

1. The set of n equations are different for each network.

2. For each network it is possible to set up a large variety of sets of n equations, all sets being different from one another and different from those of the other networks.

The *purpose of organization* of the method of analysis is to set up the equations of voltage for each mesh network for all selection of the variables in the identical form $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$.

(*b*) In order to do that, first *a reference network is picked out* of the numerous networks whose analysis is comparatively simple. *This particular reference network is called the " primitive network."* The equation of voltage of this primitive network is *easily* established as $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$. (Any other network can be used as a reference network.)

The next step is to consider *in what respect all the other networks differ from the primitive network and in what respect they are identical with it*:

1. Since the networks *differ* from the primitive network only in the manner in which their coils are interconnected, it is possible to set up a " *transformation matrix* **C** " for each network, whose components completely represent all the *essential* characteristics (manner of interconnection, manner of selection of variables, etc.) in which each particular network *differs* from the primitive network (or from any other " reference " network selected as a starting point).

2. No matter *what* the n coils are, or *how* the n coils are interconnected into n meshes, and no matter *what currents* are selected as variables, *the total instantaneous power input into each network* $\mathbf{e} \cdot \mathbf{i}$ (a linear form) *is the same, an " invariant."*

These two considerations determine the " *transformation tensor* **C** " and the " *transformation formula* " of *one* geometric object $\mathbf{e}$.

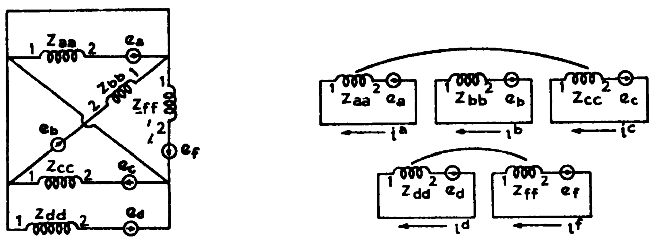
Once the components of $\mathbf{C}$ and the transformation formula of *one* geometric object $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$ have been established, the transformation formulas of all the other geometric objects follow automatically from the postulate that all networks ought to have identical equations of performance.

(*c*) It should be noted that it is not inherent in the various networks that their matric equations of performance should be the same. They possess that characteristic only as a consequence of possessing intrinsically a group of transformation matrices $\mathbf{C}$ and an " *invariant* " *power input* under all conditions of interconnections.

XIV. A SECOND EXAMPLE

As a more complicated example let the mesh network of Fig. 4.1, 6, be analyzed, which is reproduced again in Fig. 4.9*a*. The coils Z_{aa} and Z_{cc} have asymmetrical mutual inductances, similarly Z_{dd} and Z_{ff}. The direction of linkages is represented by the numerals 1–2. The coils are named in any *arbitrary* order.

(*a*) Following the steps given in Section XII, first the primitive network and its geometric objects are established as follows:



(*a*) Given network (*b*) Its primitive network

FIG. 4.9

A1. The *primitive network* is shown in Fig. 4.9*b* consisting of five coils short-circuited upon themselves.

A2. The impedance tensor $\mathbf{z}$ of the primitive network is

$$\mathbf{z} = \begin{array}{c|ccccc|} & a & b & c & d & f \\ \hline a & Z_{aa} & & X_{ac} & & \\ b & & Z_{bb} & & & \\ c & X_{ca} & & Z_{cc} & & \\ d & & & & Z_{dd} & X_{df} \\ f & & & & X_{fd} & Z_{ff} \\ \hline \end{array} \qquad 4.59$$

The zero components are not shown.

A3. The impressed voltage vector of the primitive network is

$$\begin{array}{cccccc} & a & b & c & d & f \\ \mathbf{e} = & \boxed{e_a} & \boxed{e_b} & \boxed{e_c} & \boxed{e_d} & \boxed{e_f} \end{array} \qquad 4.60$$

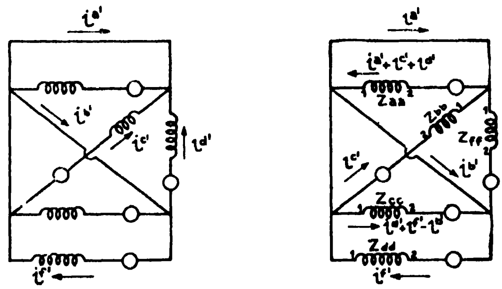
A4. The current vector is

$$\begin{array}{cccccc} & a & b & c & d & f \\ \mathbf{i} = & \boxed{i^a} & \boxed{i^b} & \boxed{i^c} & \boxed{i^d} & \boxed{i^f} \end{array} \qquad 4.61$$

(*b*) The next step is to find the transformation tensor of Fig. 4.9*a*, showing how its connection differs from that of the primitive network of Fig. 4.9*b*.

B1. Let *five* arbitrary independent currents be assumed as shown in Fig. 4.10*a*. They may be assumed to flow in any of the coils or impedanceless branches in any direction.

The five currents under B1 are *not entirely arbitrary*. They must be *independent of one another*, that is, they must be sufficient to determine all the currents flowing in the remaining branches. For instance, if, in Fig. 4.10*b*, $i^{f'}$ had been assumed to flow not in coil Z_{dd} but in the vertical impedanceless branch between Z_{aa} and Z_{cc}, then the currents in coils Z_{cc} and Z_{dd} could have not been expressed in terms of the assumed new currents.



(*a*) Assumed new currents. (*b*) Currents in each coil

FIG. 4.10

B2. The new currents flowing *in each coil* are established by Kirchhoff's first law as shown in Fig. 4.10*b*.

B3. Equate the old current and the new current flowing in each coil by inspection of Fig. 4.10*b*.

$$\begin{array}{llllll}
\text{In coil } Z_{aa} \rightarrow i^a = -i^{a'} - i^{c'} - i^{d'} & i^a = & -i^{a'} & & -i^{c'} & -i^{d'} \\
\text{"} \quad \text{"} \quad Z_{bb} \rightarrow i^b = -i^{c'} & i^b = & & & -i^{c'} & \\
\text{"} \quad \text{"} \quad Z_{cc} \rightarrow i^c = i^{d'} + i^{f'} - i^{b'} & i^c = & & -i^{b'} & & +i^{d'} + i^{f'} \\
\text{"} \quad \text{"} \quad Z_{dd} \rightarrow i^d = -i^{f'} & i^d = & & & & -i^{f'} \\
\text{"} \quad \text{"} \quad Z_{ff} \rightarrow i^f = -i^{d'} & i^f = & & & & -i^{d'}
\end{array} \qquad 4.62$$

B4. The coefficients of the new currents form the components of the transformation tensor:

$\mathbf{C} =$

	a′	b′	c′	d′	f′
a	−1		−1	−1	
b			−1		
c		1−		1	1
d					−1
f				−1	

$\mathbf{C}_t =$

	a	b	c	d	f
a′	−1				
b′			−1		
c′	−1	−1			
d′	−1		1		−1
f′			1	−1	

4.63

(*c*) The next step is the calculation of the *new* components of the various geometric objects.

C1. The new components of the impedance tensor are found in two steps. The first step is

$\mathbf{z} \cdot \mathbf{C} =$

	a′	b′	c′	d′	f′
a	$-Z_{aa}$	$-X_{ac}$	$-Z_{aa}$	$-Z_{aa}+X_{ac}$	X_{ac}
b			$-Z_{bb}$		
c	$-X_{ca}$	$-Z_{cc}$	$-X_{ca}$	$-X_{ca}+Z_{cc}$	Z_{cc}
d				$-X_{df}$	$-Z_{dd}$
f				$-Z_{ff}$	$-X_{fd}$

4.64

The second step is $\mathbf{C}_t \cdot (\mathbf{z} \cdot \mathbf{C}) =$

$\mathbf{z}' =$

	a′	b′	c′	d′	f′
a′	Z_{aa}	X_{ac}	Z_{aa}	$Z_{aa}-X_{ac}$	$-X_{ac}$
b′	X_{ca}	Z_{cc}	X_{ca}	$X_{ca}-Z_{cc}$	$-Z_{cc}$
c′	Z_{aa}	X_{ac}	$Z_{aa}+Z_{bb}$	$Z_{aa}-X_{ac}$	$-X_{ac}$
d′	$Z_{aa}-X_{ca}$	$X_{ac}-Z_{cc}$	$Z_{aa}-X_{ca}$	$Z_{aa}-X_{ac}-X_{ca}+Z_{cc}+Z_{ff}$	$-X_{ac}+Z_{cc}+X_{fd}$
f′	$-X_{ca}$	$-Z_{cc}$	$-X_{ca}$	$-X_{ca}+Z_{cc}+X_{df}$	$Z_{cc}+Z_{dd}$

4.65

If the mutual inductances are the same in both directions, that is, if $X_{ac} = X_{ca}$, etc., this impedance matrix is symmetrical. *The symmetrical form of the final matrix* $\mathbf{z}'$ *serves as a check on the correctness of the calculations.*

C2. The impressed voltages of the new system are found by:

$$\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e} = \begin{array}{c|c|} \mathbf{a}' & -e_a \\ \mathbf{b}' & -e_c \\ \mathbf{c}' & -e_a - e_b \\ \mathbf{d}' & -e_a + e_c - e_f \\ \mathbf{f}' & e_c - e_d \end{array} \downarrow \qquad 4.66$$

Each component of $\mathbf{e}'$ gives the voltages around a closed mesh. For instance, $e_{d'}$ is the impressed voltage around the mesh through impedances $-Z_{aa}$, Z_{cc}, and $-Z_{ff}$.

C3. The equation of voltage is $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$, representing five equations with five unknowns.

(*d*) The final steps are the *solution* for the unknown currents and coil voltages, if the equation of voltage is not subjected to other manipulations.

D1. To find the unknown $\mathbf{i}'$, the inverse of $\mathbf{z}'$ is calculated by the method shown in Section XIII, Chapter I, or by the labor-saving method to be shown in Chapter X. Its inverse, the admittance tensor, has the form:

$$\mathbf{y}' = \begin{array}{c|c|c|c|c|c|} & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \mathbf{d}' & \mathbf{f}' \\ \hline \mathbf{a}' & Y^{aa} & Y^{ab} & Y^{ac} & Y^{ad} & Y^{af} \\ \mathbf{b}' & Y^{ba} & Y^{bb} & Y^{bc} & Y^{bd} & Y^{bf} \\ \mathbf{c}' & Y^{ca} & Y^{cb} & Y^{cc} & Y^{cd} & Y^{df} \\ \mathbf{d}' & Y^{da} & Y^{db} & Y^{dc} & Y^{dd} & Y^{df} \\ \mathbf{f}' & Y^{fa} & Y^{fb} & Y^{bd} & Y^{fd} & Y^{ff} \\ \hline \end{array} \qquad 4.67$$

Its components are either numerals or combinations of Z — s, etc.

D2. The unknown currents are

$$\mathbf{i}' = \mathbf{y}' \cdot \mathbf{e}' = \begin{array}{c|l|} \mathbf{a}' & -Y^{aa}e_a - Y^{ab}e_c - Y^{ac}(e_a+e_b) + Y^{ad}(e_c-e_a-e_f) + Y^{af}(e_c-e_d) \\ \mathbf{b}' & -Y^{ba}e_a - Y^{bb}e_c - Y^{bc}(e_a+e_b) + Y^{bd}(e_c-e_a-e_f) + Y^{bf}(e_c-e_d) \\ \mathbf{c}' & -Y^{ca}e_a - Y^{cb}e_c - Y^{cc}(e_a+e_b) + Y^{cd}(e_c-e_a-e_f) + Y^{cf}(e_c-e_d) \\ \mathbf{d}' & -Y^{da}e_a - Y^{db}e_c - Y^{dc}(e_a+e_b) + Y^{dd}(e_c-e_a-e_f) + Y^{df}(e_c-e_d) \\ \mathbf{f}' & -Y^{fa}e_a - Y^{fb}e_c - Y^{fc}(e_a+e_b) + Y^{fd}(e_c-e_a-e_f) + Y^{ff}(e_c-e_d) \end{array} \qquad 4.68$$

Each component of $\mathbf{i}'$ represents one of the unknown currents.

D3. When the unknown currents $i^{a'}$, $i^{b'}$, etc., have been calculated, then the currents flowing in the individual coils, i^a, i^b, etc., are found, if needed, from the set of equations 4.62.

D4. The voltages induced in the individual coils are found by multiplying $\mathbf{z} \cdot \mathbf{C}$ of equation 4.64 by $\mathbf{i}'$, giving

$$\mathbf{z}\cdot\mathbf{C}\cdot\mathbf{i}' = \mathbf{e}_c = \begin{array}{c|c|} \cline{2-2} \mathbf{a} & -Z_{aa}i^{a'} - X_{ac}i^{b'} - Z_{aa}i^{c'} + (X_{ac} - Z_{aa})i^{d'} + X_{ac}i^{f'} \\ \cline{2-2} \mathbf{b} & -Z_{bb}i^{c'} \\ \cline{2-2} \mathbf{c} & -X_{ca}i^{a'} - Z_{cc}i^{b'} - X_{ca}i^{c'} + (Z_{cc} - X_{ca})i^{d'} + Z_{cc}i^{f'} \\ \cline{2-2} \mathbf{d} & -X_{df}i^{d'} - Z_{dd}i^{f'} \\ \cline{2-2} \mathbf{f} & -Z_{ff}i^{d'} - X_{fd}i^{f'} \\ \cline{2-2} \end{array} \qquad 4.69$$

CHAPTER V

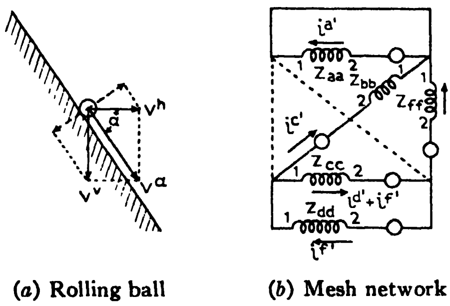
SINGULAR TRANSFORMATIONS

I. THE EQUATIONS OF CONSTRAINT

(*a*) In the transformations $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ hitherto considered, the new network had as many meshes as the primitive network and the transformation tensor **C** had as many rows as columns.

However, *in engineering practice most networks have fewer meshes than coils.* One way to consider such networks is to assume that originally they have as many meshes as coils and afterward *some or all of their impedanceless branches are open-circuited* so that no current can flow through them.

For instance, consider the network of Fig. 5.1*b*, consisting of five coils and three meshes. It may be looked upon as being derived from the all-mesh network of Fig. 4.10 by opening two of the impedanceless branches of the latter as shown by dotted lines on Fig. 5.1. Open-circuiting the branches is equivalent to making the currents in them equal to zero. That is, *the opening of the two impedanceless branches is equivalent to introducing the following two relations*:



(*a*) Rolling ball (*b*) Mesh network

FIG. 5.1.—Introduction of Constraints

$$\begin{aligned} i^{c'} + i^{d'} - i^{b'} &= 0 \\ i^{b'} &= 0 \end{aligned} \qquad 5.1$$

that must exist between the five new variables of Fig. 4.10.

(*b*) *Equations that express certain relations that exist between the variables are called in dynamics " equations of constraint." Hence the opening of branches is equivalent to introducing constraints* by preventing

the current vector i^m from having components along the opened branches. Hence as many constraints are introduced as there are opened meshes.

(c) In dynamics, a falling body being compelled to roll down on an inclined plane is an example of constraint. Introducing an inclined plane in the path of a freely falling body prevents the velocity vector v_m from having a component v^p perpendicular to the plane. That is, $v^p = 0$ is the equation of constraint (Fig. 5.1*a*); or if v^m is expressed in terms of its horizontal and vertical components v^h and v^v, then the equation of constraint is, from Fig. 5.1*a*,

$$v^v \cos \alpha - v^h \sin \alpha = 0$$

representing a relation between v^p and v^h that exists at all instants.

Of course, the opening of meshes is only one of a large variety of ways in which constraints may be introduced in network studies. Later on several other examples will be shown, like neglecting magnetizing currents, etc.

(d) *The existence of equations of constraints indicates that the number of original variables may be reduced by as many as there are equations of constraints.*

(It will be shown in Section II, Chapter XVI, that actually in place of the discarded variables another set of variables has to be introduced. At the present time this more general point of view is not considered.)

II. SINGULAR TRANSFORMATION MATRICES

(a) *With the aid of each equation of constraint it is possible to eliminate one variable by expressing it in terms of the others.* In the present case two variables, say $i^{b'}$ and $i^{c'}$, may be expressed in terms of the others as

$$\begin{aligned} i^{b'} &= 0 \\ i^{c'} &= - i^{d'} \end{aligned} \qquad 5.2$$

leaving only three new variables $i^{a'}$, $i^{d'}$, and $i^{f'}$ in place of five (corresponding to the number of new meshes existing) that are independent of one another.

(b) As a consequence, the relation $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ existing between the old and the new currents (equation 4.62) becomes, by substituting the last relations 5.2 introduced by the constraints,

$$
\begin{aligned}
i^a &= -\,i^{a'} + i^{d'} - i^{d'} &&= -\,i^{a'} \\
i^b &= \quad i^{d'} &&= \quad i^{d'} \\
i^c &= \quad i^{d'} + i^{f'} &&= \quad i^{d'} + i^{f'} \\
i^d &= -\,i^{f'} &&= \quad -\,i^{f'} \\
i^f &= -\,i^{d'} &&= \quad -\,i^{d'}
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c|ccc|}
 & \mathbf{a'} & \mathbf{d'} & \mathbf{f'} \\
\hline
\mathbf{a} & -1 & & \\
\mathbf{b} & & 1 & \\
\mathbf{c} & & 1 & 1 \\
\mathbf{d} & & & -1 \\
\mathbf{f} & & -1 & \\
\hline
\end{array}
\qquad 5.3
$$

so that *the transformation matrix now has three columns instead of five,* corresponding to the three new meshes. That is, the transformation matrix **C** is now rectangular, instead of square.

(*c*) *Transformation matrices* **C** *that are rectangular* (hence have no inverse, $\mathbf{C}^{-1}$) *are called " singular " transformation matrices,* and the corresponding transformations (of the variables) are called " *singular* " transformations. Square transformation matrices, whose inverse can be calculated, are called " *non-singular* " transformation matrices, and the corresponding transformation of the variables " non-singular " transformations.

It will be shown in Chapter XVI that all formulas hitherto developed, or to be developed later on, are equally valid for singular or non-singular transformation matrices **C**, *with the precaution that formulas in which* $\mathbf{C}^{-1}$ *occurs should not be used with singular transformation matrices* **C**. Consequently, once the singular transformation matrix **C** has been set up, the steps for setting up $\mathbf{z}'$ and $\mathbf{e}'$, and steps of finding $\mathbf{i}'$, etc., are exactly the same as in case of non-singular transformation matrices since $\mathbf{C}^{-1}$ is not needed.

It will also be shown later on that *all singular* (*rectangular*) *transformation matrices are special cases of non-singular* (*square*) *transformation matrices* whose columns have been removed because of the absence of the current variables along the open-circuited branches.

III. REPLACEMENT OF CONSTRAINTS BY SINGULAR C

(*a*) The steps of setting up the singular transformation matrix were made above in a roundabout way:

1. First, it was assumed that *five* new variables exist (as many as there are coils).

2. Then *two* of them were eliminated by means of two equations of constraint, leaving only three new variables.

However, *in case of the interconnection of electrical networks this roundabout way of setting up a singular* **C** *may be eliminated by assuming in the new network right at the start only as many new currents as there are closed meshes.*

For instance, assuming as above $i^{a'}$, $i^{d'}$, and $i^{f'}$ as the three new variables, in Fig. 5.1*b*, the new currents flowing in each individual coil may be expressed now in terms of three new currents as shown. Equating the old and the new currents flowing in each coil, the last set of equations 5.3 are again found immediately, without the roundabout reasoning.

Of course, in place of $i^{a'}$, $i^{d'}$, and $i^{f'}$, *any other three* branch currents can be assumed as the new variables, giving a transformation matrix different from equation 5.3.

(*b*) Hence *when n coils are interconnected into a network with less than n meshes, the whole method of analysis is the same as in case of a new network with n meshes, given in Chapter IV, Section XIII, except that the number of new variables assumed is the same as the number of new meshes.* That is, any mesh network with n coils may be looked upon as a network having n meshes, the current in some of the meshes, however, having been reduced to zero by opening some of the branches.

It is emphasized that it is not always possible to ignore the setting up of the equations of constraints by immediately setting up a singular **C** instead. In many cases, especially when hypothetical currents are introduced, it is necessary to set up first the equations of constraints and then only is it possible to establish a singular **C**.

(*c*) Whenever a singular transformation matrix is established it must be remembered that constraints have also been introduced into the system at the same time. There are as many constraints as there are missing columns in **C**.

IV. CALCULATIONS WITH SINGULAR C

(*a*) The primitive network of the three-meshed network of Fig. 5.1*b* and its geometric objects **z**, **e**, and **i** are the same as those of the original five-mesh network of Fig. 4.9 as given in Section XIV, Chapter IV.

The transformation matrix **C** of the three-meshed network is given in equation 5.3.

(*b*) Hence the new components of the impedance tensor **z**′ are by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$

$$\mathbf{z}\cdot\mathbf{C} = \begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \\ \mathbf{f} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{a'} & \mathbf{d'} & \mathbf{f'} \end{array} \\ \begin{array}{|c|c|c|} \hline -Z_{aa} & X_{ac} & X_{ac} \\ \hline & Z_{bb} & \\ \hline -X_{ca} & Z_{cc} & Z_{cc} \\ \hline & -X_{df} & -Z_{dd} \\ \hline & -Z_{ff} & -X_{fd} \\ \hline \end{array} \end{array} \quad 5.4 \qquad \mathbf{C}_t = \begin{array}{c} \\ \mathbf{a'} \\ \mathbf{d'} \\ \mathbf{f'} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \end{array} \\ \begin{array}{|c|c|c|c|c|} \hline -1 & & & & \\ \hline & 1 & 1 & & -1 \\ \hline & & 1 & -1 & \\ \hline \end{array} \end{array}$$

$$\mathbf{C}_t\cdot(\mathbf{z}\cdot\mathbf{C}) = \mathbf{z}' = \begin{array}{c|c|c|c} & \mathbf{a}' & \mathbf{d}' & \mathbf{f}' \\ \hline \mathbf{a}' & Z_{aa} & -X_{ac} & -X_{ac} \\ \mathbf{d}' & -X_{ca} & Z_{bb} + Z_{cc} + Z_{ff} & Z_{cc} + X_{fd} \\ \mathbf{f}' & -X_{ca} & Z_{cc} + X_{df} & Z_{cc} + Z_{dd} \end{array} \qquad 5.5$$

It should be noted that the impedance tensor $\mathbf{z}'$ of the new network has as many rows and columns as the network has meshes. If the mutual inductances are the same in the two directions, the matrix of the tensor is symmetrical.

The new impressed voltage vector is

$$\mathbf{C}_t\cdot\mathbf{e} = \mathbf{e}' = \begin{array}{c|c} \mathbf{a}' & -e_a \\ \mathbf{d}' & e_b + e_c - e_f \\ \mathbf{f}' & e_c - e_d \end{array} \qquad 5.6$$

representing the impressed voltages around three closed meshes. For instance, $e^{f'}$ is the impressed voltage around the mesh through impedances Z_{cc} and Z_{dd}, $e^{d'}$ is the voltage around the mesh through Z_{bb}, Z_{cc}, and Z_{ff}, while $e^{a'}$ is the voltage around the mesh containing Z_{aa}.

It also should be noted that *the diagonal components of* $\mathbf{z}'$ *represent the sum of the self-impedances of each of the three meshes*, around which the voltages are summed up. Also the remaining components of $\mathbf{z}'$ represent the *mutual impedances* of the three meshes with each other. The latter includes self-impedance terms of coils belonging to two meshes.

(*c*) To find the currents $\mathbf{i}'$, first the inverse of $\mathbf{z}'$ has to be calculated by the method of Section XIII, Chapter I, giving

$$\mathbf{y}' = \begin{array}{c|c|c|c} & \mathbf{a}' & \mathbf{d}' & \mathbf{f}' \\ \hline \mathbf{a}' & Y^{aa} & Y^{ad} & Y^{af} \\ \mathbf{d}' & Y^{da} & Y^{dd} & Y^{df} \\ \mathbf{f}' & Y^{fa} & Y^{fd} & Y^{ff} \end{array} \qquad 5.7$$

Hence the current vector is

$$\mathbf{i}' = \mathbf{y}'\cdot\mathbf{e}' = \begin{array}{c|c} \mathbf{a}' & -Y^{aa}e_a + Y^{ad}(e_b + e_c - e_f) + Y^{af}(e_c - e_d) \\ \mathbf{d}' & -Y^{da}e_a + Y^{dd}(e_b + e_c - e_f) + Y^{df}(e_c - e_d) \\ \mathbf{f}' & -Y^{fa}e_a + Y^{fd}(e_b + e_c - e_f) + Y^{ff}(e_c - e_d) \end{array} \qquad 5.8$$

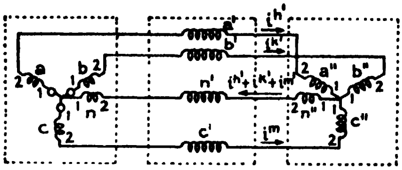
the components representing $i^{a'}$, $i^{d'}$, and $i^{f'}$, respectively.

The currents in the individual coils are found by the set of equations 5.3, representing $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$.

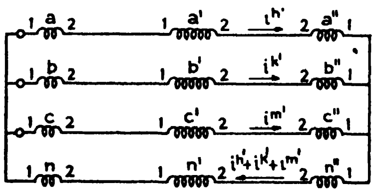
The voltages induced in the five individual coils are found by $\mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$, where $\mathbf{z} \cdot \mathbf{C}$ has already been calculated in equation 5.4, giving

$$\mathbf{e}_c = \mathbf{z}\cdot\mathbf{C}\cdot\mathbf{i}' \quad \begin{array}{c|c|} \cline{2-2} \mathbf{a} & -Z_{aa}i^{a'} + X_{ac}(i^{d'} + i^{f'}) \\ \cline{2-2} \mathbf{b} & Z_{bb}i^{d'} \\ \cline{2-2} \mathbf{c} & -X_{ca}i^{a'} + Z_{cc}(i^{d'} + i^{f'}) \\ \cline{2-2} \mathbf{d} & -X_{df}i^{d'} - Z_{dd}i^{f'} \\ \cline{2-2} \mathbf{f} & -Z_{ff}i^{d'} - X_{fd}i^{f'} \\ \cline{2-2} \end{array} \qquad 5.9$$

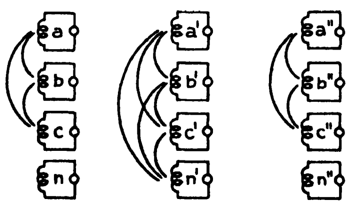
It should be noted that, although the various vectors and tensors have different numbers of rows or columns, still the *arrow rule of multiplication works smoothly*, since the *order* of the geometric objects in the various formulas automatically takes care of this inequality.



(*a*) Given network



(*b*) Simplified diagram



(*c*) The primitive network

FIG. 5.2.—Distribution Network

V. DISTRIBUTION CIRCUIT

(*a*) To show an example in which *several coils form one branch*, let a three-phase, four-wire distribution circuit be considered that is supplied from the star-connected secondary of a transformer. The load is unbalanced, as shown in Fig. 5.2*a*. (A quicker and more systematic method of solving three-phase unbalanced networks is shown in Chapter XIX.)

(*b*) The impedance tensor of the primitive network shown in Fig. 5.2*c* is $\mathbf{z} =$

	a	b	c	n	a′	b′	c′	n′	a″	b″	c″	n″
a	Z_{aa}	X_{ab}	X_{ac}									
b	X_{ba}	Z_{bb}	X_{bc}									
c	X_{ca}	X_{cb}	Z_{cc}									
n				Z_{nn}								
a′					$Z_{a'a'}$	$X_{a'b'}$	$X_{a'c'}$	$X_{a'n'}$				
b′					$X_{b'a'}$	$Z_{b'b'}$	$X_{b'c'}$	$X_{b'n'}$				
c′					$X_{c'a'}$	$X_{c'b'}$	$Z_{c'c'}$	$X_{c'n'}$				
n′					$X_{n'a'}$	$X_{n'b'}$	$X_{n'c'}$	$Z_{n'n'}$				
a″									$Z_{a''a''}$	$X_{a''b''}$	$X_{a''c''}$	
b″									$X_{b''a''}$	$Z_{b''b''}$	$X_{b''c''}$	
c″									$X_{c''a''}$	$X_{c''b''}$	$Z_{c''c''}$	
n″												$Z_{n''n''}$

The zeros may be left out of the matrix. 5.10

The impressed voltage vector is

$$
\begin{array}{c}
\\ \mathbf{e} =
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} a & b & c & n & a' & b' & c' & n' & a'' & b'' & c'' & n'' \end{array} \\
\left[\begin{array}{cccc|cccc|cccc} e_a & e_b & e_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
\end{array}
\qquad 5.11
$$

Since there are three closed circuits, the three line currents $i^{h'}$, $i^{k'}$, and $i^{m'}$ are assumed as the new currents. The neutral current is $i^{h'} + i^{k'} + i^{m'}$.

(*c*) Setting up a relation between the old currents and the new currents flowing in each coil

$$
\begin{aligned}
i^a &= i^{h'} \\
i^b &= i^{k'} \\
i^c &= i^{m'} \\
i^n &= -i^{h'} - i^{k'} - i^{m'} \\
i^{a'} &= i^{h'} \\
i^{b'} &= i^{k'} \\
i^{c'} &= i^{m'} \\
i^{n'} &= -i^{h'} - i^{k'} - i^{m'} \\
i^{a''} &= -i^{h'} \\
i^{b''} &= -i^{k'} \\
i^{c''} &= -i^{m'} \\
i^{n''} &= -i^{h'} + i^{k'} + i^{m'}
\end{aligned}
\qquad
\mathbf{C} =
\begin{array}{c|ccc|}
 & h' & k' & m' \\
\hline
a & 1 & & \\
b & & 1 & \\
c & & & 1 \\
n & -1 & -1 & -1 \\
\hline
a' & 1 & & \\
b' & & 1 & \\
c' & & & 1 \\
n' & -1 & -1 & -1 \\
\hline
a'' & -1 & & \\
b'' & & -1 & \\
c'' & & & -1 \\
n'' & -1 & -1 & -1 \\
\hline
\end{array}
\qquad 5.12
$$

The coefficients of the new currents give **C**.

(*d*) The new components of the impedance tensor $\mathbf{z}'$ are found by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ as $\mathbf{z}' =$

$$
\begin{array}{c|c|c|c|}
 & \mathbf{h}' & \mathbf{k}' & \mathbf{m}' \\
\hline
\mathbf{h}' & \begin{array}{l} Z_{aa} + Z_{nn} \\ +Z_{a'a'} - 2X_{a'n'} \\ +Z_{n'n'} \\ +Z_{a''a''} + Z_{n''n''} \end{array} & \begin{array}{l} X_{ab} + Z_{nn} \\ +X_{a'b'} - X_{n'b'} \\ -X_{a'n'} + Z_{n'n'} \\ +X_{a''b''} + Z_{n''n''} \end{array} & \begin{array}{l} X_{ac} + Z_{nn} \\ +X_{a'c'} - X_{n'c'} \\ -X_{a'n'} + Z_{n'n'} \\ +X_{a''c''} + Z_{n''n''} \end{array} \\
\hline
\mathbf{k}' & \begin{array}{l} X_{ba} + Z_{nn} \\ +X_{b'a'} - X_{n'a'} \\ -X_{b'n'} + Z_{n'n'} \\ +X_{b''a''} + Z_{n''n''} \end{array} & \begin{array}{l} Z_{bb} + Z_{nn} \\ +Z_{b'b'} - 2X_{b'n'} \\ +Z_{n'n'} \\ +Z_{b''b''} + Z_{n''n''} \end{array} & \begin{array}{l} X_{bc} + Z_{nn} \\ +X_{b'c'} - X_{n'c'} \\ -X_{b'n'} - Z_{n'n'} \\ +X_{b''c''} + Z_{n''n''} \end{array} \\
\hline
\mathbf{m}' & \begin{array}{l} X_{cc} + Z_{nn} \\ +X_{c'a'} + X_{n'a'} \\ -X_{c'n'} + Z_{n'n'} \\ +X_{c''a''} + Z_{n''n''} \end{array} & \begin{array}{l} X_{cb} + Z_{nn} \\ +X_{c'b'} - X_{n'b'} \\ -X_{c'n'} + Z_{n'n'} \\ +X_{c''b''} + Z_{n''n''} \end{array} & \begin{array}{l} Z_{cc} + Z_{nn} \\ +Z_{c'c'} - 2X_{c'n'} \\ +Z_{n'n'} \\ +Z_{c''c''} + Z_{n''n''} \end{array} \\
\hline
\end{array}
\qquad 5.13
$$

The impressed voltage vector is by $\mathbf{C}_t \cdot \mathbf{e} =$

$$
\mathbf{e}' = \begin{array}{|c|c|c|}
\multicolumn{1}{c}{\mathbf{h}'} & \multicolumn{1}{c}{\mathbf{k}'} & \multicolumn{1}{c}{\mathbf{m}'} \\
\hline
e_a & e_b & e_c \\
\hline
\end{array}
\qquad 5.14
$$

The currents are found by $\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}'$.

(*e*) The currents flowing in the individual coils are found by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ and the differences of potential appearing across the individual coils are found by $\mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$.

VI. BRIDGE CIRCUIT

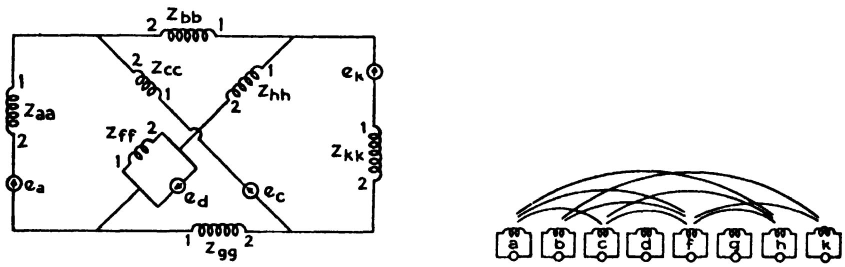
(*a*) As a more complex example consider the network of Fig. 5.3*a* containing eight coils.

When the coils cross each other it is not always obvious at a glance just how many meshes there are. In such cases the best procedure is to find first the number of *junction-pairs* (number of junctions minus number of sub-networks) in the present case $5 - 1 = 4$, then to find the number of *meshes* (number of coils minus number of junction-pairs), in the present case $8 - 4 = 4$. Hence the number of meshes is four.

(*b*) The first step is to establish the primitive network and its geometric objects.

1. The primitive network is given in Fig. 5.3*b*, showing that there is no mutual inductance between some of the coils. The impedance Z_{dd} which exists alongside the impressed voltage e_d is zero. However, its existence has to be assumed in the primitive network, since an

impedance and a voltage are always associated together in each of its meshes. (In the more general theory, to be developed in Chapter XVI, the introduction of the extra axis **d** along Z_{dd} is not needed).



(*a*) Given network (*b*) Its primitive network

FIG. 5.3.—Bridge Network

2. Its impedance tensor is

z =

	a	**b**	**c**	**d**	**f**	**g**	**h**	**k**
a	Z_{aa}	0	Z_{ac}	0	Z_{af}	0	Z_{ah}	0
b	0	Z_{bb}	0	0	Z_{bf}	0	0	Z_{bk}
c	Z_{ac}	0	Z_{cc}	0	Z_{cf}	0	Z_{ch}	0
d	0	0	0	0	0	0	0	0
f	Z_{af}	Z_{bf}	Z_{cf}	0	Z_{ff}	0	Z_{fh}	Z_{fk}
g	0	0	0	0	0	Z_{gg}	0	0
h	Z_{ah}	0	Z_{ch}	0	Z_{fh}	0	Z_{hh}	0
k	0	Z_{bk}	0	0	Z_{fk}	0	0	Z_{kk}

5.15

3. Its impressed voltage vector is

e =

a	**b**	**c**	**d**	**f**	**g**	**h**	**k**
e_a	0	e_c	e_d	0	0	0	e_k

5.16

4. Its current vector, representing the *old* variables, is

i =

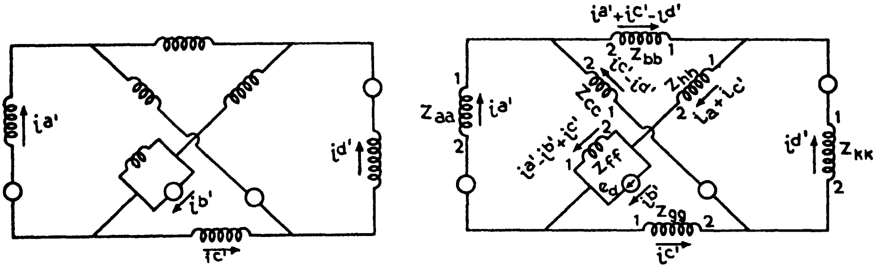
a	**b**	**c**	**d**	**f**	**g**	**h**	**k**
i^a	i^b	i^c	i^d	i^f	i^g	i^h	i^k

(*c*) The next step is to set up the transformation matrix **C**.

1. The new system has four meshes, hence *four* new currents are assumed arbitrarily as shown in Fig. 5.4*a*.

$$\mathbf{i}' = \begin{array}{cccc} a' & b' & c' & d' \\ \hline \multicolumn{1}{|c|}{i^{a'}} & \multicolumn{1}{c|}{i^{b'}} & \multicolumn{1}{c|}{i^{c'}} & \multicolumn{1}{c|}{i^{d'}} \\ \hline \end{array}$$

2. The currents flowing in each coil are expressed in terms of $i^{a'}$, $i^{b'}$, $i^{c'}$, and $i^{d'}$ as shown in Fig. 5.4*b*.



(*a*) The assumed new currents

(*b*) The new currents in each coil

FIG. 5.4.

3. Equating the old and the new currents flowing in each individual coil, the following eight equations can be set up:

$$\begin{aligned}
i^a &= -i^{a'} \\
i^b &= -i^{a'} - i^{c'} + i^{d'} \\
i^c &= i^{c'} - i^{d'} \\
i^d &= i^{b'} \\
i^f &= -i^{a'} + i^{b'} - i^{c'} \\
i^g &= i^{c'} \\
i^h &= i^{a'} + i^{c'} \\
i^k &= -i^{d'}
\end{aligned}$$

$$\mathbf{C} = \begin{array}{c} \\ \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \\ \mathrm{f} \\ \mathrm{g} \\ \mathrm{h} \\ \mathrm{k} \end{array} \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{a'} & \multicolumn{1}{c}{b'} & \multicolumn{1}{c}{c'} & \multicolumn{1}{c}{d'} \\ \hline -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline \end{array} \qquad 5.17$$

The coefficients of the *new* currents form the transformation matrix.

(*d*) Once the transformation matrix **C** has been established the new components of the geometric objects follow automatically. In particular:

1. The new components of the impedance tensor are found by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ as

$\mathbf{z}' =$

	a′	b′	c′	d′
a′	$Z_{aa}+Z_{af}-Z_{ah}+Z_{bb}+Z_{bf}+Z_{af}+Z_{bf}+Z_{ff}-Z_{fh}-Z_{ah}-Z_{fh}+Z_{hh}$	$-Z_{af}-Z_{bf}-Z_{ff}+Z_{fh}$	$Z_{bb}+Z_{bf}-Z_{ac}+Z_{af}+Z_{bf}+Z_{ff}-Z_{fh}-Z_{ah}-Z_{fh}+Z_{hh}-Z_{cf}+Z_{ch}$	$-Z_{bb}-Z_{bf}+Z_{ac}+Z_{bk}+Z_{fk}+Z_{cf}-Z_{ch}$
b′	$-Z_{af}-Z_{bf}-Z_{ff}+Z_{fh}$	Z_{ff}	$-Z_{bf}+Z_{cf}-Z_{ff}+Z_{fh}$	$Z_{bf}-Z_{cf}-Z_{fk}$
c′	$-Z_{ac}+Z_{af}-Z_{ah}+Z_{bb}+Z_{bf}+Z_{bf}-Z_{cf}+Z_{ff}-Z_{fh}+Z_{ch}-Z_{fh}+Z_{hh}$	$-Z_{bf}+Z_{cf}-Z_{ff}+Z_{fh}$	$Z_{bb}+Z_{bf}+Z_{cc}-Z_{cf}+Z_{ch}+Z_{bf}-Z_{cf}+Z_{ff}-Z_{fh}+Z_{gg}+Z_{ch}-Z_{fh}+Z_{hh}$	$-Z_{bb}-Z_{bf}-Z_{cc}+Z_{cf}-Z_{ch}+Z_{bk}+Z_{fk}$
d′	$Z_{ac}-Z_{bb}+Z_{bk}-Z_{bf}+Z_{cf}+Z_{fk}-Z_{ch}$	$Z_{bf}-Z_{cf}-Z_{fk}$	$-Z_{bb}+Z_{bk}-Z_{cc}-Z_{bf}+Z_{cf}+Z_{fk}-Z_{ch}$	$Z_{bb}-Z_{bk}+Z_{cc}-Z_{bk}+Z_{kk}$

5.18

This final impedance tensor may be written shortly as

$\mathbf{z}' =$

	a′	b′	c′	d′
a′	$Z_{a'a'}$	$Z_{a'b'}$	$Z_{a'c'}$	$Z_{a'd'}$
b′	$Z_{a'b'}$	$Z_{b'b'}$	$Z_{b'c'}$	$Z_{b'd'}$
c′	$Z_{a'c'}$	$Z_{b'c'}$	$Z_{c'c'}$	$Z_{c'd'}$
d′	$Z_{a'd'}$	$Z_{b'd'}$	$Z_{c'd'}$	$Z_{d'd'}$

5.19

2. The impressed voltage vector $\mathbf{e}'$, representing the mesh voltages, is by $\mathbf{C}_t \cdot \mathbf{e} =$

$\mathbf{e}' =$

a′	b′	c′	d′
$-e_a$	e_d	e_c	$-e_c - e_k$

5.20

If it is assumed that in series with each coil an impressed voltage exists, that is, if none of the components of $\mathbf{e}$ given in equation 5.16 are zero, then

$\mathbf{e}' =$

a′	b′	c′	d′
$-e_a - e_b - e_f + e_h$	$e_d + e_f$	$-e_b + e_c - e_f + e_g + e_h$	$e_b - e_c - e_k$

5.21

showing the four meshes of Fig. 5.5, whose equations of voltage are given by $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$. The signs of the voltages in $\mathbf{e}'$ also show the direction of travel around each closed circuit. The equation of voltage may be subjected to various types of subdivisions and manipulations.

(*e*) The currents $\mathbf{i}'$ are found by finding first the inverse of $\mathbf{z}'$, then multiplying it by $\mathbf{e}'$ as $\mathbf{z}'^{-1} \cdot \mathbf{e}'$.

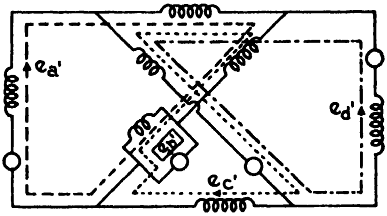
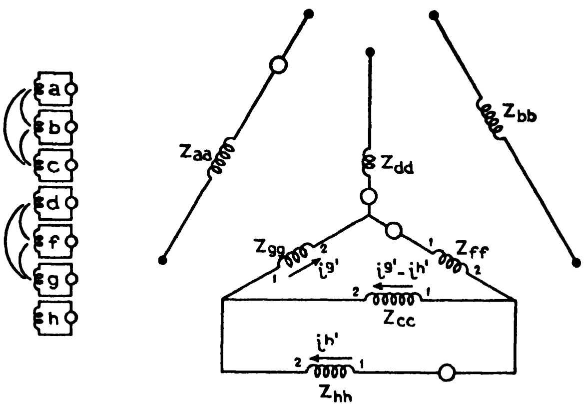


FIG. 5.5.—Voltages Around the Meshes

The currents in the individual coils (if desired) are found by calculating $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ as given in equations 5.17. The voltages in the individual coils are found by $\mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$.

VII. POTENTIALS ACROSS OPEN-CIRCUITED COILS

(*a*) If the potential difference appearing across open-circuited coils is desired, then *the primitive network should include also the coils that are left open-circuited after the interconnection. The procedure is exactly the same as where all coils have currents flowing through them.*



(*a*) Primitive network

(*b*) Given network

FIG. 5.6.—Open-circuited Coils

(*b*) For instance let the circuit of Fig. 5.6*b* be analyzed. The impedance tensor and impressed voltage of the primitive network (Fig. 5.6*a*) are

$$\mathbf{z} = \begin{array}{c|c|c|c|c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} & \mathbf{g} & \mathbf{h} \\ \hline \mathbf{a} & Z_{aa} & X_{ab} & X_{ac} & & & & \\ \hline \mathbf{b} & X_{ab} & Z_{bb} & X_{bc} & & & & \\ \hline \mathbf{c} & X_{ac} & X_{bc} & Z_{cc} & & & & \\ \hline \mathbf{d} & & & & Z_{dd} & X_{df} & X_{dg} & \\ \hline \mathbf{f} & & & & X_{df} & Z_{ff} & X_{fg} & \\ \hline \mathbf{g} & & & & X_{dg} & X_{fg} & Z_{gg} & \\ \hline \mathbf{h} & & & & & & & Z_{hh} \\ \hline \end{array} \qquad 5.22$$

$$\mathbf{e} = \begin{array}{|c|c|c|c|c|c|c|} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} & \mathbf{g} & \mathbf{h} \\ \hline e_a & & & e_d & e_f & & e_h \\ \hline \end{array} \qquad 5.23$$

In the interconnected system there are only *two* closed meshes, hence two new currents are assumed, say $i^{h'}$ and $i^{g'}$. The transformation tensor is

$$\begin{aligned} i^a &= 0 \\ i^b &= 0 \\ i^c &= i^{g'} - i^{h'} \\ i^d &= 0 \\ i^f &= i^{g'} \\ i^g &= i^{g'} \\ i^h &= i^{h'} \end{aligned} \qquad \mathbf{C} = \begin{array}{c|c|c|} & \mathbf{g}' & \mathbf{h}' \\ \hline \mathbf{a} & 0 & 0 \\ \hline \mathbf{b} & 0 & 0 \\ \hline \mathbf{c} & 1 & -1 \\ \hline \mathbf{d} & 0 & 0 \\ \hline \mathbf{f} & 1 & 0 \\ \hline \mathbf{g} & 1 & 0 \\ \hline \mathbf{h} & 0 & 1 \\ \hline \end{array} \qquad 5.24$$

It should be noted that *in the rows of* **C**, *corresponding to the open-circuited coils* **a**, **b**, *and* **d**, *all components are zero.*

The impedance tensor $\mathbf{z}'$ is by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} =$

$$\mathbf{z}' = \begin{array}{c|c|c|} & \mathbf{g}' & \mathbf{h}' \\ \hline \mathbf{g}' & Z_{cc} + Z_{ff} + Z_{gg} + 2X_{fg} & -Z_{cc} \\ \hline \mathbf{h}' & -Z_{cc} & Z_{cc} + Z_{hh} \\ \hline \end{array} \qquad 5.25$$

(*c*) The currents are found by $\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}'$, where $\mathbf{C}_t \cdot \mathbf{e} =$

$$\mathbf{e}' = \begin{array}{|c|c|} \mathbf{g}' & \mathbf{h}' \\ \hline e_f & e_h \\ \hline \end{array} \qquad 5.26$$

The currents $\mathbf{i}$ flowing in the individual coils are calculated by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$. *The voltages appearing across all individual coils, open or closed, are*

$$\mathbf{e}_c = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}' = \begin{array}{c} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \\ \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} \end{array} \begin{array}{|c|} \hline X_{ac}(i^{g\prime} - i^{h\prime}) \\ \hline X_{bc}(i^{g\prime} - i^{h\prime}) \\ \hline Z_{cc}(i^{g\prime} - i^{h\prime}) \\ \hline (X_{df} + X_{dg})i^{g\prime} \\ \hline (Z_{ff} + X_{fg})i^{g\prime} \\ \hline (Z_{gg} + X_{fg})i^{g\prime} \\ \hline Z_{hh}i^{h\prime} \\ \hline \end{array} \qquad 5.27$$

VIII. THE ADMITTANCE TENSOR OF INDIVIDUAL COILS

(*a*) In the method of reasoning hitherto followed three steps are required to find the current $\mathbf{i}$ flowing in *each* coil due to a voltage $\mathbf{e}$ impressed in *another* coil. These three steps, starting with the known *coil* voltage $\mathbf{e}$, are:

1. Find the *mesh* voltage $\mathbf{e}'$ by $\mathbf{C}_t \cdot \mathbf{e}$.
2. Find the *mesh* current $\mathbf{i}'$ by $\mathbf{z}'^{-1} \cdot \mathbf{e}'$.
3. Find the *coil* current $\mathbf{i}$ by $\mathbf{C} \cdot \mathbf{i}'$.

That is, to find a *coil* current in terms of known *coil* voltage, first the two *mesh* quantities have to be evaluated as intermediary steps.

(*b*) Many mesh problems may be formulated as follows: "Given a voltage e_a in series with coil $\mathbf{a}$. What is the current i^b in another coil b? Formulated in another way the problem is: *What are the self- and mutual admittances of the individual coils when they are interconnected?* This tensor should have as many rows and columns as there are coils.

Of course *before* the coils are interconnected their self- and mutual admittances are found by solving the equation $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$, giving $\mathbf{y} = \mathbf{z}^{-1}$. However, this admittance tensor is *not* valid when the n coils are interconnected into a network with *less* than n meshes, even though it has as many rows and columns as there are coils.

It should be noted that if the n coils are interconnected into a mesh network with n meshes, then the two admittance tensors (expressed along the individual coils) are the same. With less than n meshes constraints exist in the system, hence the two $\mathbf{y}$'s (calculated with and without constraints) have to be different.

The self- and mutual admittances of the individual coils $\mathbf{y}_c$ may be found by *one* formula by combining the above three steps into one

step. *By combining them it will not be necessary to evaluate the mesh voltages* $\mathbf{e}'$ *and currents* $\mathbf{i}'$ *as intermediary steps.* Hence:

Let the equation of the primitive network be

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i} \qquad \Big| \qquad e_m = z_{mn} i^n$$

Replacing $\mathbf{e}$ by $\mathbf{C}_t^{-1} \cdot \mathbf{e}'$ and $\mathbf{i}$ by $\mathbf{C} \cdot \mathbf{i}'$

$$\mathbf{C}_t^{-1} \cdot \mathbf{e}' = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}' \qquad \Big| \qquad C_m^{m'} e_{m'} = Z_{mn} C_{n'}^n i^{n'}$$

Multiplying both sides by $\mathbf{C}_t$ the mesh equation is

$$\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}' \qquad \Big| \qquad e_{m'} = z_{mn} C_{m'}^m C_{n'}^n i^{n'}$$

Solving for $\mathbf{i}'$

$$\mathbf{i}' = (\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C})^{-1} \cdot \mathbf{e}' \qquad \Big| \qquad i^{n'} = y^{n'm'} e_{m'} \qquad 5.28$$

This equation has already been established.

(*c*) Now replacing $\mathbf{i}'$ by $\mathbf{C}^{-1} \cdot \mathbf{i}$ and $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$ in order to express the equation along the axes of the individual coils

$$\mathbf{C}^{-1} \cdot \mathbf{i} = (\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C})^{-1} \cdot \mathbf{C}_t \cdot \mathbf{e} \qquad \Big| \qquad C_n^{n'} i^n = y^{n'm'} C_{m'}^m e_m$$

Multiplying both sides by $\mathbf{C}$

$$\mathbf{i} = \mathbf{C} \cdot (\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C})^{-1} \cdot \mathbf{C}_t \cdot \mathbf{e} \qquad \Big| \qquad i^n = y^{n'm'} C_{n'}^n C_{m'}^m e_m$$

Since the equation has the form

$$\mathbf{i} = \mathbf{y}_c \cdot \mathbf{e} \qquad \Big| \qquad i^n = y^{nm} e_m$$

therefore *the individual admittance tensor is*

$$\boxed{\mathbf{y}_c = \mathbf{C} \cdot (\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C})^{-1} \cdot \mathbf{C}_t} \qquad 5.29$$

$$\boxed{\mathbf{y}_c = \mathbf{C} \cdot \mathbf{y}' \cdot \mathbf{C}_t} \qquad \Big| \qquad \boxed{y^{nm} = y^{n'm'} C_{n'}^n C_{m'}^m} \qquad 5.30$$

That is, *the individual coil admittance tensor is found from the mesh admittance tensor* $\mathbf{y}'$ (calculated in the routine manner) *by simply transforming it again with the aid of* $\mathbf{C}$. That is, the steps are:

1. Transform $\mathbf{z}$ of the primitive network to $\mathbf{z}'$ by $\mathbf{C}$ as $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$.
2. Find the inverse of $\mathbf{z}'$ as $\mathbf{y}'$.
3. Transform $\mathbf{y}'$ back by $\mathbf{C}$ as $\mathbf{C} \cdot \mathbf{y}' \cdot \mathbf{C}_t$.

It should be noted that in transforming from $\mathbf{z}$ to $\mathbf{z}'$ first $\mathbf{C}_t$ then $\mathbf{C}$ occurs; however, in transforming from $\mathbf{y}'$ to $\mathbf{y}_c$ first $\mathbf{C}$ occurs and then $\mathbf{C}_t$.

(*d*) If the transformation tensor $\mathbf{C}$ is square (that is, if the new

network still has n meshes) then $\mathbf{C}^{-1}$ *can be calculated* and the equation may be written as

$$\mathbf{y}_c = \mathbf{C} \cdot (\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C})^{-1} \cdot \mathbf{C}_t = \mathbf{C} \cdot \mathbf{C}^{-1} \cdot \mathbf{z}^{-1} \cdot \mathbf{C}_t^{-1} \cdot \mathbf{C}_t = \mathbf{z}^{-1}$$

That is, the admittance tensor of the new network is the same as that of the primitive network, as mentioned above.

IX. EXAMPLE OF $\mathbf{y}_c$ FOR INDIVIDUAL COILS

(*a*) Let for instance the admittance tensor $\mathbf{y}'$ of the network of Fig. 5.7 (containing six coils and three meshes) be

$$\mathbf{y}' = \begin{array}{c|ccc} & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' \\ \hline \mathbf{a}' & y^{a'a'} & y^{a'b'} & y^{a'c'} \\ \mathbf{b}' & y^{b'a'} & y^{b'b'} & y^{b'c'} \\ \mathbf{c}' & y^{c'a'} & y^{c'b'} & y^{c'c'} \end{array} \qquad 5.31$$

$$\mathbf{C} = \begin{array}{c|ccc} & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' \\ \hline \mathbf{a} & -1 & & \\ \mathbf{b} & 1 & 1 & 1 \\ \mathbf{c} & & 1 & 1 \\ \mathbf{d} & -1 & & -1 \\ \mathbf{f} & & & -1 \\ \mathbf{g} & & 1 & \end{array} \qquad 5.32$$

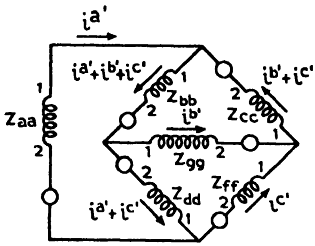


FIG. 5.7

The equation $\mathbf{i}' = \mathbf{y}' \cdot \mathbf{e}'$ gives *three* of the coil-currents if the *mesh* voltages $\mathbf{e}'$ are known. However, it is intended to find another tensor $\mathbf{y}_c$ that gives any one of the *six* coil currents if any one of the individual voltages in series with the coils are known. It should be noted that a single impressed coil voltage may belong to several meshes.

(*b*) The desired $\mathbf{y}_c$ is found from the above $\mathbf{y}'$ by $\mathbf{C} \cdot \mathbf{y}' \cdot \mathbf{C}_t = \mathbf{y}_c =$

	a	b	c	d	f	g
a	$y^{a'a'}$	$-y^{a'a'}-y^{a'b'}-y^{a'c'}$	$-y^{a'b'}-y^{a'c'}$	$y^{a'a'}+y^{a'c'}$	$y^{a'c'}$	$-y^{a'b'}$
b	$-y^{a'a'}$ $-y^{b'a'}$ $-y^{c'a'}$	$y^{a'a'}+y^{a'b'}+y^{a'c'}$ $+y^{b'a'}+y^{b'b'}+y^{b'c'}$ $+y^{c'a'}+y^{c'b'}+y^{c'c'}$	$y^{a'b'}+y^{a'c'}$ $+y^{b'b'}+y^{b'c'}$ $+y^{c'b'}+y^{c'c'}$	$-y^{a'a'}-y^{a'c'}$ $-y^{b'a'}-y^{b'c'}$ $-y^{c'a'}-y^{c'c'}$	$-y^{a'c'}$ $-y^{b'c'}$ $-y^{c'c'}$	$y^{a'b'}$ $+y^{b'b'}$ $+y^{c'b'}$
c	$-y^{b'a'}$ $-y^{c'a'}$	$y^{b'a'}+y^{b'b'}+y^{b'c'}$ $+y^{c'a'}+y^{c'b'}+y^{c'c'}$	$y^{b'b'}+y^{b'c'}$ $+y^{c'b'}+y^{c'c'}$	$-y^{b'a'}-y^{b'c'}$ $-y^{c'a'}-y^{c'c'}$	$-y^{b'c'}$ $-y^{c'c'}$	$y^{b'b'}$ $+y^{c'b'}$
d	$y^{a'a'}$ $+y^{c'a'}$	$-y^{a'a'}-y^{a'b'}-y^{a'c'}$ $-y^{c'a'}-y^{c'b'}-y^{c'c'}$	$-y^{a'b'}-y^{a'c'}$ $-y^{c'b'}-y^{c'c'}$	$y^{a'a'}+y^{a'b'}$ $+y^{c'a'}+y^{c'c'}$	$y^{a'c'}$ $+y^{c'c'}$	$-y^{a'b'}$ $-y^{c'b'}$
f	$y^{c'a'}$	$-y^{c'a'}-y^{c'b'}-y^{c'c'}$	$-y^{c'b'}-y^{c'c'}$	$y^{c'a'}+y^{c'c'}$	$y^{c'c'}$	$-y^{c'b'}$
g	$-y^{b'a'}$	$y^{b'a'}+y^{b'b'}+y^{b'c'}$	$y^{b'b'}+y^{b'c'}$	$-y^{b'a'}-y^{b'c'}$	$-y^{b'c'}$	$y^{b'b'}$

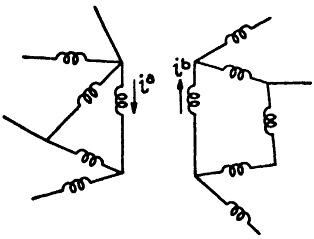
5.33

For instance, the current i^f in coil **f** due to an impressed voltage e_d in coil **d** is $i^f = (Y^{c'a'} + Y^{c'c'})e_d$.

X. INTERCONNECTION OF SEVERAL NETWORKS

(*a*) Instead of interconnecting *individual coils* into a network, *it is possible to take several individual networks and interconnect them into one larger system with the aid of a transformation tensor* $\mathbf{C} = C^m_{m'}$. This procedure will be found to be a very powerful engineering tool, inasmuch as *the individual apparatus to be interconnected are not limited to those of the same type.* The individual apparatus to be interconnected may be stationary or rotating, also electrical, mechanical, acoustical, optical, etc., apparatus. The resultant system may be some instrument, device, or machine utilizing several types of energies.

(*b*) The interconnection of *mesh networks* consists of opening up one (or more) branches of each network and interconnecting the two points of entry of each network (A–A' and B–B') *in pairs* as shown in Fig. 5.8*b*. The effect of the interconnection is to change the two



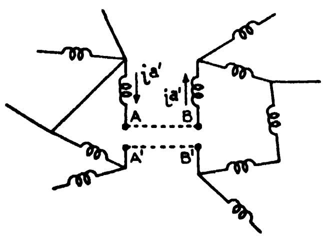


FIG. 5.8.—The Interconnection of Mesh Networks

coil currents i^a and i^b into one current, leaving all the other currents of each network unchanged.

The interconnection of *junction networks* is shown in Chapter XIV. The types of interconnection of physical apparatus are, however, of far greater variety than shown here.

(*c*) Often a system can be analyzed by building it up as usual by interconnecting individual coils. However, owing to some symmetry in parts of the system, the impedance tensors of these parts could be built up much more quickly if the disturbing asymmetrical parts were absent. In such cases *the system can be divided arbitrarily into any number of symmetrical and asymmetrical parts in any manner by opening up branches, then each part is analyzed separately, and finally recombined again by a* **C**. For instance, in fault studies of three-phase systems the

various places where faults occur changing the symmetry of the circuits may be removed, and the impedance tensors of the system and the faults (any number of them) may be set up separately, then recombined into the actual system.

It also happens that a system is built up from several parts where the components of the geometric objects **z**, etc., of some or all of the individual parts have already been calculated *on previous occasions.* In such cases the system again can be divided into familiar and unfamiliar parts, or into individual familiar parts, then recombined again with the aid of a **C**. That is, *many of the results arrived at previously in the analysis of the individual systems can be utilized again in the analysis of the resultant system.*

In interconnecting networks into a larger system the transformation tensor again represents a mathematical photograph of the manner in which the networks are interconnected. The performance of the combined system again can be found automatically by a *routine* transformation.

(*d*) In interconnecting entire networks the same steps are followed as in interconnecting individual coils, namely:

1. Find the geometric objects e_m, z_{mn}, etc., of the primitive system (consisting of the individual networks without interconnection).

2. Find the transformation tensor $C^m_{m'}$ showing the interconnection of the individual networks by which the primitive system has been changed into the actual system.

3. Find the new components of the geometric objects $e_{m'}$, $z_{m'n'}$, then $y^{n'm'}$, $i^{n'}$, etc., of the new system.

XI. THE PRIMITIVE SYSTEM

(*a*) *The aggregate of the individual networks without any interconnection between them is called the " primitive system."* The *fixed* indices of the various networks are different from one another so that *the number of axes of the primitive system is the sum of the axes of the individual systems.*

(*b*) Let, for instance, two independent stationary networks be given with the impedance tensors

$$\mathbf{z}_1 = \begin{array}{c|ccc} & a & b & c \\ \hline a & Z_{aa} & Z_{ab} & Z_{ac} \\ b & Z_{ab} & Z_{bb} & Z_{bc} \\ c & Z_{ac} & Z_{bc} & Z_{cc} \end{array} \qquad 5.34 \qquad\qquad \mathbf{z}_2 = \begin{array}{c|cc} & d & f \\ \hline d & Z_{dd} & Z_{df} \\ f & Z_{df} & Z_{ff} \end{array} \qquad 5.35$$

If the two networks are to be interconnected, they are considered as parts of a " primitive system " with the impedance tensor

$$\mathbf{z}_1 + \mathbf{z}_2 = \mathbf{z} = \begin{array}{c|ccccc} & a & b & c & d & f \\ \hline a & Z_{aa} & Z_{ab} & Z_{ac} & 0 & 0 \\ b & Z_{ab} & Z_{bb} & Z_{bc} & 0 & 0 \\ c & Z_{ac} & Z_{bc} & Z_{cc} & 0 & 0 \\ d & 0 & 0 & 0 & Z_{dd} & Z_{df} \\ f & 0 & 0 & 0 & Z_{df} & Z_{ff} \end{array} \qquad 5.36$$

having $3 + 2 = 5$ axes.

It should be noted that by adding the two impedance tensors $\mathbf{z}_1$ and $\mathbf{z}_2$, as shown in equation 5.36, no change whatever has been made in the physical system. *The addition of $\mathbf{z}_1$ and $\mathbf{z}_2$ is equivalent to the recognition that the two networks are independent parts of a more general system, the so-called " primitive system."* That is, the addition of $\mathbf{z}_1$ and $\mathbf{z}_2$ is equivalent to writing the original $\mathbf{z}_1$ and $\mathbf{z}_2$ each with five rows and columns instead of three or two respectively, by *filling out the components of the missing rows and columns by zero*, as

$$\mathbf{z}_1 = \begin{array}{c|ccccc} & a & b & c & d & f \\ \hline a & Z_{aa} & Z_{ab} & Z_{ac} & 0 & 0 \\ b & Z_{ab} & Z_{bb} & Z_{bc} & 0 & 0 \\ c & Z_{ac} & Z_{bc} & Z_{cc} & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 \end{array} \qquad 5.37 \qquad \mathbf{z}_2 = \begin{array}{c|ccccc} & a & b & c & d & f \\ \hline a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & Z_{dd} & Z_{df} \\ f & 0 & 0 & 0 & Z_{df} & Z_{ff} \end{array} \qquad 5.38$$

(*c*) Similarly if the impressed voltages of the individual networks are

$$\mathbf{e}_1 = \begin{array}{ccc} a & b & c \\ \hline e_a & e_b & e_c \end{array} \qquad 5.39 \qquad \mathbf{e}_2 = \begin{array}{cc} d & f \\ \hline e_d & e_f \end{array} \qquad 5.40$$

the impressed voltage vector of the primitive system is

$$\mathbf{e}_1 + \mathbf{e}_2 = \mathbf{e} = \begin{array}{ccccc} a & b & c & d & f \\ \hline e_a & e_b & e_c & e_d & e_f \end{array} \qquad 5.41$$

assuming that the impressed voltages of the component networks actually are

$$\mathbf{e}_1 = \begin{array}{ccccc} a & b & c & d & f \\ \hline e_a & e_b & e_c & 0 & 0 \end{array} \qquad 5.42 \qquad \mathbf{e}_2 = \begin{array}{ccccc} a & b & c & d & f \\ \hline 0 & 0 & 0 & e_d & e_f \end{array} \qquad 5.43$$

XII. THE RESULTANT SYSTEM

(*a*) Once the geometric objects of the primitive system have been set up by simple addition, the remaining work is identical with the analysis of any other system.

If the manner of interconnection of the component networks with one another is represented by $\mathbf{C}$, *the impedance tensor of the resultant system is*

$$\mathbf{z}' = \mathbf{C}_t \cdot (\mathbf{z}_1 + \mathbf{z}_2) \cdot \mathbf{C} \qquad 5.44$$

or with several component networks

$$\boxed{\mathbf{z}' = \mathbf{C}_t \cdot (\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3 + \cdots) \cdot \mathbf{C}} \qquad 5.45$$

The impressed voltage vector of the resultant system is

$$\boxed{\mathbf{e}' = \mathbf{C}_t \cdot (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \cdots)} \qquad 5.46$$

and its equation of voltage

$$\boxed{\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'} \qquad 5.47$$

(*b*) If the equation is not subdivided into several component equations, it may be solved for the unknowns as $\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}'$, that is, by finding $\mathbf{y}'$, the inverse of $\mathbf{z}'$.

(*c*) It is emphasized that this method of reasoning should be used not only when actually two component networks are given and then they are to be interconnected. The method of reasoning given can be used also when a complex system *already built* is to be analyzed and it is difficult to consider the resultant system as a whole. *Complex systems can usually be divided into several component systems that individually are easy to analyze and their geometric objects* $\mathbf{z}_1$, $\mathbf{z}_2$, *etc., are easily established.* Once each component system has been treated as one unit then their interconnection into one unit is a far easier procedure than the analysis of the original system.

One example of such a complex system is a vacuum-tube circuit where several multielectrode tubes are interconnected with static networks. Their analysis is treated in Chapter XV. Another example is a three-phase transmission system, where numerous three-phase apparatus (generators, transformers, loads, etc.) are interconnected in various manner. They are treated in detail in Chapter XIX.

(*d*) If the 2-matrix of the geometric object $\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3$ has a

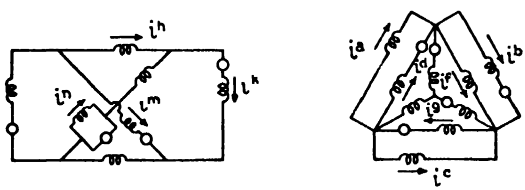
cumbersome number of rows and columns, the resultant **z**, equation 5.45, *can be calculated in steps* since it may be written as

$$\mathbf{z}' = \mathbf{C}_t \cdot \mathbf{z}_1 \cdot \mathbf{C} + \mathbf{C}_t \cdot \mathbf{z}_2 \cdot \mathbf{C} + \mathbf{C}_t \cdot \mathbf{z}_3 \cdot \mathbf{C} + \cdots \qquad 5.48$$

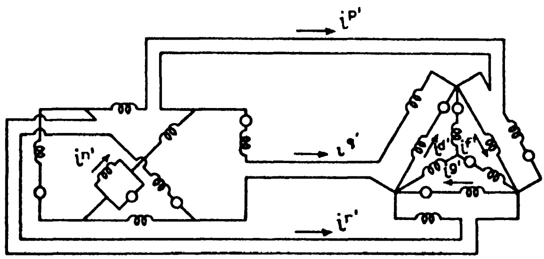
showing that the final impedance tensor may be represented as the sum of several tensors, each tensor being the contribution of one of the component systems.

XIII. EXAMPLE OF INTERCONNECTION OF TWO MESH NETWORKS

(*a*) Let two mesh networks be given as shown in Fig. 5.9*a*. It is assumed that their geometric objects have already been established



(*a*) The primitive system



(*b*) Resultant system

FIG. 5.9.—The Interconnection of Several Networks

by selecting the shown four and six currents respectively as variables. In particular their impedance tensors are

$$\mathbf{z}_1 = \begin{array}{c|cccc|} & m & n & h & k \\ \hline m & Z_{mm} & Z_{mn} & Z_{mh} & Z_{mk} \\ n & Z_{mn} & Z_{nn} & Z_{nh} & Z_{nk} \\ h & Z_{mh} & Z_{nh} & Z_{hh} & Z_{hk} \\ k & Z_{mk} & Z_{nk} & Z_{hk} & Z_{kk} \\ \hline \end{array} \quad 5.49 \qquad \mathbf{z}_2 = \begin{array}{c|cccccc|} & a & b & c & d & f & g \\ \hline a & Z_{aa} & Z_{ab} & Z_{ac} & Z_{ad} & Z_{af} & Z_{ag} \\ b & Z_{ab} & Z_{bb} & Z_{bc} & Z_{bd} & Z_{bf} & Z_{bg} \\ c & Z_{ac} & Z_{bc} & Z_{cc} & Z_{cd} & Z_{cf} & Z_{cg} \\ d & Z_{ad} & Z_{bd} & Z_{cd} & Z_{dd} & Z_{df} & Z_{dg} \\ f & Z_{af} & Z_{bf} & Z_{cf} & Z_{df} & Z_{ff} & Z_{fg} \\ g & Z_{ag} & Z_{bg} & Z_{cg} & Z_{dg} & Z_{fg} & Z_{gg} \\ \hline \end{array} \quad 5.50$$

and their impressed voltage vectors are

$$\mathbf{e}_1 = \begin{array}{cccc} m & n & h & k \\ \hline \multicolumn{1}{|c|}{e_m} & \multicolumn{1}{c|}{e_n} & \multicolumn{1}{c|}{e_h} & \multicolumn{1}{c|}{e_k} \\ \hline \end{array} \quad 5.51 \qquad \mathbf{e}_2 = \begin{array}{cccccc} a & b & c & d & f & g \\ \hline \multicolumn{1}{|c|}{e_a} & \multicolumn{1}{c|}{e_b} & \multicolumn{1}{c|}{e_c} & \multicolumn{1}{c|}{e_d} & \multicolumn{1}{c|}{e_f} & \multicolumn{1}{c|}{e_g} \\ \hline \end{array} \quad 5.52$$

(*b*) If the two networks of Fig. 5.9*a* are to be interconnected into one system in any manner they may be considered as forming already *one* system, the primitive system whose impedance tensor is the sum of the impedance tensors $\mathbf{z}_1$ and $\mathbf{z}_2$ of the individual systems shown in equations 5.49 and 5.50. That is, $\mathbf{z}_1 + \mathbf{z}_2 =$

$$\mathbf{z} = \begin{array}{c|cccccccccc|} & m & n & h & k & a & b & c & d & f & g \\ \hline m & Z_{mm} & Z_{mn} & Z_{mh} & Z_{mk} & & & & & & \\ n & Z_{mn} & Z_{nn} & Z_{nh} & Z_{nk} & & & & & & \\ h & Z_{mh} & Z_{nh} & Z_{hh} & Z_{hk} & & & & & & \\ k & Z_{mk} & Z_{nk} & Z_{hk} & Z_{kk} & & & & & & \\ a & & & & & Z_{aa} & Z_{ab} & Z_{ac} & Z_{ad} & Z_{af} & Z_{ag} \\ b & & & & & Z_{ab} & Z_{bb} & Z_{bc} & Z_{bd} & Z_{bf} & Z_{bg} \\ c & & & & & Z_{ac} & Z_{bc} & Z_{cc} & Z_{cd} & Z_{cf} & Z_{cg} \\ d & & & & & Z_{ad} & Z_{bd} & Z_{cd} & Z_{dd} & Z_{df} & Z_{dg} \\ f & & & & & Z_{af} & Z_{bf} & Z_{cf} & Z_{df} & Z_{ff} & Z_{fg} \\ g & & & & & Z_{ag} & Z_{bg} & Z_{cg} & Z_{dg} & Z_{fg} & Z_{gg} \\ \hline \end{array} \qquad 5.53$$

The impressed voltage vector of the primitive system is $\mathbf{e}_1 + \mathbf{e}_2 =$

$$\mathbf{e} = \begin{array}{cccccccccc} m & n & h & k & a & b & c & d & f & g \\ \hline \multicolumn{1}{|c|}{e_m} & \multicolumn{1}{c|}{e_n} & \multicolumn{1}{c|}{e_h} & \multicolumn{1}{c|}{e_k} & \multicolumn{1}{c|}{e_a} & \multicolumn{1}{c|}{e_b} & \multicolumn{1}{c|}{e_c} & \multicolumn{1}{c|}{e_d} & \multicolumn{1}{c|}{e_f} & \multicolumn{1}{c|}{e_g} \\ \hline \end{array} \qquad 5.54$$

(*c*) From Fig. 5.9*b* the interconnection of the two systems consists of *connecting* i^h *in series with* i^b, also i^k in series with i^a, and finally i^m in series with i^c, leaving all the other four currents i^n, i^d, i^f, and i^g unchanged. The currents flowing in the series connections may be denoted by three new letters $i^{p\prime}$, $i^{q\prime}$, and $i^{r\prime}$ while the four unchanged currents may be primed.

Hence the relation between the old and the new currents $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ and the transformation tensor $\mathbf{C}$ are

$$
\begin{aligned}
i^m &= -i^{r'} = -i^{r'} \\
i^n &= i^{n'} = i^{n'} \\
i^h &= i^{p'} = i^{p'} \\
i^k &= i^{q'} = i^{q'} \\
i^a &= i^{q'} = i^{q'} \\
i^b &= i^{p'} = i^{p'} \\
i^c &= -i^{r'} = -i^{r'} \\
i^d &= i^{d'} = i^{d'} \\
i^f &= i^{f'} = i^{f'} \\
i^g &= i^{g'} = i^{g'}
\end{aligned}
$$

C =

	p′	q′	r′	n′	d′	f′	g′
m			−1				
n				1			
h	1						
k		1					
a		1					
b	1						
c			−1				
d					1		
f						1	
g							1

5.55

(*d*) The impedance tensor of the resultant system is by $\mathbf{C}_t \cdot (\mathbf{z}_1 + \mathbf{z}_2) \cdot \mathbf{C} =$

z′ =

	p′	q′	r′	n′	d′	f′	g′
p′	$Z_{hh}+Z_{bb}$	$Z_{hk}+Z_{ab}$	$-Z_{mh}-Z_{bc}$	Z_{nh}	Z_{bd}	Z_{bf}	Z_{bg}
q′	$Z_{hk}+Z_{ab}$	$Z_{kk}+Z_{aa}$	$-Z_{mk}-Z_{ac}$	Z_{nk}	Z_{ad}	Z_{af}	Z_{ag}
r′	$-Z_{mh}-Z_{bc}$	$-Z_{mk}-Z_{ac}$	$Z_{mm}+Z_{cc}$	$-Z_{nm}$	Z_{cd}	$-Z_{cf}$	$-Z_{cg}$
n′	Z_{nh}	Z_{nk}	$-Z_{mn}$	Z_{nn}	0	0	0
d′	Z_{bd}	Z_{ad}	$-Z_{cd}$	0	Z_{dd}	Z_{df}	Z_{dg}
f′	Z_{bf}	Z_{af}	$-Z_{cf}$	0	Z_{df}	Z_{ff}	Z_{fg}
g′	Z_{bg}	Z_{ag}	$-Z_{cg}$	0	Z_{dg}	Z_{fg}	Z_{gg}

5.56

The impressed voltage vector is by $\mathbf{C}_t \cdot (\mathbf{e}_1 + \mathbf{e}_2) =$

e′ =

p′	q′	r′	n′	d′	f′	g′
$e_h + e_b$	$e_k + e_a$	$-e_m - e_c$	e_n	e_d	e_f	e_g

5.57

The currents are found by $\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}'$. In finding the inverse of $\mathbf{z}'$ the labor-saving procedure of Chapter X may be used.

XIV. SELECTION OF THE VARIABLES

The interconnection of systems shows clearly the desirability of selecting particular *branch currents* as the variables in particular problems. In interconnecting networks *those branch currents of the component networks that are connected in series should appear among the variables.*

When a needed current is not one of the variables, then first the variables of the component network should be changed to another set of variables by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$, *so that among the new variables all needed currents may appear.* In changing the variables of course both $\mathbf{e}$ and $\mathbf{z}$ of the system have also to be changed. (Such a change of variables where the physical system is not changed will be shown in Section IV, Chapter VI.)

In other words, when the available $\mathbf{z}_1$ or $\mathbf{z}_2$ (or both) do not contain as variables the currents that are interconnected, then $\mathbf{z}_1$ or $\mathbf{z}_2$ (or both) are changed first to another $\mathbf{z}'_1$ or $\mathbf{z}'_2$ by $\mathbf{C}_1$ or $\mathbf{C}_2$ and then only are they considered to be interconnected into one system by $\mathbf{C}$. In such cases the impedance tensor $\mathbf{z}'$, of the resultant system is found by

$$\mathbf{z}' = \mathbf{C}_t \cdot [\mathbf{z}'_1 + \mathbf{z}'_2 + \cdots] \cdot \mathbf{C} \qquad 5.58$$

$$\mathbf{z}' = \mathbf{C}_t \cdot [(\mathbf{C}_{1t} \cdot \mathbf{z}_1 \cdot \mathbf{C}_1) + (\mathbf{C}_{2t} \cdot \mathbf{z}_2 \cdot \mathbf{C}_2) + \cdots] \cdot \mathbf{C} \qquad 5.59$$

CHAPTER VI

EXAMPLES OF INVARIANT TRANSFORMATIONS

I. TYPES OF TRANSFORMATIONS

(*a*) *The addition of interconnections between coils is only one of a large variety of other changes that may be represented mathematically by a transformation tensor* **C**. *The idea of transformation involves only the replacement of a set of variables* (*here the currents*) *by another set of variables* irrespective of whether this replacement corresponds to a physical change in the system or not. Various examples will be shown now where the transformation of the currents involves changes of various types either in the *physical* set-up of the system or in the selection of the variables.

The following types of transformations will be shown in this chapter:

1. Instead of the primitive mesh network, *any other* n-mesh network of n coils is used as a reference network, to be transformed to a given n-mesh, n-coil network.
2. A set of currets flowing in the *branches* of a network is replaced by another set of currents flowing in different *branches* of the same network.
3. A set of currents flowing in the *branches* of a network is replaced by hypothetical currents assumed to flow in the *meshes* of the network.
4. A set of *mesh* currents is replaced by a set of *branch* currents.
5. The *number of turns* of several coils is changed.
6. The " *magnetizing currents* " are neglected.
7. The *actual* currents are replaced by a set of hypothetical " *load* " currents and " *magnetizing* " currents.
8. The *opening of meshes* is considered as a transformation.
9. The *order* of the variables is changed by a transformation.
10. The role of the commutator of a d-c. machine is considered as a transformation changing the *magnetic order* of the conductors to their *electrical order*.

Quite a large number of assumptions or labor-saving devices which the engineer introduces to organize and solve his problem may be considered

as a "transformation" representable by a transformation tensor $C_{m'}^{m}$. Hence a great variety of transformation is possible and those given in this chapter are only examples of the simplest type. Many more of them will be introduced subsequently.

(*b*) These numerous types of transformations, however, have one feature in common. *All these transformations leave the power input,* $e_m i^m$, (a linear form) *invariant.* As a consequence, in spite of the use of such a variety of transformations, all the actual and hypothetical *mesh* networks have the same equation of performance, namely:

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i} \qquad \Big| \qquad e_m = z_{mn} i^n$$

and for any particular set-up the components $\mathbf{e}$ and $\mathbf{z}$ are found from those of any reference network by routine transformation formulas.

Transformations $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ that leave a "form" invariant will be called "*invariant transformations.*" Later other types of transformations will also be introduced that do not leave a "form" invariant.

(*c*) It may be mentioned that these transformations rarely occur alone as presented in this chapter. *Most engineering problems consist of the simultaneous application of several transformations*, and often it is quite difficult to unscramble the resultant complicated transformation into the successive application of a series of simpler transformations.

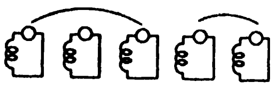
The study of the successive application of *several* transformations will be undertaken in Chapter XI.

II. THE TRANSFORMATION OF AN N-MESH NETWORK INTO ANOTHER N-MESH NETWORK

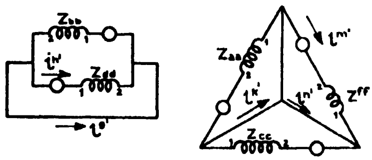
(*a*) In order to set up the equation of performance of a network *quickly* and *with the least amount of reasoning*, the "primitive network" is used as a reference network. However, *for the analysis of an n-coil network any other n-coil network may be used as a reference network.* Quite often the use of some n-coil network with a special structure as the starting point leads to a quicker answer than the use of the primitive network.

It will be shown in Chapter XVI, Section XVII, that both the given network and the reference network may have different number of meshes and junction-pairs as long as they have the same number of coils. However, both networks have to be considered as *orthogonal* networks in order to be able to pass from one to the other and back by a transformation tensor $\mathbf{C}$ and its inverse $\mathbf{C}^{-1}$.

(*b*) Since an n-coil, n-mesh network is also an orthogonal network (having zero junction-pairs), *let an example be worked out in which the given network and the reference network both have n coils and n meshes,* but different numbers of sub-networks and different numbers of branches with zero impedance. For instance, let the equation of voltage $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$ of the network of Fig. 6.1*a* be set up if the equation of voltage $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ of the reference network of Fig. 6.2 is already



(*b*) Its primitive network



(*a*) Given network

FIG. 6.1

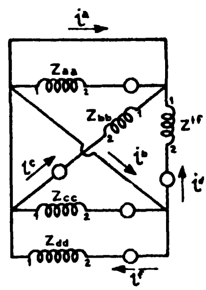


FIG. 6.2.—Reference Network

known. The latter has already been calculated in Chapter III, Section XIV.

Since the given network of Fig. 6.1*a* has five meshes, five arbitrary branch currents, $i^{g'}$, $i^{h'}$, $i^{k'}$, $i^{m'}$, and $i^{n'}$ are assumed as variables.

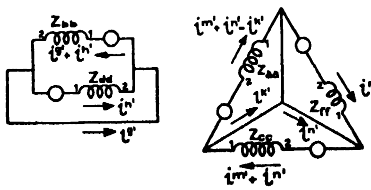


FIG. 6.3.—Given Network

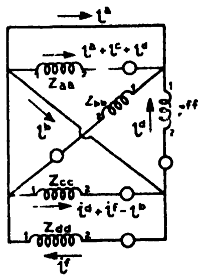


FIG. 6.4.—Reference Network

(*c*) In order to set up the transformation tensor $\mathbf{C}$ the first step is to write in both networks *along each individual coil* the currents flowing through them as shown in Figs. 6.3 and 6.4.

The next step is to *equate* the old and the new currents flowing in each coil as follows:

In coil

$$
\begin{array}{ll}
Z_{aa} & -i^a - i^c - i^d = i^{k'} - i^{m'} - i^{n'} \\
Z_{bb} & - i^c = i^{g'} + i^{h'} \\
Z_{cc} & i^d + i^f - i^b = - i^{m'} - i^{n'} \\
Z_{dd} & - i^f = i^{h'} \\
Z_{ff} & - i^d = - i^{m'}
\end{array}
\qquad 6.1
$$

This set of equations always can be solved for the old currents as linear functions of the new currents as

$$
\begin{aligned}
i^a &= i^{g'} + i^{h'} - i^{k'} + i^{n'} \\
i^b &= - i^{h'} + 2i^{m'} + i^{n'} \\
i^c &= - i^{g'} - i^{h'} \\
i^d &= i^{m'} \\
i^f &= - i^{h'}
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c|ccccc}
 & g' & h' & k' & m' & n' \\
\hline
a & 1 & 1 & -1 & & 1 \\
b & & -1 & & 2 & 1 \\
c & -1 & -1 & & & \\
d & & & & 1 & \\
f & & -1 & & &
\end{array}
\qquad 6.2
$$

The coefficients of the new currents represent the transformation tensor **C**, changing the network of Fig. 6.3 to the network of Fig. 6.4. *It should be noted that one of the components in* **C** *is* 2, *an integer also.*

(*d*) The new components of the impedance tensor **z′** for the given network are found from that of the reference network, given in equation 4.65, by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ as

$$
\mathbf{z}' = \begin{array}{c|ccccc}
 & g' & h' & k' & m' & n' \\
\hline
g' & Z_{bb} & Z_{bb} & & & \\
h' & Z_{bb} & Z_{bb}+Z_{dd} & & -X_{df} & \\
k' & & & Z_{aa} & -Z_{aa}-X_{ac} & -Z_{aa}-X_{ac} \\
m' & & -X_{fd} & -Z_{aa}-X_{ca} & Z_{aa}+Z_{cc}+Z_{ff}+X_{ac}+X_{ca} & Z_{aa}+Z_{cc}+X_{ca}+X_{ac} \\
n' & & & -Z_{aa}-X_{ca} & Z_{aa}+Z_{cc}+X_{ac}+X_{ca} & Z_{aa}+Z_{cc}+X_{ac}+X_{ca}
\end{array}
\qquad 6.3
$$

The impressed voltage vector **e′** is by $\mathbf{C}_t \cdot \mathbf{e}$, where **e** is given in equation 4.66, as

$$
\mathbf{e}' = \begin{array}{|c|c|c|c|c|}
g' & h' & k' & m' & n' \\
\hline
e_b & e_b + e_d & e_a & - e_a - e_c - e_f & - e_a - e_c
\end{array}
\qquad 6.4
$$

III. A CHECK ON THE TRANSFORMATION

(*a*) The correctness of the tensors **z**′ and **e**′ of Fig. 6.3 may be checked by calculating the components of the same tensors from the primitive network given in Fig. 6.1*b* instead of Fig. 6.2.

The **z** and **e** tensors of the primitive mesh network are

z =

	a	b	c	d	f
a	Z_{aa}		X_{ac}		
b		Z_{bb}			
c	X_{ca}		Z_{cc}		
d				Z_{dd}	X_{df}
f				X_{fd}	Z_{ff}

6.5

e =

a	b	c	d	f
e_a	e_b	e_c	e_d	e_g

6.6

The transformation tensor **C**′ of the given network is from Fig. 6.3, by equating the old and the new currents in Fig. 6.3 and 6.1*b*.

$$
\begin{aligned}
i^a &= i^{k'} - i^{m'} - i^{n'} \\
i^b &= i^{g'} + i^{h'} \\
i^c &= - i^{m'} - i^{n'} \\
i^d &= i^{h'} \\
i^f &= - i^{m'}
\end{aligned}
$$

C′ =

	g′	h′	k′	m′	n′
a			1	−1	−1
b	1	1			
c				−1	−1
d		1			
f				−1	

6.7

(*b*) The impedance tensor **z**′ is by $\mathbf{C}'_t \cdot \mathbf{z} \cdot \mathbf{C}'$ =

z′ =

	g′	h′	k′	m′	n′
g′	Z_{bb}	Z_{bb}			
h′	Z_{bb}	$Z_{bb}+Z_{dd}$		$-X_{df}$	
k′			Z_{aa}	$-Z_{aa}-X_{ac}$	$-Z_{aa}-X_{ac}$
m′		$-X_{fd}$	$-Z_{aa}-X_{ca}$	$Z_{aa}+Z_{cc}+Z_{ff}+X_{ac}+X_{ca}$	$Z_{aa}+Z_{cc}+X_{ac}+X_{ca}$
n′			$-Z_{aa}-X_{ca}$	$Z_{aa}+Z_{cc}+X_{ac}+X_{ca}$	$Z_{aa}+Z_{cc}+X_{ac}+X_{ca}$

6.8

This tensor checks equation 6.3.

The impressed-voltage vector $\mathbf{e}'$ is by $\mathbf{C}'_t \cdot \mathbf{e}$ =

$$\mathbf{e}' = \begin{array}{ccccc} \mathbf{g}' & \mathbf{h}' & \mathbf{k}' & \mathbf{m}' & \mathbf{n}' \\ \hline \multicolumn{1}{|c|}{e_b} & \multicolumn{1}{c|}{e_b + e_d} & \multicolumn{1}{c|}{e_a} & \multicolumn{1}{c|}{-e_a - e_c - e_f} & \multicolumn{1}{c|}{-e_a - e_c} \\ \hline \end{array} \qquad 6.9$$

This vector checks equation 6.4.

IV. TRANSFORMATION OF BRANCH CURRENTS

(*a*) Let it be assumed that, for Fig. 5.4, $\mathbf{e}'$, $\mathbf{i}'$, and $\mathbf{z}'$ have already been calculated by considering the four currents $i^{a'}$, $i^{b'}$, $i^{c'}$, and $i^{d'}$ as the variables. Fig. 5.4 is reproduced again in Fig. 6.5.

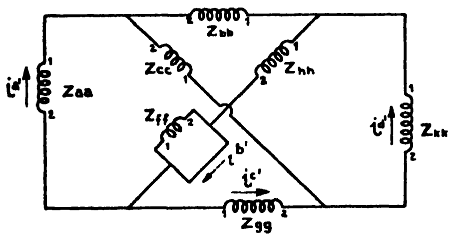


FIG. 6.5.—The Four Old Currents

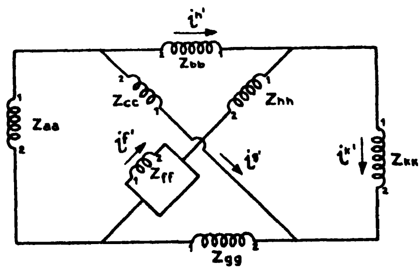


FIG. 6.6.—The Four New Currents

Let it also be assumed that for some reason *the performance of the system needs to be expressed in terms of four other currents*, say $i^{f'}$, $i^{g'}$, $i^{h'}$, and $i^{k'}$ as variables, shown in Fig. 6.6. Two procedures are available:

1. The calculation may be started all over again by removing all interconnections, in order to set up again the original primitive network and a new transformation tensor $\mathbf{C}$.

2. The already calculated $\mathbf{e}'$, $\mathbf{z}'$, and $\mathbf{i}'$ are replaced by another set $\mathbf{e}''$, $\mathbf{z}''$, $\mathbf{i}''$ with the aid of a new transformation matrix $\mathbf{C}'$ that sets up a relation between $\mathbf{i}'$ and $\mathbf{i}''$, where

$$\mathbf{i}' = \begin{array}{cccc} \mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \mathbf{d}' \\ \hline \multicolumn{1}{|c|}{i^{a'}} & \multicolumn{1}{c|}{i^{b'}} & \multicolumn{1}{c|}{i^{c'}} & \multicolumn{1}{c|}{i^{d'}} \\ \hline \end{array} \qquad \mathbf{i}'' = \begin{array}{cccc} \mathbf{f}' & \mathbf{g}' & \mathbf{h}' & \mathbf{k}' \\ \hline \multicolumn{1}{|c|}{i^{f'}} & \multicolumn{1}{c|}{i^{g'}} & \multicolumn{1}{c|}{i^{h'}} & \multicolumn{1}{c|}{i^{k'}} \\ \hline \end{array}$$

(*b*) *Following the second procedure* the role of the *individual coils* is replaced by that of the *branches* in which the old currents flow. The steps are as follows:

1. The currents in those branches in which the old currents

$i^{a'}$, $i^{b'}$, $i^{c'}$, and $i^{d'}$ flow are expressed in terms of the new currents as shown in Fig. 6.7.

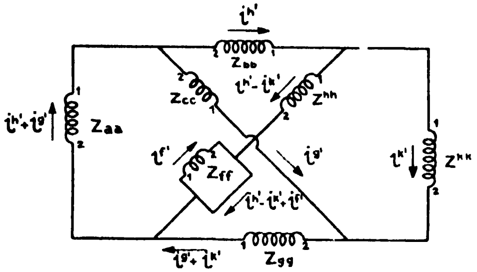


FIG. 6.7.—The New Currents in the Individual Coils

2. Equate the old and the new currents flowing in the four *old* branches as

$$
\begin{aligned}
i^{a'} &= i^{h'} + i^{g'} &&= i^{g'} + i^{h'} \\
i^{b'} &= i^{h'} - i^{k'} + i^{f'} &&= i^{f'} + i^{h'} - i^{k'} \\
i^{c'} &= -(i^{g'} + i^{k'}) &&= -i^{g'} - i^{k'} \\
i^{d'} &= -i^{k'} &&= -i^{k'}
\end{aligned}
\qquad
\mathbf{C}' = \begin{array}{c|cccc}
 & \mathbf{f}' & \mathbf{g}' & \mathbf{h}' & \mathbf{k}' \\
\hline
\mathbf{a}' & & 1 & 1 & \\
\mathbf{b}' & 1 & & 1 & -1 \\
\mathbf{c}' & & -1 & & -1 \\
\mathbf{d}' & & & & -1
\end{array}
\qquad 6.10
$$

The coefficients of the new currents give the new transformation tensor:

(*c*) With the knowledge of $\mathbf{C}'$ the new components $\mathbf{z}''$ of the impedance tensor are found by $\mathbf{C}'_t \cdot \mathbf{z}' \cdot \mathbf{C}$, where $\mathbf{z}'$ is given in equation 5.19, so that

$\mathbf{Z}'' =$ (6.11)

	$\mathbf{f}'$	$\mathbf{g}'$	$\mathbf{h}'$	$\mathbf{k}'$
$\mathbf{f}'$	$Z_{b'b'}$	$Z_{a'b'}-Z_{b'c'}$	$Z_{a'b'}+Z_{b'c'}$	$-Z_{ba'b'}-Z_{b'c'}-Z_{b'd'}$
$\mathbf{g}'$	$Z_{a'b'}-Z_{b'c'}$	$Z_{a'a'}-Z_{a'c'}$ $-Z_{a'c'}+Z_{c'c'}$	$Z_{a'a'}-Z_{a'c'}$ $+Z_{a'b'}-Z_{b'c'}$	$-Z_{a'b'}+Z_{b'c'}-Z_{a'c'}$ $+Z_{c'c'}-Z_{a'd'}+Z_{c'd'}$
$\mathbf{h}'$	$Z_{a'b'}+Z_{b'b'}$	$Z_{a'a'}+Z_{a'b'}$ $-Z_{a'c'}-Z_{b'c'}$	$Z_{a'a'}+Z_{a'b'}$ $+Z_{a'b'}+Z_{b'b'}$	$-Z_{a'b'}-Z_{b'b'}-Z_{a'c'}$ $-Z_{b'c'}-Z_{a'd'}-Z_{b'd'}$
$\mathbf{k}'$	$-Z_{b'b'}-Z_{b'c'}$ $-Z_{b'd'}$	$-Z_{a'b'}-Z_{a'c'}$ $-Z_{a'd'}$ $+Z_{b'c'}+Z_{c'c'}$ $+Z_{c'd'}$	$-Z_{a'b'}-Z_{a'c'}$ $-Z_{a'd'}$ $-Z_{b'b'}-Z_{b'c'}$ $-Z_{b'd'}$	$Z_{b'b'}+Z_{b'c'}+Z_{b'd'}$ $+Z_{b'c'}+Z_{c'c'}+Z_{c'd'}$ $+Z_{b'd'}+Z_{c'd'}+Z_{d'd'}$

The values of $Z_{a'a'}$, etc., in terms of Z_{aa}, X_{ab}, etc., may be substituted from equation 5.18 to express $\mathbf{z}''$ in terms of the individual impedances $Z_{aa}, X_{ab} \cdots$.

The impressed voltage vector is by $\mathbf{C}'_t \cdot \mathbf{e}' =$

$$\mathbf{e}'' = \begin{array}{cccc} \mathbf{f} & \mathbf{g} & \mathbf{h}' & \mathbf{k}' \\ \hline \multicolumn{1}{|c|}{e_{b'}} & \multicolumn{1}{c|}{e_{a'} - e_{c'}} & \multicolumn{1}{c|}{e_{a'} + e_{b'}} & \multicolumn{1}{c|}{-e_{b'} - e_{c'} - e_{d'}} \\ \hline \end{array}$$

In terms of the actual impressed voltages, using equation 5.20

$$\mathbf{e}'' = \begin{array}{cccc} \mathbf{f} & \mathbf{g} & \mathbf{h}' & \mathbf{k}' \\ \hline \multicolumn{1}{|c|}{e_{d}} & \multicolumn{1}{c|}{-e_{a} - e_{c}} & \multicolumn{1}{c|}{-e_{a} + e_{d}} & \multicolumn{1}{c|}{-e_{d} + e_{k}} \\ \hline \end{array}$$

The voltage equation $\mathbf{e}'' = \mathbf{z}'' \cdot \mathbf{i}''$ expresses the performance of the network of Fig. 6.6 in terms of $i^{f'}$, $i^{g'}$, $i^{h'}$, and $i^{k'}$ as it was desired.

It should be noted that in general there are a large (though not infinite) number of ways to select the new variables $\mathbf{i}''$.

(*d*) In assuming new mesh-currents $\mathbf{i}''$ by $\mathbf{i}' = \mathbf{C}' \cdot \mathbf{i}''$ *it is possible to have the voltages* $\mathbf{e}'$ *expressed around the old set of meshes.* In that case the voltage equation becomes

$$\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}' = \mathbf{z}' \cdot \mathbf{C}' \cdot \mathbf{i}'' = \mathbf{z}''' \cdot \mathbf{i}''$$

$$e_{m'} = z_{m'n'} i^{n'} = z_{m'n'} C^{n'}_{n''} i^{n''} = z_{m'n''} i^{n''} \qquad 6.12$$

where

$$\boxed{z_{m'n''} = z_{m'n'} C^{n'}_{n''}} \qquad \Bigg| \qquad \boxed{\mathbf{z}''' = \mathbf{z}' \cdot \mathbf{C}'} \qquad 6.13$$

representing the impedance tensor of a network in which the voltages are expressed around a different set of meshes than the currents.

Multiplying matrix 5.19 by 6.10

$$\mathbf{z}''' = \begin{array}{c|c|c|c|c|} & \mathbf{f}' & \mathbf{g}' & \mathbf{h}' & \mathbf{k}' \\ \hline \mathbf{a}' & Z_{a'b'} & Z_{a'a'} - Z_{a'c'} & Z_{a'a'} + Z_{a'b'} & -Z_{a'b'} - Z_{a'c'} - Z_{a'd'} \\ \hline \mathbf{b}' & Z_{b'b'} & Z_{a'b'} - Z_{b'c'} & Z_{a'b'} + Z_{b'b'} & -Z_{b'b'} - Z_{b'c'} - Z_{b'd'} \\ \hline \mathbf{c}' & Z_{b'c'} & Z_{a'c'} - Z_{c'c'} & Z_{a'c'} + Z_{b'c'} & -Z_{b'c'} - Z_{c'c'} - Z_{c'd'} \\ \hline \mathbf{d}' & Z_{b'd'} & Z_{a'd'} - Z_{c'd'} & Z_{a'd'} + Z_{b'd'} & -Z_{b'd'} - Z_{c'd'} - Z_{d'd'} \\ \hline \end{array} \qquad 6.14$$

It should be noted that $\mathbf{z}'''$ has two sets of reference frames, ***the meshes of the currents being different from the meshes of the voltages.***

V. THE HYPOTHETICAL " MESH " CURRENTS

(*a*) In all problems so far considered the new currents i' always were *currents* actually flowing in some branch of the network. However, *the idea of " transformation " involves a replacement of a set of currents by another set, irrespective of whether either set of currents is physically existing or is only some hypothetical set.* That is, either the old or the new or both sets of currents may be hypothetical currents.

There are, of course, a large number of ways in which hypothetical currents may be introduced in electrical engineering. In fact, most of the analysis of engineering problems involves hypothetical currents, such as the method of symmetrical components, the cross-field and revolving field theories, traveling waves, and in fact any resolution of resultant currents into harmonics (space or time harmonics).

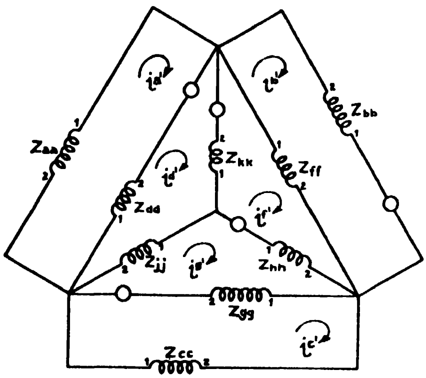


FIG. 6.8.—The Six Mesh-currents as Variables

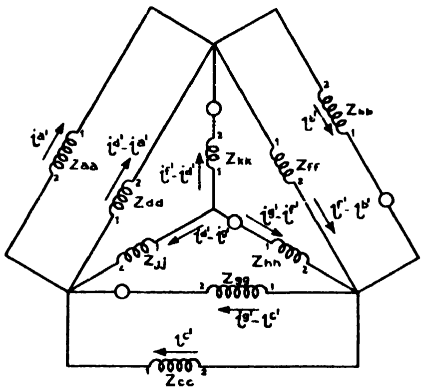


FIG. 6.10.—The Mesh-currents in the Individual Coils

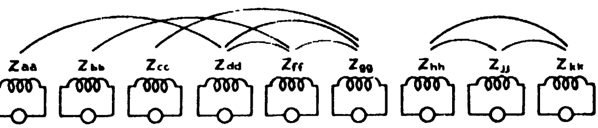


FIG. 6.9.—The Primitive Network

(*b*) In stationary-network analysis it is customary to assume hypothetical " *mesh* " currents (or " Maxwell " currents) in each closed mesh, so that the sum of these hypothetical currents in each branch is the actual current flowing in that branch. For instance, in Fig. 6.8 the actual current flowing through coil Z_{dd} from 1 to 2 is $i^{d'} - i^{a'}$. In Fig. 6.8 out of the six mesh currents three are also actually existing branch currents, namely $i^{a'}$, $i^{b'}$, and $i^{c'}$, but the three other mesh currents $i^{d'}$, $i^{f'}$, and $i^{g'}$ are hypothetical currents, having no actual physical existence.

The method of attack is of course the same as with actually existing currents. The primitive network is shown in Fig. 6.9. Its impedance tensor is

$$\mathbf{z} = \begin{array}{c|ccccccccc|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} & \mathbf{g} & \mathbf{h} & \mathbf{j} & \mathbf{k} \\ \hline \mathbf{a} & Z_{aa} & & & X_{ad} & & & & & \\ \mathbf{b} & & Z_{bb} & & & X_{bf} & & & & \\ \mathbf{c} & & & Z_{cc} & & & X_{cg} & & & \\ \mathbf{d} & X_{ad} & & & Z_{dd} & X_{df} & X_{dg} & & & \\ \mathbf{f} & & X_{bf} & & X_{df} & Z_{ff} & X_{fg} & & & \\ \mathbf{g} & & & X_{cg} & X_{dg} & X_{fg} & Z_{gg} & & & \\ \mathbf{h} & & & & & & & Z_{hh} & X_{hj} & X_{hk} \\ \mathbf{j} & & & & & & & X_{hj} & Z_{jj} & X_{jk} \\ \mathbf{k} & & & & & & & X_{hk} & X_{jk} & Z_{kk} \\ \hline \end{array} \qquad 6.15$$

The impressed voltage vector is

$$\mathbf{e} = \begin{array}{|ccccccccc|} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} & \mathbf{g} & \mathbf{h} & \mathbf{j} & \mathbf{k} \\ \hline 0 & e_b & 0 & -e_d & 0 & e_g & -e_h & 0 & e_k \\ \hline \end{array}$$

(*c*) In setting up the transformation tensor the first step is to express the actual currents flowing through each coil in terms of the new mesh currents as shown in Fig. 6.10. The second step is to equate the old and the new currents flowing through each coil, the old currents being those flowing in the primitive network.

$$\begin{aligned} i^a &= -i^{a'} \\ i^b &= -i^{b'} \\ i^c &= -i^{c'} \\ i^d &= i^{d'} - i^{a'} \\ i^f &= i^{f'} - i^{b'} \\ i^g &= i^{g'} - i^{c'} \\ i^h &= i^{g'} - i^{f'} \\ i^j &= i^{d'} - i^{g'} \\ i^k &= i^{f'} - i^{d'} \end{aligned} \qquad \mathbf{C} = \begin{array}{c|cccccc|} & \mathbf{a'} & \mathbf{b'} & \mathbf{c'} & \mathbf{d'} & \mathbf{f'} & \mathbf{g'} \\ \hline \mathbf{a} & -1 & & & & & \\ \mathbf{b} & & -1 & & & & \\ \mathbf{c} & & & -1 & & & \\ \mathbf{d} & -1 & & & 1 & & \\ \mathbf{f} & & -1 & & & 1 & \\ \mathbf{g} & & & -1 & & & 1 \\ \mathbf{h} & & & & & -1 & 1 \\ \mathbf{j} & & & & 1 & & -1 \\ \mathbf{k} & & & & -1 & 1 & \\ \hline \end{array} \qquad 6.16$$

The coefficients of the new currents form the transformation tensor.

(*d*) The new components of the impedance tensor are by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} = \mathbf{z}' =$

$\mathbf{z}' =$	a′	b′	c′	d′	f′	g′
a′	$Z_{aa}+Z_{dd}$ $+2X_{ad}$	X_{df}	X_{dg}	$-X_{hd}-Z_{dd}$	$-X_{df}$	$-X_{dg}$
b′	X_{df}	$2X_{bf}+X_{bb}$ $+Z_{ff}$	X_{fg}	$-X_{df}$	$-X_{bf}-Z_{ff}$	$-X_{fg}$
c′	X_{dg}	X_{fg}	$2X_{cg}+X_{cc}$ $+Z_{gg}$	$-X_{dg}$	$-X_{fg}$	$-X_{cg}-Z_{gg}$
d′	$-X_{ad}-Z_{dd}$	$-X_{df}$	$-X_{dg}$	$Z_{dd}+Z_{jj}-X_{jk}$ $-X_{jk}+Z_{kk}$	$X_{df}-X_{hj}+X_{hk}$ $+X_{jk}-Z_{kk}$	$X_{dg}+X_{hj}-X_{hk}$ $-Z_{jj}+Z_{jk}$
f′	$-X_{df}$	$-X_{bf}-Z_{ff}$	$-X_{jg}$	$X_{df}-X_{hj}+X_{jk}$ $+X_{hk}-Z_{kk}$	$Z_{ff}+Z_{hh}-X_{hk}$ $-X_{hk}+Z_{kk}$	$X_{fg}-Z_{hh}+X_{hk}$ $+X_{hj}-X_{jk}$
g′	$-X_{dg}$	$-X_{fg}$	$-X_{cg}$ $-Z_{gg}$	$X_{dg}+X_{hj}-Z_{jj}$ $-X_{hk}+X_{jk}$	$X_{fg}-Z_{hh}+X_{hj}$ $+X_{hh}-X_{jk}$	$Z_{gg}+Z_{hh}-X_{hj}$ $-X_{hj}+Z_{jj}$

6.17

The impressed voltage vector is by $\mathbf{C}_t \cdot \mathbf{e} =$

	a′	b′	c′	d′	f′	g′
$\mathbf{e}' =$	e_d	$-e_b$	$-e_g$	$-e_d-e_k$	e_h+e_k	e_g-e_h

6.18

The mesh currents are found by $\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}'$.

VI. REPLACING MESH CURRENTS BY BRANCH CURRENTS

Just as *actual* branch currents $\mathbf{i}$ may be replaced by *hypothetical* mesh currents $\mathbf{i}'$, the *inverse step* may also be made, that is, hypothetical mesh currents may be replaced by actual branch currents.

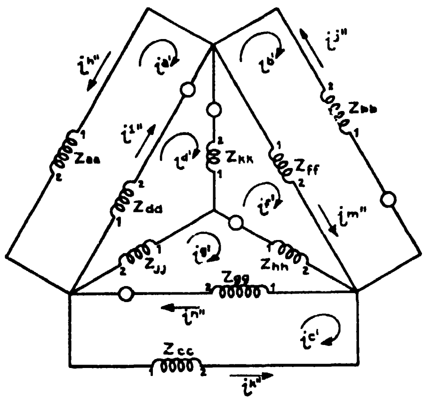


FIG. 6.11.—Changing Mesh Currents to Branch Currents

Using the mesh currents $\mathbf{i}'$ of the previous section as starting points, let them be transformed to the branch currents $\mathbf{i}''$ shown in Fig. 6.11.

Instead of setting up the relation $\mathbf{i}' = \mathbf{C}' \cdot \mathbf{i}''$, *it will be found easier to set up first the inverse relation* $\mathbf{i}'' = \mathbf{C}'^{-1} \cdot \mathbf{i}'$. Then either the equations are solved for the actual currents, or the inverse of $\mathbf{C}'^{-1}$ is found. (If the matrix of $\mathbf{C}$ involves many zeros it

is easier to find its inverse by solving a set of linear equations than by following the standard procedure.)

That is, the relation between the new (actual) currents i″ and the old (mesh) currents are:

$$
\begin{aligned}
i^{h''} &= -\, i^{a'} \\
i^{j''} &= -\, i^{b'} \\
i^{k''} &= -\, i^{c'} \\
i^{l''} &= i^{d'} - i^{a'} \\
i^{m''} &= i^{f'} - i^{b'} \\
i^{n''} &= i^{g'} - i^{c'}
\end{aligned}
\qquad
\mathbf{C}'^{-1} \quad
\begin{array}{c|cccccc|}
 & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \mathbf{d}' & \mathbf{f}' & \mathbf{g}' \\
\hline
\mathbf{h}'' & -1 & & & & & \\
\mathbf{j}'' & & -1 & & & & \\
\mathbf{k}'' & & & -1 & & & \\
\mathbf{l}'' & -1 & & & 1 & & \\
\mathbf{m}'' & & -1 & & & 1 & \\
\mathbf{n}'' & & & -1 & & & 1 \\
\hline
\end{array}
\qquad 6.19
$$

Solving the equations for the old currents

$$
\begin{aligned}
i^{a'} &= -\, i^{h''} & &= -\, i^{h''} \\
i^{b'} &= -\, i^{j''} & &= \quad -i^{j''} \\
i^{c'} &= -\, i^{k''} & &= \quad\quad -\, i^{k''} \\
i^{d'} &= \quad i^{l''} + i^{a'} = i^{l''} - i^{h''} & &= -\, i^{h''} \quad\quad + i^{l''} \\
i^{f'} &= \quad i^{m''} + i^{b'} = i^{m''} - i^{j''} & &= \quad -\, i^{j''} \quad\quad + i^{m''} \\
i^{g'} &= \quad i^{n''} + i^{c'} = i^{n''} - i^{k''} & &= \quad\quad -\, i^{k''} \quad\quad + i^{n''}
\end{aligned}
\qquad 6.20
$$

Hence the transformation tensor is

$$
\mathbf{C}' =
\begin{array}{c|cccccc|}
 & \mathbf{h}'' & \mathbf{j}'' & \mathbf{k}'' & \mathbf{l}'' & \mathbf{m}'' & \mathbf{n}'' \\
\hline
\mathbf{a}' & -1 & & & & & \\
\mathbf{b}' & & -1 & & & & \\
\mathbf{c}' & & & -1 & & & \\
\mathbf{d}' & -1 & & & 1 & & \\
\mathbf{f}' & & -1 & & & 1 & \\
\mathbf{g}' & & & -1 & & & 1 \\
\hline
\end{array}
\qquad 6.21
$$

The new impedance tensor is found by $\mathbf{z}'' = \mathbf{C}'_t \cdot \mathbf{z}' \cdot \mathbf{C}'$ and the new impressed voltage vector by $\mathbf{e}'' = \mathbf{C}'_t \cdot \mathbf{e}'$.

VII. EQUIVALENCE OF BRANCH AND MESH CURRENTS

(*a*) Consider again the network of Fig. 6.7 as reproduced in Fig. 6.12, in which the variables are the four actual branch currents i^f, i^g, i^h, and i^k. The actual currents in the other branches are expressed in terms of these four currents in some arbitrary order.

(*b*) If each of the four currents, i^f, i^g, i^h, and i^k is traced out in the

various branches, it is found that each of them forms a closed circuit. For instance, i^k can be traced through the coils Z_k, Z_g, Z_d, and Z_h so that it may be assumed that i^k flows through this mesh.

That is, *a branch current may be assumed as a mesh current that has an actual existence in that particular branch.* However, a mesh current may not necessarily exist as an independent branch current, as $i^{d'}$ in Fig. 6.11 shows.

Comparing Figs. 6.12 and 6.13 it is found that the closed meshes in which the voltages are added up by $\mathbf{C}_t \cdot \mathbf{e}$ are identical with the

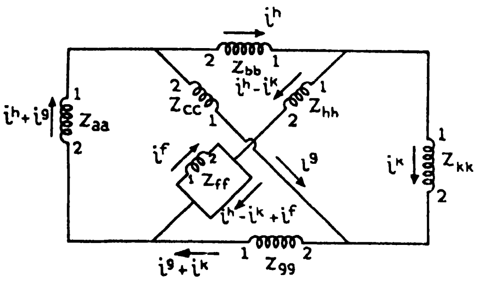


FIG. 6.12.—Currents Flowing in Branches

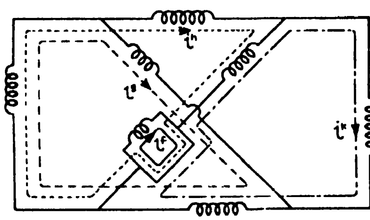


FIG. 6.13.—Currents Flowing in Closed Meshes

closed meshes in which the branch currents have been assumed to flow in setting up Fig. 6.12.

Hence *from a mathematical point of view the components of* $\mathbf{i}'$ *and* $\mathbf{e}'$ *(found after the coils are interconnected) may be looked upon also as representing currents and voltages existing in closed meshes, hence the fixed indices f, g, h, k may be considered as representing the closed meshes.* In particular, e^f represents the sum of the impressed voltages in mesh f and i^f represents the hypothetical current flowing in mesh f. The closed meshes themselves may be either actual (to be traceable on the network diagram along the coils) or fictitious (not traceable through actual conductors).

(*c*) Just as the hypothetical mesh current $\mathbf{i}'$ may be replaced by an actually existing branch current, similarly the hypothetical *mesh voltage* $\mathbf{e}'$ (representing the sum of the actually impressed voltages around a mesh) may be replaced by a single voltage actually existing in a branch, in particular in that branch in which the mesh current has an actual existence as shown in Fig. 6.14. Hence *a mesh current and voltage may also be physically represented as a branch current and voltage.* Mathematically both physical representations have the same form and the fixed indices *f, g, h, k* may be considered to represent either closed meshes or branches.

In most practical problems the branch currents have to be selected according to the need of that particular problem, while the closed meshes are assumed purely accidentally in expressing the currents flowing through the various coils in terms of the needed branch currents, as in Fig. 6.13. Hence *it is possible to select either the n branches or the n closed meshes arbitrarily*; then the selection of the others becomes restricted, although some freedom is still available in their choice because of the step from branches to coils.

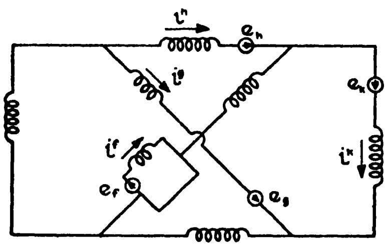


FIG. 6.14.—Branch Currents and Branch Voltages

VIII. CHANGING THE NUMBER OF TURNS

(*a*) Another example of an invariant transformation is the step of changing the number of turns of coils.

In network and rotating machine problems the reactances of the individual coils or windings are usually calculated by assuming *each coil to have the same number of turns.* In the actual interconnected network the available copper space is utilized by winding the coils with different numbers of turns, thereby changing the calculated self- and mutual inductances of the individual coils.

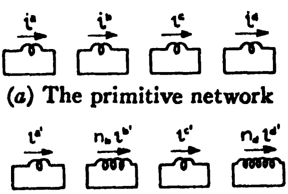
The change of the reactances due to the change in the number of turns can be taken care of in two different manners:

1. The primitive network itself is assumed to contain coils with *different* numbers of turns, and its components X_{aa}, X_{ab}, etc., are calculated with the coils having different numbers of turns.

2. The coils of the primitive network are assumed to contain *identical* numbers of turns (usually one), and the components of **z** are thus calculated. Then a transformation tensor is set up showing how the number of turns of the individual coils had changed.

(*b*) That is, let the impedance tensor of the primitive mesh network of Fig. 6.15 be

$$\mathbf{z} = \begin{array}{c|cccc|} & \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} \\ \hline \mathrm{a} & Z_{aa} & X_{ab} & X_{ac} & X_{ad} \\ \mathrm{b} & X_{ab} & Z_{bb} & X_{bc} & X_{bd} \\ \mathrm{c} & X_{ac} & X_{bc} & Z_{cc} & X_{cd} \\ \mathrm{d} & X_{ad} & X_{bd} & X_{cd} & Z_{dd} \\ \hline \end{array} \qquad 6.22$$



(*a*) The primitive network

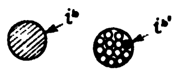
(*b*) The actual network

FIG. 6.15.—Changing the Number of Turns

where each coil has the same number of turns.

Let now the number of turns of the coils be changed. In particular if the number of turns of coils a and c is unity, then that of coil b is n_b, and that of coil d is n_d. *The changing of the number of turns in a particular cross-section of copper from one to n is equivalent to replacing the original current i^b flowing through the whole cross-section by 1/nth of its value, by $i^{b'} = i^b/n$ flowing through the smaller cross-section*, Fig. 6.16. That is, the *old* currents are the currents flowing through the larger cross-section and the *new* ones are those flowing through the smaller cross-section.

Just as the introduction of *short conductors to interconnect coils* does not contribute any additional resistance or inductance to the system, leaving thereby the power input $\mathbf{e} \cdot \mathbf{i}$ invariant, similarly *the introduction of thin insulation to separate strands of copper does not introduce any additional reluctance to the system* and hence it is a transformation that leaves the power input $\mathbf{e} \cdot \mathbf{i}$ invariant.



(a) 1 turn (b) n_b turn

FIG. 6.16.—The Introduction of Insulation

(c) The relation between the currents flowing through the *larger* cross-sections and those through the *smaller* cross-section can be set up by expressing the currents flowing in each coil in terms of the new currents (Fig. 6.16) and equating the old and the new currents (through the same cross-section) in each coil as

$$\begin{aligned} i^a &= i^{a'} = i^{a'} \\ i^b &= n_b i^{b'} = n_b i^{b'} \\ i^c &= i^{c'} = i^{c'} \\ i^d &= n_d i^{d'} = n_d i^{d'} \end{aligned} \qquad \mathbf{C} = \begin{array}{c|cccc|} & a' & b' & c' & d' \\ \hline a & 1 & & & \\ b & & n_b & & \\ c & & & 1 & \\ d & & & & n_d \\ \hline \end{array} \tag{6.23}$$

The coefficients of the new variables give the transformation matrix $\mathbf{C}$.

The transformation matrix contains only diagonal components, each representing the number of turns of a particular coil (or the number of turns of the coils in terms of those of some particular coil). *It should be expressly noted that the components of* $\mathbf{C}$ *may now be fractions, not only integers.*

(d) The impedance tensor of the network containing coils with various turns is $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} =$

$$\mathbf{z} = \begin{array}{c|cccc|} & a' & b' & c' & d' \\ \hline a' & Z_{aa} & n_b X_{ab} & X_{ac} & n_d X_{ad} \\ b' & n_b X_{ab} & n_b^2 Z_{bb} & n_b X_{bc} & n_b n_d X_{bd} \\ c' & X_{ac} & n_b X_{bc} & Z_{cc} & n_d X_{cd} \\ d' & n_d X_{ad} & n_b n_d X_{bd} & n_d X_{cd} & n_d^2 X_{dd} \\ \hline \end{array} \tag{6.24}$$

The impressed voltage vector is by $\mathbf{C}_t \cdot \mathbf{e} =$

$$\begin{array}{cc} & \begin{array}{cccc} \mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \mathbf{d}' \end{array} \\ \mathbf{e}' = & \begin{array}{|c|c|c|c|} \hline e_a & n_b e_b & e_c & n_d e_d \\ \hline \end{array} \end{array} \qquad 6.25$$

This present transformation of changing the number of turns rarely occurs alone; usually the coils are also interconnected at the same time.

IX. NEGLECTING THE "MAGNETIZING CURRENTS" AS CONSTRAINTS

(*a*) In a multiwinding transformer several coils are wound around the same iron core as shown in Fig. 6.17. The larger number of flux lines, the so-called "*core fluxes*," go around the core linking all the coils, while the remaining flux lines, the so-called "*leakage fluxes*," also travel in the air gap between the coils.

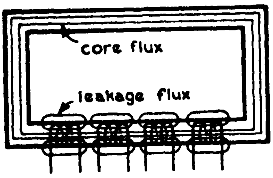


FIG. 6.17.—Fluxes in Multiwinding Transformers

The magnetomotive force M, producing the core flux, is due to all the currents that are linked by the core flux; that is, in a multi-winding transformer with four coils the m.m.f. producing the core flux is

$$M = n_1 i^1 + n_2 i^2 + n_3 i^3 + n_4 i^4$$

The leakage fluxes are produced by m.m.f.'s due to single and several currents. The value of these m.m.f.'s is of no further interest now.

It is customary to assume that the "*core flux*" linking all the coils is caused by a so-called hypothetical "*magnetizing current*" i^m flowing in one of coils, while the remaining "*leakage fluxes*" are produced by the remaining currents, the "*load currents*." That is, the artificial division of flux lines into "core flux" (path in iron) and "leakage flux" (path in air) leads also to the artificial division of the currents into "magnetizing current" and "load current".

(*b*) Since in a transformer the reluctance of the core-flux path is small, *the m.m.f. producing the core flux may be assumed to be zero*, or, in engineering parlance, *the "magnetizing current" may be neglected.* This assumption is equivalent to the equation:

$$n_1 i^1 + n_2 i^2 + n_3 i^3 + n_4 i^4 = 0 \qquad 6.26$$

that is, *the sum of m.m.f.'s around the magnetic circuit is zero.*

This last equation sets up a relation between the currents of a multiwinding transformer, hence it is an "*equation of constraint*"

(Section XV, Chapter IV). In the previous case, constraints on the currents were set up by controlling the *electric* circuit (by opening it); here, constraints on the currents are set up by controlling the *magnetic* circuit.

When a network contains *several* closed magnetic circuits in which the m.m.f.'s producing the core fluxes may be neglected, *for each magnetic circuit a separate " equation of constraint " may be set up*, making the sum of the m.m.f.'s acting around each magnetic circuit equal to zero.

X. CONSTRAINTS AS TRANSFORMATIONS

(*a*) In the previous occasion (Section XV, Chapter IV) equations of constraint were introduced by opening up some of the circuits. They were taken care of in *two* different ways:

1. The equations were substituted into the already established relation $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ in order to reduce the number of new variables and thereby *the old* **C** *was changed to a new* **C**$'$ having less columns than rows. The number of variables eliminated was the same as the number of the equations of constraint.

2. The already established *old* **C** *was discarded,* and another, singular **C** was established all over again by assuming smaller number of new variables. This is the general method used in all networks.

(*b*) In the present case a third method of taking care of the equation of constraint will be used, since the second method is quite cumbersome with the present type of constraint. *A separate transformation tensor* **C** *will be set up so that the relation* $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ *will be equivalent to the equations of constraint themselves.* That is, *the introduction of constraint will be considered as a separate transformation.*

In any general problem before the introduction of constraint there are n variables. The effect of the introduction of k equations of constraint is to introduce by the use of a transformation tensor **C** $n - k$ new variables in place of the n old variables.

(*c*) The steps in setting up a transformation tensor when the set of k equations of constraint is given are the following:

1. Express *any* k of the variables in the equations of constraint as functions of the remaining $n - k$ variables.

2. Leave the remaining $n - k$ variables unchanged.

The above set of n equations represents the relation between the " old " and the " new " variables (*the " new " variables are simply the " old " variables reduced in number*), hence the transformation tensor is found by taking the coefficients of the new currents as usual.

(*d*) As an example let the last equation of constraint be given in the form

$$n_1 i^1 + n_2 i^2 + n_3 i^3 + n_4 i^4 = 0 \tag{6.27}$$

stating that in the four-winding transformer the resultant m.m.f. is zero. Expressing, say, i^1 in terms of the other three (in order to eliminate i^1)

$$i^1 = -\frac{n_2}{n_1} i^2 - \frac{n_3}{n_1} i^3 - \frac{n_4}{n_1} i^4 \tag{6.28}$$

and leaving the remaining three unchanged

$$\begin{aligned} i^2 &= i^2 \\ i^3 &= i^3 \\ i^4 &= i^4 \end{aligned} \tag{6.29}$$

the resulting four equations change the original *four* currents i^1, i^2, i^3, and i^4 into *three* currents i^1, i^2, and i^3. To avoid any possible confusion the three currents on the right-hand side may be primed, so that the set of equations $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ (representing the equations of constraint) are

$$\begin{aligned} i^1 &= -\frac{n_2}{n_1} i^{2'} - \frac{n_3}{n_1} i^{3'} - \frac{n_4}{n_1} i^{4'} \\ i^2 &= i^{2'} \\ i^3 &= i^{3'} \\ i^4 &= i^{4'} \end{aligned} \qquad \mathbf{C} = \begin{array}{c|ccc|} & 2' & 3' & 4' \\ \hline 1 & -\frac{n_2}{n_1} & -\frac{n_3}{n_1} & -\frac{n_4}{n_1} \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ \hline \end{array} \tag{6.30}$$

Hence the role of the equation of constraint (6.27) has been replaced by the transformation tensor (6.30).

XI. MULTIWINDING TRANSFORMERS

Let the impedance tensor of the four-winding transformer be

$$\mathbf{z} = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & Z_{11} & X_{12} & X_{13} & X_{14} \\ 2 & X_{12} & Z_{22} & X_{23} & X_{24} \\ 3 & X_{13} & X_{23} & Z_{33} & X_{34} \\ 4 & X_{14} & X_{34} & X_{34} & Z_{44} \\ \hline \end{array} \tag{6.31}$$

each impedance being calculated with its actual number of turns.

Its magnetizing current is neglected by transforming **z** with the aid of **C** (equation 6.30), giving $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} =$

$$\mathbf{z}' = \begin{array}{c|c|c|c|} & 2' & 3' & 4' \\ \hline 2' & Z_{22} - 2\frac{n_2}{n_1}X_{12} + \left(\frac{n_2}{n_1}\right)^2 Z_{11} & X_{23} - \frac{n_2}{n_1}X_{13} - \frac{n_3}{n_1}X_{12} + \frac{n_2}{n_1}\frac{n_3}{n_1}Z_{11} & X_{24} - \frac{n_2}{n_1}X_{14} - \frac{n_4}{n_1}X_{12} + \frac{n_2}{n_1}\frac{n_4}{n_1}Z_{11} \\ \hline 3' & X_{23} - \frac{n_2}{n_1}X_{13} - \frac{n_3}{n_1}X_{12} + \frac{n_2}{n_1}\frac{n_3}{n_1}Z_{11} & Z_{33} - 2\frac{n_3}{n_1}X_{13} + \left(\frac{n_3}{n_1}\right)^2 Z_{11} & X_{34} - \frac{n_3}{n_1}X_{14} - \frac{n_4}{n_1}X_{13} + \frac{n_3}{n_1}\frac{n_4}{n_1}Z_{11} \\ \hline 4' & X_{24} - \frac{n_2}{n_1}X_{14} - \frac{n_4}{n_1}X_{12} + \frac{n_2}{n_1}\frac{n_4}{n_1}Z_{11} & X_{34} - \frac{n_3}{n_1}X_{14} - \frac{n_4}{n_1}X_{13} + \frac{n_3}{n_1}\frac{n_4}{n_1}Z_{11} & Z_{44} - 2\frac{n_4}{n_1}X_{14} + \left(\frac{n_4}{n_1}\right)^2 Z_{11} \\ \hline \end{array} \qquad 6.32$$

Hence the effect of neglecting the magnetizing current of a multiwinding transformer is to reduce the number of rows and columns of its impedance tensor by one by a singular transformation tensor **C**, that is *the four windings are replaced by three other (hypothetical) windings.*

The impressed voltage vector $\mathbf{e}'$, after neglecting the magnetizing current, is $\mathbf{C}_t \cdot \mathbf{e} =$

$$\mathbf{e}' = \begin{array}{|c|c|c|} \multicolumn{1}{c}{2'} & \multicolumn{1}{c}{3'} & \multicolumn{1}{c}{4'} \\ \hline e_2 - \frac{n_2}{n_1}e_1 & e_3 - \frac{n_3}{n_1}e_1 & e_4 - \frac{n_4}{n_1}e_1 \\ \hline \end{array} \qquad 6.33$$

That is, *the difference of potential between two windings* (reduced to one-to-one ratio) *is taken.* One of the two windings is the one whose current has been eliminated.

The load currents $\mathbf{i}'$ in windings 2, 3, and 4 are found by $\mathbf{z}'^{-1} \cdot \mathbf{e}'$. The currents in the original coils are found in terms of $\mathbf{i}'$ by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ and the voltages induced in each coil by $\mathbf{e} = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$.

A detailed analysis of multiwinding transformer *circuits* is given in Chapter X.

XII. MAGNETIZING AND LOAD CURRENTS

(*a*) Instead of assuming that the hypothetical magnetizing current i^m producing the core flux is zero, it may be assumed that it actually exists in one of the coils, say the one connected to the line.

In that case this coil is considered as a return circuit for the other

" load " currents producing the flux lines in the air. That is in a four-winding transformer the " equation of m.m.f." is

$$n_1 i^1 + n_2 i^2 + n_3 i^3 + n_4 i^4 = n_1 i^m = M \qquad 6.34$$

where i^m is assumed to flow in coil 1. Hence i^1 is replaced by the following relation

$$i^1 = i^m - \frac{n_2}{n_1} i^2 - \frac{n_3}{n_1} i^3 - \frac{n_4}{n_1} i^4 \qquad 6.35$$

while the other " load " currents remain unchanged

$$\begin{aligned} i^2 &= i^2 \\ i^3 &= i^3 \\ i^4 &= i^4 \end{aligned} \qquad 5.36$$

These four equations may be considered to transform the original four variables i^1, i^2, i^3, and i^4 to four other variables i^m, $i^{2'}$, $i^{3'}$, and $i^{4'}$ Hence the set of equations of transformations $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ are

$$\begin{aligned} i^1 &= i^m - \frac{n_2}{n_1} i^{2'} - \frac{n_3}{n_1} i^{3'} - \frac{n_4}{n_1} i^{4'} \\ i^2 &= i^{2'} \\ i^3 &= i^{3'} \\ i^4 &= i^{4'} \end{aligned} \qquad \mathbf{C} = \begin{array}{c|cccc} & m & 2' & 3' & 4' \\ \hline 1 & 1 & -\frac{n_2}{n_1} & -\frac{n_3}{n_1} & -\frac{n_4}{n_1} \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array} \qquad 6.37$$

The coefficients of the new currents form the *non-singular* (square) transformation tensor **C**.

(*b*) This transformation changes not only the currents flowing in the coils but also the physical set-up of the network by introducing fictitious connections between the coil of i^m and the other coils. The fictitious meshes may be established by finding the new voltage vector $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e} =$

$$\mathbf{e}' = \begin{array}{cccc} m & 2' & 3' & 4' \\ \hline e_1 & e_2 - \frac{n_2}{n_1} e_1 & e_3 - \frac{n_3}{n_1} e_1 & e_4 - \frac{n_4}{n_1} e_1 \end{array} \qquad 6.38$$

showing that each load circuit is connected in series opposing with the magnetizing current.

(*c*) It should be noted that *the singular* **C** *used to neglect the magnetizing current is a part of a non-singular* **C**. *It is found from the latter by leaving out the columns corresponding to the magnetizing currents.*

In general, whenever a singular **C** is set up to represent a set of equations of constraint, *it is always possible to construct a non-singular (square)* **C** *by introducing additional new variables* (in the present case i^m) *of which the singular* **C** *forms a part.*

XIII. TWO-WINDING TRANSFORMER

(*a*) Let the performance of a two-winding transformer be expressed in terms of its " magnetizing " and " load " current in place of its " primary " and " secondary " current. The original components of the impedance tensor **z** and the new transformation tensor **C** are respectively

$$\mathbf{z} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & Z_{11} & X_{12} \\ 2 & X_{12} & Z_{22} \end{array} \qquad 6.39 \qquad \mathbf{C} = \begin{array}{c|cc} & m & 2' \\ \hline 1 & 1 & -\dfrac{n_2}{n_1} \\ 2 & 0 & 1 \end{array} \qquad 6.40$$

The product $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ gives the new components of the impedance tensor

$$\mathbf{z}' = \begin{array}{c|cc} & m & 2' \\ \hline m & Z_{11} & X_{12} - \dfrac{n_2}{n_1} Z_{11} \\ 2' & X_{12} - \dfrac{n_2}{n_1} Z_{11} & Z_{22} - 2\dfrac{n_2}{n_1} X_{12} + \left(\dfrac{n_2}{n_1}\right)^2 Z_{11} \end{array} \qquad 6.41$$

The admittance tensor is

$$\mathbf{y}' = \begin{array}{c|cc} & m & 2' \\ \hline m & \left[Z_{22} - 2\dfrac{n_2}{n_1} X_{12} + \left(\dfrac{n_2}{n_1}\right)^2 Z_{11}\right]/D & \left(\dfrac{n_2}{n_1} Z_{11} - X_{12}\right)/D \\ 2' & \left(\dfrac{n_2}{n_1}\right) Z_{11} - X_{12}\Big)/D & Z_{11}/D \end{array} \qquad 6.42$$

where $D = Z_{11}Z_{22} - (X_{12})^2$. The impressed-voltage vector is

$$\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e} = \begin{array}{cc} m & 2' \\ \hline e_1 & e_2 - \dfrac{n_2}{n_1} e_1 \end{array} \qquad 6.43$$

(*b*) If no impressed voltage exists on the secondary, then $e_2 = 0$ and

$$\mathbf{i}' = \begin{array}{c|c|} \hline \mathrm{m} & \left[\left(Z_{22} - \frac{n_2}{n_1} X_{12}\right)/D\right] e_1 \\ \hline 2' & [-X_{12}/D] e_1 \\ \hline \end{array} \qquad 6.44$$

giving the " magnetizing " and " load " currents.

(*c*) The currents in the two windings in terms of $\mathbf{i}'$ are found by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$.

$$\mathbf{i} = \begin{array}{|c|c|} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} \\ \hline i^m - \frac{n_2}{n_1} i^{2'} & i^{2'} \\ \hline \end{array} \qquad 6.45$$

That is, the primary current i^1 is the sum of the magnetizing current i^m and the load current $-i^{2'}(n_2/n_1)$.

The voltages induced in the two coils are found by $\mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$ as

$$\mathbf{e}_c = \begin{array}{c|c|} \hline 1 & Z_{11} i^m + \left(X_{12} - \frac{n_2}{n_1} Z_{11}\right) i^{2'} \\ \hline 2 & X_{12} i^m + \left(Z_{22} - \frac{n_2}{n_1} X_{12}\right) i^{2'} \\ \hline \end{array} \qquad 6.46$$

XIV. THE OPENING OF CIRCUITS AS TRANSFORMATIONS

(*a*) As another example to show that a set of equations of constraint may be replaced by a singular transformation matrix $\mathbf{C}$ let some of the meshes of a network (whose equation $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ has already been established) be opened up, without, however, removing any of the coils.

For instance, let the network of Fig. 6.18 be considered having six meshes with the six variables i^a, i^b, i^c, i^d, i^f, and i^g. Its impedance tensor is given in equation 6.17.

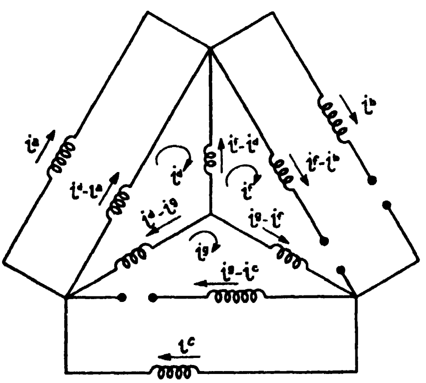


FIG. 6.18.—The Opening of Branches

(*b*) Now let the branches be opened, in which the following currents flow, i^b, $i^g - i^c$, and $i^f - i^b$, as shown in Fig. 6.18. These openings are equivalent to the following *equations of constraint*

$$\begin{aligned} i^b &= 0 \\ i^g - i^c &= 0 \\ i^f - i^b &= 0 \end{aligned} \qquad 6.47$$

The effect of these constraints is to reduce the number of variables by three, namely from six to three. Selecting arbitrarily as the remaining variables i^a, i^c, and i^f as shown in Fig. 6.19, the following relation $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ between the old and the new variables is equivalent to the equations of constraint

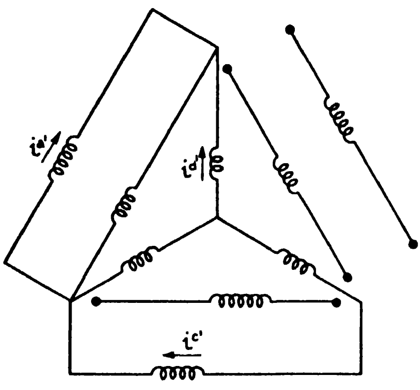


FIG. 6.19.—The New Variables

$$
\begin{aligned}
i^a &= i^{a'} \\
i^b &= 0 \\
i^c &= i^{c'} \\
i^d &= i^{d'} \\
i^f &= 0 \\
i^g &= i^{c'}
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c} \\ \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \\ \mathrm{f} \\ \mathrm{g} \end{array}
\begin{array}{c}
\begin{array}{ccc} \mathrm{a'} & \mathrm{c'} & \mathrm{d'} \end{array} \\
\begin{array}{|c|c|c|}
\hline
1 & & \\ \hline
 & & \\ \hline
 & 1 & \\ \hline
 & & 1 \\ \hline
 & & \\ \hline
 & 1 & \\ \hline
\end{array}
\end{array}
\qquad 6.48
$$

(*c*) Hence the impedance tensor of the reduced network of Fig. 6.19 is found from equation 6.17 by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} =$

$$
\mathbf{z}' = \begin{array}{c|c|c|c|}
 & \mathrm{a'} & \mathrm{c'} & \mathrm{d'} \\ \hline
\mathrm{a'} & Z_{aa}+Z_{dd}+2X_{ad} & 0 & -X_{ad}-Z_{dd} \\ \hline
\mathrm{c'} & 0 & Z_{cc}+Z_{hh}+Z_{jj}-2X_{hj} & X_{hj}-X_{hk}+X_{jk}-Z_{jj} \\ \hline
\mathrm{d'} & -X_{ad}-Z_{dd} & X_{hj}-X_{hk}+X_{jk}-Z_{jj} & Z_{dd}+Z_{jj}+Z_{kk}-2X_{jk} \\ \hline
\end{array}
\qquad 6.49
$$

The same result could have been found, of course, for Fig. 6.19 by starting with the primitive network instead of with Fig. 6.18 and by setting up a new C containing the same three new variables.

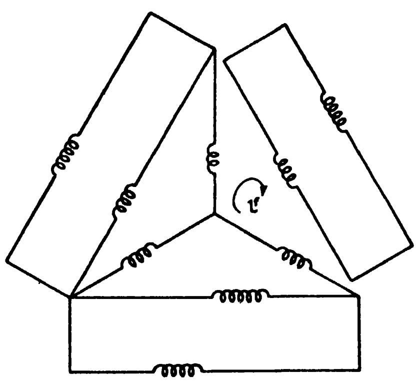


FIG. 6.20.—The Opening of a Mesh

(*d*) Instead of putting an actually existing current flowing in a branch equal to zero, it is possible to equate a *hypothetical* mesh current to zero such as i^f in Fig. 6.18. This is equivalent of removing the mesh f by opening it at some junctions, as shown in Fig. 6.20.

(*e*) When the equations of constraint have the simple form, $i^b = 0$, $i^d = 0$, $i^f = 0$ (Fig. 6.21), then the operations $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ consist

of simply removing the rows and columns of **b**, **d**, **f** from the tensor 6.17. The transformation tensor **C** is

$$\mathbf{C} = \begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \\ \mathbf{f} \\ \mathbf{g} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{a}' & \mathbf{c}' & \mathbf{g}' \end{array} \\ \left[\begin{array}{c|c|c} 1 & & \\ \hline & & \\ \hline & 1 & \\ \hline & & \\ \hline & & \\ \hline & & 1 \end{array}\right] \end{array} \qquad 6.50$$

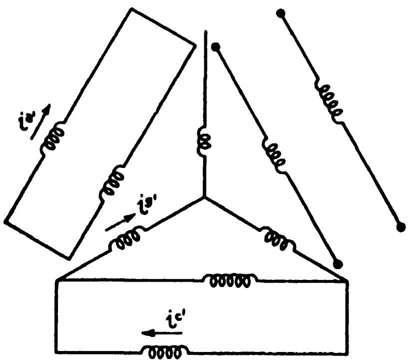


FIG. 6.21.—The Opening of Branches and Meshes

The impedance tensor **z**′ of the network of Fig. 6.21 is found by inspection from the tensor 6.17, representing $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ as

$$\mathbf{z}' = \begin{array}{c} \\ \mathbf{a}' \\ \mathbf{c}' \\ \mathbf{g}' \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{a}' & \mathbf{c}' & \mathbf{g}' \end{array} \\ \left[\begin{array}{c|c|c} Z_{aa} + Z_{dd} + 2X_{ad} & X_{dg} & -X_{dg} \\ \hline X_{dg} & Z_{cc} + Z_{gg} + 2X_{cg} & -X_{cg} - Z_{gg} \\ \hline -X_{dg} & -X_{cg} - Z_{gg} & Z_{gg} + Z_{hh} + Z_{jj} - 2X_{hj} \end{array}\right] \end{array} \qquad 6.51$$

(*f*) *The voltages in the individual coils cannot be found by* $\mathbf{e} = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$, *since* **z** *is not the primitive network.* This formula gives the voltages induced around a mesh (whether a mesh is open or closed), since the original equations that are transformed by **C** represent *mesh* equations of the network of Fig. 6.18. If the voltages induced in the *individual* open-circuited coils are desired, then the starting **z** should represent the impedance tensor of the primitive network.

XV. PERMUTATIONS AS TRANSFORMATIONS

(*a*) The transformations considered up to this point represent either a rearrangement of the physical system or the selection of different variables. In many cases the transformation consists of simply changing the *order* in which the variables appear. *Such a change in the order of the variables is called a "permutation."* For instance, in several three-phase systems sometimes it is desired to place side by side those axes of each system that belong to the first, second, and third phase respectively. Or when the coils of a d-c. winding are connected to a commutator *the effect of the commutator is to change the order in which the voltages generated in the individual coils are added up into the*

terminal voltage at any one instant. Or the order in which a *switch* turns on a series of lamps represents a permutation, and so on. Such examples of permutation that occur in engineering work can be continued indefinitely.

(*b*) As a simple example let **e**, **i**, and **z** of some system be calculated as

e = (6.52)

g	i	a	j	k	b	h	c	d	f
e_g	e_i	e_a	e_j	e_k	e_b	e_h	e_c	e_d	e_f

i = (6.53)

g	i	a	j	k	b	h	c	d	f
i^g	i^i	i^a	i^j	i^k	i^b	i^h	i^c	i^d	i^f

z = (6.54)

	g	i	a	j	k	b	h	c	d	f
g	Z_1	Z_2			Z_3	Z_4			Z_5	Z_6
i	Z_7	Z_8	Z_9	Z_{10}				Z_{11}		Z_{12}
a			Z_{13}	Z_{14}			Z_{15}		Z_{16}	
j		Z_{17}	Z_{18}	Z_{19}		Z_{20}		Z_{21}		Z_{22}
k	Z_{23}	Z_{24}			Z_{25}		Z_{26}			Z_{27}
b	Z_{28}			Z_{29}		Z_{30}	Z_{31}		Z_{32}	
h			Z_{33}		Z_{34}	Z_{35}	Z_{36}	Z_{37}		Z_{38}
c		Z_{39}		Z_{40}			Z_{41}	Z_{42}		Z_{43}
d	Z_{44}		Z_{45}			Z_{46}			Z_{47}	Z_{48}
f	Z_{49}	Z_{50}		Z_{51}	Z_{52}		Z_{53}	Z_{54}	Z_{55}	Z_{56}

(*c*) Now let the *order* of the axes change from **g**, **i**, **a**, **j**, **k**, **b**, **h**, **c**, **d**, **f** to **g**, **j**, **h**, **f**, **a**, **b**, **d**, **i**, **k**, **c**. That is, *let every fourth axis be the neighboring axis.* Such a change in order is equivalent to the transformation

$$\begin{array}{llllllllllll}
i^g = & i^{g'} & & & & & & & & & \\
i^i = & & & & & & & & i^{i'} & & \\
i^a = & & & & & i^{a'} & & & & & \\
i^j = & & i^{j'} & & & & & & & & \\
i^k = & & & & & & & & & i^{k'} & \\
i^b = & & & & & & i^{b'} & & & & \\
i^h = & & & i^{h'} & & & & & & & \\
i^c = & & & & & & & & & & i^{c'} \\
i^d = & & & & & & & i^{d'} & & & \\
i^f = & & & & i^{f'} & & & & & &
\end{array} \qquad 6.55$$

C = (6.56)

	g′	j′	h′	f′	a′	b′	d′	i′	k′	c′
g	1									
i								1		
a					1					
j		1								
k									1	
b						1				
h			1							
c										1
d							1			
f				1						

where

i′ = (6.57)

g′	j′	h′	f′	a′	b′	d′	i′	k′	c′
i^g	i^j	i^h	i^f	i^a	i^b	i^d	i^i	i^k	i^c

That is, the currents themselves remain the same; only the order in which their axes appear on top of the transformation tensor changes as shown. It should be noted that all components of the transformation tensor C of a permutation are zero except *one* in each row and in each column.

The new components of the voltage vector are by $\mathbf{C}_t \cdot \mathbf{e} = \mathbf{e}' =$

e′ = (6.58)

g′	j′	h′	f′	a′	b′	d′	i′	k′	c′
e_g	e_j	e_h	e_f	e_a	e_b	e_d	e_i	e_k	e_c

The new components of the impedance tensor are found by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ as

z′ = (6.59)

	g′	j′	h′	f′	a′	b′	d′	i′	k′	c′
g′	Z_1			Z_6		Z_4	Z_5	Z_2	Z_3	
j′		Z_{19}		Z_{22}	Z_{18}	Z_{20}		Z_{17}		Z_{21}
h′			Z_{36}	Z_{38}	Z_{33}	Z_{35}			Z_{34}	Z_{37}
f′	Z_{49}	Z_{51}	Z_{53}	Z_{56}			Z_{55}	Z_{50}	Z_{52}	Z_{54}
a′		Z_{14}	Z_{15}		Z_{13}		Z_{16}			
b′	Z_{28}	Z_{29}	Z_{31}			Z_{30}	Z_{32}			
d′	Z_{44}			Z_{48}	Z_{45}	Z_{46}	Z_{47}			
i′	Z_7	Z_{10}		Z_{12}	Z_9			Z_8		Z_{11}
k′	Z_{23}		Z_{26}	Z_{27}				Z_{24}	Z_{25}	
c′		Z_{40}	Z_{41}	Z_{43}				Z_{39}		Z_{42}

The permutation **C** *on* **z** *may be performed more quickly by first interchanging the columns of* **z**, (giving $\mathbf{z} \cdot \mathbf{C}$), then *interchanging the rows of* $\mathbf{z} \cdot \mathbf{C}$, giving $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$. The steps $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ are shown only to serve as introduction to more complicated permutations.

XVI. CLOSED WINDINGS AS PERMUTATIONS

(*a*) More complicated examples of permutations are the closed d-c. armature windings, in which the conductors are cut by the field-pole flux in a different order from that in which the currents flow through them.

As a simple example consider sixteen conductors placed in sixteen slots as shown in Fig. 6.22.

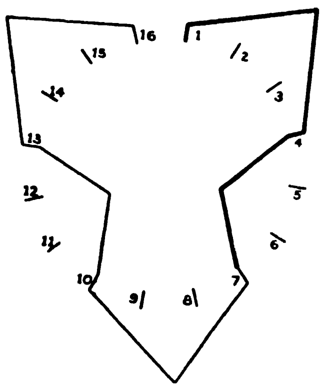


FIG. 6.22.—Magnetic Order in a Closed Winding

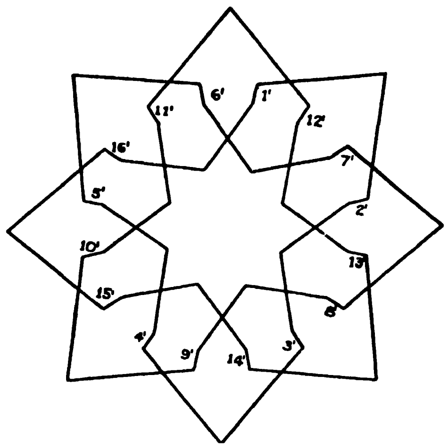


FIG. 6.23.—Electrical Order in a Closed Winding

The numbering of the conductors shows the order in which they cut an outside flux as they rotate.

(*b*) Now let the sixteen conductors be connected into a continuous closed winding by interconnecting, say, *every fourth conductor* in opposing series in the order, 1, -4, 7, -10, 13, -16, etc., until all conductors are covered, as shown in Fig. 6.23.

Because of this interconnection a current starting in conductor 1 will flow next through conductor -4 (in opposite direction through 4) then through 7, and so on. That is, *the effect of this type of continuous interconnection is to change the order of the conductors as far as the currents are concerned.* The new numbering of the conductors is shown in Fig. 6.23.

That is, the numbering of Fig. 6.22 shows the order in which

an outside *flux* cuts the conductors as they rotate and the numbering of Fig. 6.23 shows the order in which a *current* flows through the conductors.

(*c*) This *change in the order* of the conductors as viewed from a *flux* or from a *current* may be expressed as a transformation $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$

$$\begin{aligned} i^1 &= i^{1'} \\ i^2 &= -i^{12'} \\ i^3 &= i^{7'} \\ i^4 &= -i^{2'} \\ i^5 &= i^{13'} \\ i^6 &= -i^{8'} \\ i^7 &= i^{3'} \\ i^8 &= -i^{14'} \\ i^9 &= i^{9'} \\ i^{10} &= -i^{4'} \\ i^{11} &= i^{15'} \\ i^{12} &= -i^{10'} \\ i^{13} &= i^{5'} \\ i^{14} &= -i^{16'} \\ i^{15} &= i^{11'} \\ i^{16} &= -i^{6'} \end{aligned} \qquad \mathbf{C} = \begin{array}{c|cccccccccccccccc|} & 1' & 2' & 3' & 4' & 5' & 6' & 7' & 8' & 9' & 10' & 11' & 12' & 13' & 14' & 15' & 16' \\ \hline 1 & 1 & & & & & & & & & & & & & & & \\ 2 & & & & & & & & & & & & -1 & & & & \\ 3 & & & & & & & 1 & & & & & & & & & \\ 4 & & -1 & & & & & & & & & & & & & & \\ 5 & & & & & & & & & & & & & 1 & & & \\ 6 & & & & & & & & -1 & & & & & & & & \\ 7 & & & 1 & & & & & & & & & & & & & \\ 8 & & & & & & & & & & & & & & -1 & & \\ 9 & & & & & & & & & 1 & & & & & & & \\ 10 & & & & -1 & & & & & & & & & & & & \\ 11 & & & & & & & & & & & & & & & 1 & \\ 12 & & & & & & & & & & -1 & & & & & & \\ 13 & & & & & 1 & & & & & & & & & & & \\ 14 & & & & & & & & & & & & & & & & -1 \\ 15 & & & & & & & & & & & 1 & & & & & \\ 16 & & & & & & -1 & & & & & & & & & & \\ \hline \end{array} \qquad (6.60)$$

This transformation tensor changes the *electrical* order of the conductors (primed numbers) to their *magnetic* order. It is easily established as follows: starting in the upper left-hand corner, *the numbers* 1 *and* -1 *are placed in a corresponding row in the same order as the conductors are interconnected* (1, -4, 7, -10, 13, -16, *etc.*), *each time shifting the column by one to the right.*

It should be noted that in each row and in each column only one non-zero component exists. Also just as the winding goes *three times* around the circumference of the armature before it closes, similarly the numbers $\pm$ 1 cover the matrix three times in slant lines before they cover all columns.

XVII. CONTINUOUS D-C. WINDINGS

(*a*) As a more complex example of a d-c. winding, let the sixteen slots contain thirty-two conductors, two in a slot, one above the other, as shown in Fig. 6.24, where the end view of the conductors is shown.

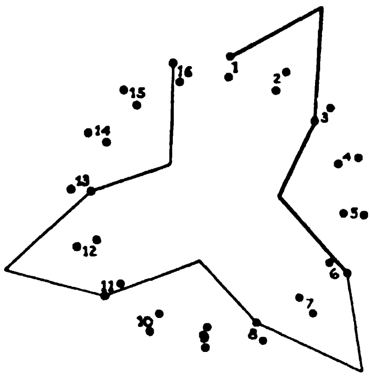


FIG. 6.24.—Magnetic Order

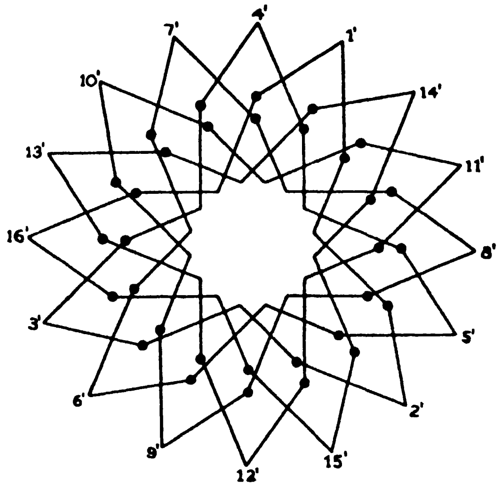


FIG. 6.25.—Electrical Order

From the point of view of an outside *flux*, there are still sixteen conductors, since a top and a bottom conductor of a slot are cut by the same flux at the same instant.

(*b*) Let the thirty-two conductors be interconnected as follows: top of 1, bottom of -3; top of 6, bottom of -8, and so on as shown in Fig. 6.25. Considering two neighboring interconnected conductors as forming a *coil* (say top of 1 and bottom of -3), the conductors are connected into coils and the coils into a winding in the following manner:

1. The two conductors of a coil cover two slots, that is, the " coil pitch " is $1 - 3$.
2. Every sixth coil is connected in series, that is, the " commutator pitch " is $1 - 6$.

Viewed from the *flux* a top and a bottom conductor form the smallest magnetic unit (altogether sixteen units), and viewed from the *current* the two conductors of a coil form the smallest electric unit (altogether sixteen units).

(*c*) Hence the transformation tensor, changing the magnetic order to the electrical order, contains sixteen rows and columns, as:

$$
\begin{aligned}
i^{1} &= i^{1'} - i^{7'} \\
i^{2} &= i^{14'} - i^{4'} \\
i^{3} &= i^{11'} - i^{1'} \\
i^{4} &= i^{8'} - i^{14'} \\
i^{5} &= i^{5'} - i^{11'} \\
i^{6} &= i^{2'} - i^{8'} \\
i^{7} &= i^{15'} - i^{5'} \\
i^{8} &= i^{12'} - i^{2'} \\
i^{9} &= i^{9'} - i^{15'} \\
i^{10} &= i^{6'} - i^{12'} \\
i^{11} &= i^{3'} - i^{9'} \\
i^{12} &= i^{16'} - i^{6'} \\
i^{13} &= i^{13'} - i^{3'} \\
i^{14} &= i^{10'} - i^{16'} \\
i^{15} &= i^{7'} - i^{13'} \\
i^{16} &= i^{4'} - i^{10'}
\end{aligned}
$$

C =

	1′	2′	3′	4′	5′	6′	7′	8′	9′	10′	11′	12′	13′	14′	15′	16′
1	1						−1									
2				−1										1		
3	−1										1					
4								1						−1		
5					1						−1					
6		1						−1								
7					−1										1	
8		−1										1				
9									1						−1	
10						1						−1				
11			1						−1							
12						−1										1
13			−1										1			
14										1						−1
15							1						−1			
16				1						−1						

6.61

It is easily established by starting in the upper left-hand corner; *the numbers* 1 *and* −1 *are placed into a corresponding row in the same order as the conductors are interconnected* (1, −3, 6, −8, 11, −13, etc.), *shifting, however, the columns to the right only for a* +1. In each column a +1 and a −1 exist representing the two conductors cut by the same flux (the magnetic units). That is, now in each row and column *two* non-zero components exist.

The use of this transformation tensor **C** in d-c. winding calculations is shown in Chapter XI, Section XV. In d-c. windings **C** occurs of course with still more complex form.

(*d*) The left-hand side of **C** represents the actual spatial order of the coils as they lie in the slots. **C** may be looked upon as transforming discrete points of the actual space into a hypothetical space in which these discrete points are arranged in a different order, corresponding to the order in which electric charges flow through them.

It is possible to leave empty rows in **C** to represent the *teeth* on the armature along which no conductors lie, as shown in Fig. 6.26 for a lap

winding with 70 coils. Now the left-hand side represents still more accurately the spatial order and position of the conductors in the slots with respect to the field poles as shown on the left-hand side of Fig. 6.26.

(*e*) It is of interest to point out *in passing* that, if brushes are put on top of C covering several commutator bars, they short-circuit the

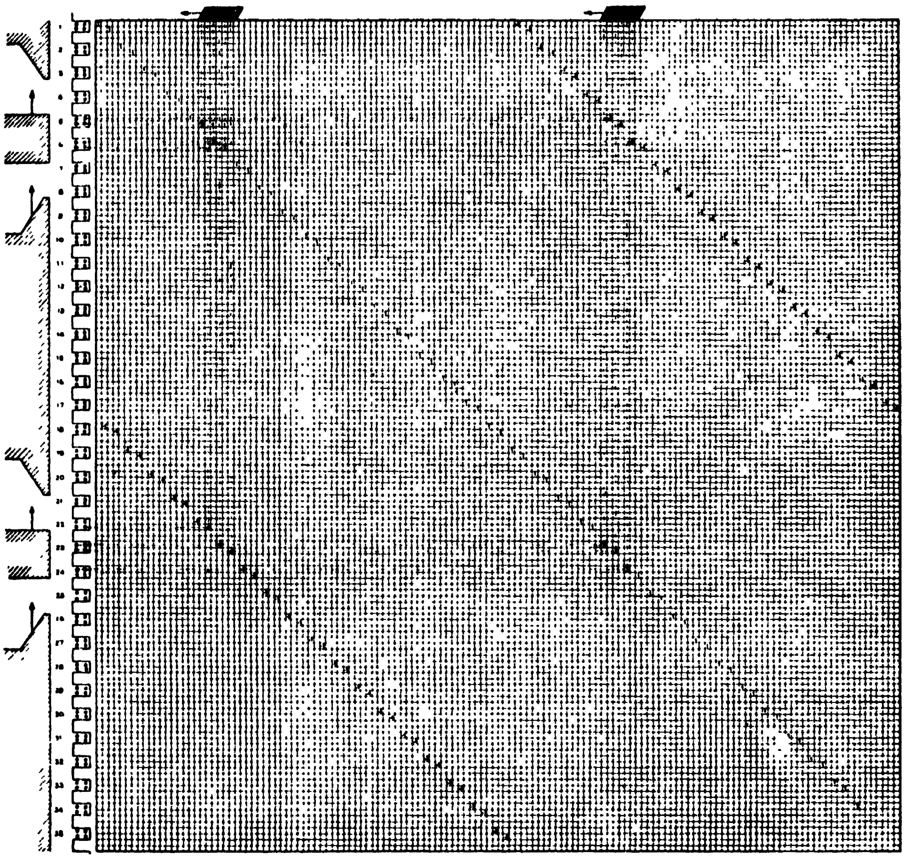


FIG. 6.26.—Transformation Matrix C of a Lap-winding with 70 Coils, showing from instant to instant the coils short-circuited by the brushes and their instantaneous positions along the poles

coils represented by +1 and −1 lying directly under them. The actual space position of the short-circuited coils with respect to the poles is shown on the left-hand side.

The rotation of the armature may be represented by moving the brushes to the left and the field poles upward simultaneously. As the brushes move, each time they make or break a contact with a column

they short-circuit or open-circuit a coil. Hence *it is possible to visualize on the transformation tensor the coils as they are short-circuited by the moving brushes at each instant and their instantaneous position with respect to the field poles*, as shown on Fig. 6.26 for one particular instant.

The subject of commutation is not undertaken in these pages.

XVIII. PHYSICAL INTERPRETATION OF THE TRANSFORMATION TENSOR

(*a*) *When the transformation consists of changing physically one network into another network by interconnections, the components of the transformation tensor* **C** (*containing only integers*) *completely determine the physical set-up of the change.* That is, the transformation tensor represents a mathematical photograph of the diagram of connection. In particular:

1. The *columns* of **C** enumerate the old meshes, whose coils build up the new meshes.

2. The *rows* of **C** enumerate the new meshes whose coils build up the old meshes.

It is interesting that some meshes may be enumerated several times (see equation 6.2).

(*b*) *When the "old" network is the "primitive" network and each old mesh is an individual coil, then and only then,*

1. *The columns of* **C** *enumerate the coils that build up the new meshes.*

2. The *rows* of **C** enumerate the new meshes to which each coil belongs.

These relations serve as powerful aids in determining the mesh network if its transformation tensor is known. In problems of network synthesis the transformation tensor **C** is known, giving a network having a desired performance characteristic, and the problem is to find such a network.

(*c*) The components of **C** completely determine also the electromagnetic relations between the two networks. In particular:

1. The *columns* of **C** enumerate the old meshes whose voltages **e** add up to form the new mesh voltages **e**′.

2. The *rows* of **C** enumerate the new meshes whose currents **i**′ add up to form the old mesh currents **i**.

(*d*) When the old network is the primitive network, then

1. The columns of **C** enumerate the coils whose voltages **e** add up to form the mesh voltages **e**′.

2. The rows of **C** enumerate the mesh currents **i**′ that add up to form the coil currents **i**.

(*e*) The study of such transformation tensors **C** *containing only integers* is undertaken in a branch of geometry called "*topology.*"

When the transformation consists of other types of changes and the components of **C** may be fractions also, then other interpretation has to be assigned to the components of **C**. In general, *each group of transformation tensors has a different physical interpretation.*

CHAPTER VII

COVARIANT AND CONTRAVARIANT INDICES

I. UPPER AND LOWER INDICES

(*a*) In equations 4.32 and 4.33 the following transformation formulae were introduced for the current and voltage vectors:

$\mathbf{i}' = \mathbf{C}^{-1}\cdot\mathbf{i}$	$i^{\alpha'} = C^{\alpha'}_{\alpha}\, i^{\alpha}$	6.1
$\mathbf{e}' = \mathbf{C}_t\cdot\mathbf{e}$	$e_{\alpha'} = C^{\alpha}_{\alpha'}\, e_{\alpha}$	6.2

It should be noted that in changing the old components of the *voltage* vector e_α to the new components $e_{\alpha'}$, the transformation tensor is $C^{\alpha}_{\alpha'}$ (in index notation $\mathbf{C}$ and $\mathbf{C}_t$ are both represented by $C^{\alpha}_{\alpha'}$). On the other hand, in changing the old components of the *current* vector i^α to the new components $i^{\alpha'}$ the transformation tensor is $C^{\alpha'}_{\alpha}$, that is, the *inverse* of $C^{\alpha}_{\alpha'}$. In other words, *in changing over to a new reference frame (or system) the current vector does not behave the same way as the voltage vector.* For instance, in connecting two coils *in series*, the currents through the two coils remain identical, while the voltages are added, on the other hand, connecting the two coils *in shunt* the currents are added and the voltages are identical.

(*b*) To represent this difference in the behavior of the voltage and current vectors during a transformation, in index notation *the voltage vector always has a lower index, as e_α, while the current vector always has an upper index, as i^α.* (In direct notation the distinction between the two types of vectors is not made.)

The lower indices are called " covariant " indices and the upper ones " contravariant " indices. Accordingly e_α is called a " covariant vector " and i^α a " contravariant vector." By convention the variable representing a velocity, in this case i^α, always has an upper index.

It is emphasized that the voltage vector is not an " inverse " of the current vector. The voltage vector *behaves* in an inverse manner to the current vector *only when* a new reference frame is introduced. When an equation refers to only *one* particular reference frame, the distinction between the current and the voltage vectors disappears.

(*c*) Just as vectors have lower or upper indices, similarly the indices

of other geometric objects also are lower or upper. In particular, *since a geometric object is multiplied by $C^{\alpha}_{\alpha'}$ or by its inverse $C^{\alpha'}_{\alpha}$ as many times as it has indices*, each index that is multiplied by the transformation tensor $\mathbf{C} = C^{\alpha}_{\alpha'}$ (or by $\mathbf{C}_t$) is written as a lower (covariant) index, while an index that is multiplied by its inverse $\mathbf{C}^{-1} = C^{\alpha'}_{\alpha}$ (or by $\mathbf{C}_t^{-1}$) is written as an upper (contravariant) index in order to balance the indices.

For instance, since the transformation formula of $\mathbf{z}$ is

$$\mathbf{z}' = \mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} \qquad \Big| \qquad z_{\alpha'\beta'} = z_{\alpha\beta} C^{\alpha}_{\alpha'} C^{\beta}_{\beta'} \tag{7.3}$$

both indices of $z_{\alpha\beta}$ are lower (covariant) indices. With this arrangement one of the dummy indices is an upper, the other is a lower index.

(*d*) To find the transformation formula of $\mathbf{y} = \mathbf{z}^{-1}$, consider the current equation

$$\mathbf{i} = \mathbf{y} \cdot \mathbf{e} \qquad \Big| \qquad i^{\alpha} = y^{\alpha\beta} e_{\beta} \tag{7.4}$$

Replacing $\mathbf{i}$ and $\mathbf{e}$ by their value along the new reference frame

$$\mathbf{C} \cdot \mathbf{i}' = \mathbf{y} \cdot (\mathbf{C}_t^{-1} \cdot \mathbf{e}') \qquad \Big| \qquad C^{\alpha}_{\alpha'} i^{\alpha'} = y^{\alpha\beta} C^{\beta'}_{\beta} e_{\beta'}$$

Multiplying both sides by $\mathbf{C}^{-1}$ on the left

$$\mathbf{i}' = \mathbf{C}^{-1} \cdot \mathbf{y} \cdot \mathbf{C}_t^{-1} \cdot \mathbf{e}' \qquad \Big| \qquad i^{\alpha'} = y^{\alpha\beta} C^{\alpha'}_{\alpha} C^{\beta'}_{\beta} e_{\beta'}$$

Since along the new reference frame the current equation 6.4 has the same form as it has along the old reference frame, namely

$$\mathbf{i}' = \mathbf{y}' \cdot \mathbf{e}' \qquad \Big| \qquad i^{\alpha'} = y^{\alpha'\beta'} e_{\beta'} \tag{7.5}$$

it follows that the transformation formula of $\mathbf{y}$ is

$$\boxed{\mathbf{y}' = \mathbf{C}^{-1} \cdot \mathbf{y} \cdot \mathbf{C}_t^{-1}} \qquad \Big| \qquad \boxed{y^{\alpha'\beta'} = y^{\alpha\beta} C^{\alpha'}_{\alpha} C^{\beta'}_{\beta}} \tag{7.6}$$

Hence $y^{\alpha\beta}$ is written with two *upper* (contravariant) indices showing that in its transformation formula the inverse of $\mathbf{C} = C^{\alpha}_{\alpha'}$, namely $C^{\alpha'}_{\alpha}$, occurs twice.

II. GEOMETRIC OBJECTS

(*a*) *All equations of tensor analysis are so formulated that they should be valid for an infinite number of reference frames or systems.* Consequently, there is a constant need to find the components of geometric objects in various reference frames with the aid of their transformation formulas.

Hence *with every geometric object used in tensor analysis a permanent "formula of transformation" is associated, which remains unchanged*

throughout the whole analysis. The possession of a permanent "formula of transformation" differentiates the components of a "geometric object" from an "n-dimensional matrix" and endows the object with an existence as a separate mathematical and physical entity.

(*b*) *In tensor analysis every separate symbol (or "base" letter) is a "geometric object," that is an "n-dimensional matrix" plus a "transformation tensor" plus a "formula of transformation."* On the other hand, matrix analysis deals with n-dimensional matrices that have no formula of transformation permanently attached to them. In other words, *matrix analysis deals with n-matrices that are valid for one particular reference frame or one particular system only.* Any transformation that may be used in matrix algebra is incidental, and usually no "form" remains invariant under the transformation. On the other hand, tensor analysis is the study of only such equations and their properties that remain invariant under some group of transformation matrices **C**. Hence "tensor analysis" and the study of "invariant transformations" are synonymous.

(*c*) When *incidentally* in tensor analysis matrices of various dimensions occur that have no formulas of transformation associated with them, *their indices will be enclosed in parentheses as* $A_{(\alpha)}$ or $A_{(\alpha)(\beta)}$, *indicating that the indices have no covariant or contravariant significance*, and that the components of the n-matrix are arranged in a row, or in a square, etc.

Hence so far the following types of indices have been introduced:

1. *Fixed and variable* (e_a, e_α).
2. *Free and dummy* ($A_{\alpha\beta}i^\beta$).
3. *Open and closed* ($A_{\alpha\beta}$, $A_{(\alpha)(\beta)}$).
4. *Upper and lower* (i^α, e_α).

(*d*) Summarizing the new concepts, *in these pages so far three different types of numbers have been introduced*:

1. *Sets = a set of numbers arranged in some order*, in a row, or a square, or a cube, etc.

2. *n-Way matrices = sets plus rules of manipulations.* Studied in matrix algebra, etc.

3. *Geometric objects = n-way matrices plus a group of transformations plus formulas of transformation.* Studied in tensor analysis.

n-Matrices have nothing whatever to do with transformations; on the other hand, geometric objects are created by means of formulas of transformations permanently attached to them. If the transformation formula is removed from a geometric object, its unity disappears and *it disinte-*

grates into its component parts, that is, into an infinite number of n-way matrices.

Expressed in another way, the existence of a group of transformation matrices **C** shows that an available series of n-matrices (say the calculated impedance matrices of a large number of networks each containing the same n coils) are not independent of each other, but represent different cross-sections of one and the same physical entity, " the impedance tensor $z_{\alpha\beta}$." The interrelation of the various matrices is shown by the fact that it is possible to derive any one from any other one with the aid of a $C^{\alpha}_{\alpha'}$ by the transformation formula of $z_{\alpha\beta}$. *If the group of transformation matrices $C^{\alpha}_{\alpha'}$ showing the manner of connection of the coils of the various networks is not available, then the various impedance matrices are independent of one another and do not form part of one single object.*

It is again emphasized that it is not necessary to represent the components of a geometric object along some particular reference frame in the form of an n-way matrix. The components may be arranged in any arbitrary manner since the indices take care of their correct manipulation anyway. The arrangement of the components of a geometric object of valence n into an n-dimensional matrix was found to be convenient in routine calculations since many manipulations with the *components* of a geometric object and with an n-way matrix happen to be identical.

(*e*) The distinction between a geometric object and an n-matrix is of the same nature as the distinction between a 2-matrix and a determinant. Now *both a 2-matrix and a determinant appear the same as far as the eye is concerned, still they are different types of concepts.* The determinant is of a more restricted type, since *a 2-matrix has a determinant but a determinant does not have a matrix.* The 2-matrix consists of k^2 entities, but the determinant is one entity.

Similarly *a geometric object of valence n and an n-matrix appear the same as far as the eye is concerned,* but yet they are different types of concepts. The n-matrix is of a more restricted type since *a geometric object has an n-matrix (a different one along each reference frame) but an n-matrix does not have a geometric object.* The fixed indices do not belong to the n-matrix, but they do belong to the geometric object. The geometric object is one entity; the n-matrix consists of k^n entities.

That is, a determinant, a 2-matrix, and a geometric object of valence two appear the same to the eye, all three consisting of a set of numbers arranged in a rectangle, but still intrinsically they are three different types of concepts, like a rough sketch of a statue, the photograph of the statue, and the marble statue itself. The single entity,

the statue, has a large number of photographs, each taken from a different view.

Generally speaking, the distinction between a geometric object and an n-matrix is of the same nature as that between an n-matrix and one of its components. An n-matrix is only a component of a geometric object. Of course, n-matrices may exist apart from geometric objects.

(*f*) *It cannot be sufficiently emphasized that n-matrices do play an important part in the manipulation and solution of the equations of tensor analysis just as determinants do,* and the greater command the engineer has of the theory of matrices and the theory of determinants the better equipped he is to profit by the concepts and methods of tensor analysis.

III. INTERMEDIARY GEOMETRIC OBJECTS

(*a*) A geometric object may contain indices that belong to two (or more) different reference frames such as $z_{\alpha\beta'}$. They are called "*intermediary geometric objects.*"

An example of such a geometric object is the transformation tensor $C^{\alpha}_{\alpha'}$. Another example has already occurred in the calculation of the differences of potential existing in the individual coils of a network by $\mathbf{e} = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$. The combination $\mathbf{z} \cdot \mathbf{C} = z_{mn}C^{n}_{n'}$ may be written by one symbol as

$$z_{mn'} = z_{mn}C^{n}_{n'} \qquad \Big| \qquad \mathbf{z}_1 = \mathbf{z} \cdot \mathbf{C} \qquad 7.7$$

representing a new form of the impedance tensor.

(*b*) Just as geometric objects may belong to several reference frames, similarly the equations of tensor analysis may contain various geometric objects that individually or with their individual indices belong to several reference frames. That is e_m may belong to one, $i^{m'}$ to another reference frame, but still both occur in the same equation. Such an example is the equation giving the voltages along the *old* axes in terms of the *new* currents as

$$\boxed{e_m = z_{mn'}i^{n'}} \qquad \Big| \qquad \boxed{\mathbf{e} = \mathbf{z}_1 \cdot \mathbf{i}'} \qquad 7.8$$

Such an equation occurs whenever the impressed voltages are expressed along different meshes than the currents are, as was shown in equation 6.12.

Another example is the equation of transformation

$$i^{m} = C^{m}_{m'}i^{m'} \qquad \Big| \qquad \mathbf{i} = \mathbf{C} \cdot \mathbf{i}' \qquad 7.9$$

Equations with intermediary geometric objects play an important

part in electrical engineering since the currents, voltages, impedances, etc., may be expressed in several types of reference frames, each in a different one, and it is rarely desired to reduce all of them to the same reference frame.

(*c*) Instead of saying that $z_{\alpha\beta'}$ is a geometric object of valence *two*, it is said that $z_{\alpha\beta'}$ is a geometric object of valence *one* in the α reference frame and a geometric object of valence one in the α' reference frame.

This view is necessitated by the fact that when the α reference frame is changed to α'' by $C^{\alpha}_{\alpha''}$ (leaving the α' reference frame unchanged) then $z_{\alpha\beta'}$ is multiplied by $C^{\alpha}_{\alpha''}$ *only once* as $z_{\alpha\beta'}C^{\alpha}_{\alpha''}$, giving $z_{\alpha''\beta'}$. A similar situation arises in changing the α' reference frame.

IV. ASSOCIATED GEOMETRIC OBJECTS

(*a*) Since a geometric object may have upper or lower indices, depending upon whether an index has to be multiplied by $C^{\alpha}_{\alpha'}$ or by its inverse $C^{\alpha'}_{\alpha}$, *there are several types of geometric objects of the same valence having the same base letter.* In particular there are:

1. Two types of valence one: A_{α}, A^{α}.
2. Four types of valence two: $A_{\alpha\beta}$, $A^{\alpha\beta}$, $A_{\alpha}^{\cdot\beta}$, $A^{\alpha}_{\cdot\beta}$.
3. Eight types of valence three: $A_{\alpha\beta\gamma}$, $A^{\alpha\beta\gamma}$, $A_{\alpha}^{\cdot\beta\gamma}$, $A^{\alpha}_{\cdot\beta\gamma}$, $A^{\alpha\cdot\gamma}_{\cdot\beta}$, $A_{\alpha\cdot\gamma}^{\cdot\beta}$, $A^{\alpha\beta}_{\cdot\cdot\gamma}$, $A_{\alpha\beta}^{\cdot\cdot\gamma}$.

Each of these represents the same physical entity, but in a different manifestation.

In order to show clearly whether an index is an upper or a lower index, and to indicate the correct order of the indices, *it is customary to fill with a dot the empty space opposite an upper or a lower index* as in $A_{\alpha\cdot\gamma}^{\cdot\beta}$ in order that β should not be written, for instance, above γ, in which case it could not be decided whether β is the second or the third index.

(*b*) Geometric objects of the same valence that have the same base letter but indices in different positions are called "*associated*" geometric objects.

$A_{\alpha\beta}$ is called a "doubly covariant" geometric object of valence two, $A^{\alpha\beta}$ is "doubly contravariant," $A_{\alpha}^{\cdot\beta}$ is a "once covariant and once contravariant" geometric object, and so on. In general, one with upper and lower indices is called a "*mixed*" geometric object.

Once the position of the indices has been fixed by the requirement. of the problem, *their position and order cannot be disturbed in general.* Later on, rules will be given for raising or lowering an index in special cases.

(*c*) Because associated objects of the same valence, say $A_{\alpha\beta}$ and

$A_{\alpha}^{\cdot\beta}$, are different types of representations of the same entity A, *associated objects with different indices cannot be added,* even though they have the same valence. For instance, $A_{\alpha\beta}$ and $A_{\alpha}^{\cdot\beta}$ cannot be added.

V. LINEAR AND FUNCTIONAL TRANSFORMATIONS

(*a*) The most important geometric object is the "*transformation tensor*" $C_{\alpha'}^{\alpha}$. *It is the key to tensor analysis.*

The transformation tensor $C_{\alpha'}^{\alpha}$ differs in many respects from other geometric objects of valence two. It has one upper and one lower index, but it is not shown whether α is the first or the second index. Also, one of its indices α belongs to the *old* reference frame while its other index α' belongs to the *new* reference frame. That is, it is an "intermediary geometric object." The indices of most geometric objects hitherto introduced all belong to the same reference frame as $z_{\alpha\beta}$ or $z_{\alpha'\beta'}$.

(*b*) *The transformation tensor* $C_{\alpha'}^{\alpha}$ *is found by setting up a relation between the old and the new contravariant variables.* Two cases will be distinguished:

1. A *linear* relation can be set up between the variables (as in all cases so far considered) $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$.

2. A *functional* relation only can be set up between the variables $\mathbf{i} = f(\mathbf{i}')$.

(*c*) When a *linear* relation can be set up between the old and the new variables, *the transformation tensor is found by taking the coefficients of the new variables as*

$$x^{\alpha} = C_{\alpha'}^{\alpha} x^{\alpha'} \quad \text{or} \quad i^{\alpha} = C_{\alpha'}^{\alpha} i^{\alpha'} \tag{7.9}$$

Only in simpler problems (such as networks, where the variables are the currents or voltages) is it possible to set up a *linear* relation between the old and the new variables. In such cases *the components of* $C_{\alpha'}^{\alpha}$ *are all constants.*

(*d*) In more complex problems (in field problems with curvilinear coordinate axes or in rotating electrical machines, where the variables x^{α} are not the currents, but the charges and rotor displacement) no linear relation can be set up between the old and the new variables. *In such general cases the transformation tensor* $C_{\alpha'}^{\alpha}$ *is found by setting up a relation between the "differentials"* dx^{α} (*or currents* dx^{α}/dt) *of the variables* x^{α} and taking the coefficients of the new differentials as

$$dx^{\alpha} = C_{\alpha'}^{\alpha}\, dx^{\alpha'} \quad \text{or} \quad \frac{dx^{\alpha}}{dt} = C_{\alpha'}^{\alpha} \frac{dx^{\alpha'}}{dt} \tag{7.10}$$

In these cases the components of $C^{\alpha}_{\alpha'}$ are not constants, but are functions of the variables. Such a transformation tensor is for instance

$$C^{\alpha}_{\alpha'} = \begin{array}{c|ccccc} \alpha \backslash \alpha' & a & b & c & d & f \\ \hline a & 1 & 0 & 0 & 0 & 0 \\ b & 0 & \cos x^f & -\sin x^f & 0 & 0 \\ c & 0 & \sin x^f & \cos x^f & 0 & 0 \\ d & 0 & 0 & 0 & 1 & 0 \\ f & 0 & 0 & 0 & 0 & 1 \end{array}$$

that occurs in electrical machine studies. Some of the components are functions of the variable x^f (there are five variables x^a, x^b, x^c, x^d, and x^f).

VI. HOLONOMIC AND NON-HOLONOMIC TRANSFORMATIONS

(*a*) The relation between the old and the new differentials is established in two different manners:

1. If the relation between the old and the new variables $x^{\alpha} = f(x^{\alpha'})$ is known, then the differential of both sides is taken as

$$\boxed{dx^{\alpha} = \frac{\partial f(x^{\alpha'})}{\partial x^{\alpha'}} dx^{\alpha'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} dx^{\alpha'}} \qquad 7.11$$

so that the transformation tensor is defined as

$$\boxed{C^{\alpha}_{\alpha'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}} \qquad 7.12$$

Such transformations are called " holonomic transformations."

2. If no relation can be established between the old and the new variables x^{α} and $x^{\alpha'}$ then the relation between the old and the new differentials dx^{α} and $dx^{\alpha'}$ is usually determined by the problem itself in the form

$$\boxed{dx^{\alpha} = C^{\alpha}_{\alpha'} dx^{\alpha'}} \qquad 7.13$$

Since $C^{\alpha}_{\alpha'}$ *cannot be written as* $\partial x^{\alpha}/\partial x^{\alpha'}$, such transformations are called " *non-holonomic transformations.*"

As soon as spatial motion enters into electrodynamics, as in rotating electrical machines, by far the largest number of problems involve non-holonomic transformations.

(*b*) Although the transformation tensor is defined by a linear relation between the old and new *contravariant variables* or their differentials, it does not follow, however, that in each problem the transformation tensor $C^{\alpha}_{\alpha'}$ must be established in the manner of equations 7.10 or 7.11. In some problems it is easier to determine first the *inverse* of $C^{\alpha}_{\alpha'}$ by the relations

$$x^{\alpha'} = C^{\alpha'}_{\alpha}\, x^{\alpha} \quad \text{or} \quad dx^{\alpha'} = C^{\alpha'}_{\alpha}\, dx^{\alpha} \qquad 7.14$$

or by the relations between the *covariant variables*

$$E_{\alpha} = C^{\alpha'}_{\alpha}\, E_{\alpha'} \quad \text{or} \quad dE_{\alpha} = C^{\alpha'}_{\alpha}\, dE_{\alpha'} \qquad 7.15$$

In other problems it is easier to set up a relation, say, between the old and the new impressed voltages e_{α}, instead of the response currents i^{α}, and the transformation tensor may be determined from the set of equations

$$e_{\alpha} = C^{\alpha'}_{\alpha} e_{\alpha'} \quad \text{or} \quad e_{\alpha'} = C^{\alpha}_{\alpha'}\, e_{\alpha} \qquad 7.16$$

(*c*) The more advanced concepts of tensor analysis appear only when the components of the transformation tensor $C^{\alpha}_{\alpha'}$ are functions of the variables x^{α}.

VII. THE FORMULAS OF TRANSFORMATION

(*a*) *Each time a new geometric object is introduced into the analysis, the first step is to find its transformation formula,* in order to assign to it upper and lower indices, etc. Once the indices have been attached to it (a lower index for each $C^{\alpha}_{\alpha'}$ and an upper index for each $C^{\alpha'}_{\alpha}$), any time throughout the analysis *its transformation formula* (*or at least a part of it*) *can be found, when it is needed, automatically from the position of the indices* since in multiplying the geometric object with $C^{\alpha}_{\alpha'}$ or with $C^{\alpha'}_{\alpha}$ one of the dummy indices is an upper, the other a lower, index. For instance,

$i^{\alpha'} = i^{\alpha} C^{\alpha'}_{\alpha}$	$\mathbf{i}' = \mathbf{C}^{-1} \cdot \mathbf{i}$	
$e_{\alpha'} = e_{\alpha} C^{\alpha}_{\alpha'}$	$\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$	
$z_{\alpha'\beta'} = z_{\alpha\beta} C^{\alpha}_{\alpha'} C^{\beta}_{\beta'}$	$\mathbf{z}' = \mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$	7.17
$y^{\alpha'\beta'} = y^{\alpha\beta} C^{\alpha'}_{\alpha} C^{\beta'}_{\beta}$	$\mathbf{y}' = \mathbf{C}^{-1} \cdot \mathbf{y} \cdot \mathbf{C}_t^{-1}$	
$A_{\alpha'\beta'}^{\cdot\cdot\gamma'} = A_{\alpha\beta}^{\cdot\cdot\gamma} C^{\alpha}_{\alpha'} C^{\beta}_{\beta'} C^{\gamma'}_{\gamma}$	$\mathbf{A}'$ = cannot be represented	

It should be noted that the indices of the *new* reference frame are primed and that on both sides of the equations *the free indices are identical and occupy the same position.* Also it should be noted that

the direct notation needs additional symbols when three or more dimensional geometric objects as $A_{\alpha\beta}^{\cdot\cdot\gamma}$ are used.

The transformation formulas of geometric objects in general, however, are more complicated than that shown; they contain also additional terms.

(*b*) In case of a geometric object the number of dimensions in which its components are arranged is the same as its number of indices. Also its number of indices is the same as the number of transformation tensors with which it is successively multiplied. In order to emphasize that *the number of dimensions in which its components are arranged depends solely on the number of transformation tensors it attracts*, an "n-dimensional geometric object" is called instead a "*geometric object of valence n*," suggesting an analogy to the chemical valence of molecules. The expression "n-dimensional matrix" or "n-way matrix" is retained, however, since an n-matrix attracts no transformation tensors.

(*c*) In index notation in a term the *order* of the geometric objects is immaterial, if their components do not contain operators. Also both $\mathbf{C}$ and $\mathbf{C}_t$ are represented by $C^{\alpha}_{\alpha'}$, and both $\mathbf{C}^{-1}$ and $\mathbf{C}_t^{-1}$ by $C^{\alpha'}_{\alpha}$.

Whereas in direct notation it has to be constantly remembered whether $\mathbf{C}$, $\mathbf{C}_t$, $\mathbf{C}^{-1}$ or $\mathbf{C}_t^{-1}$ is to be used and in what order, no such difficulty exists in index notation. On the other hand, in index notation the *position* of the indices has to be remembered.

VIII. THE FORMULAS OF TRANSFORMATION OF INTERMEDIARY GEOMETRIC OBJECTS

(*a*) When the indices of a geometric object, say $z_{\alpha\beta'}$ or $C^{\alpha}_{\alpha'}$, refer to two different reference frames, both frames may be replaced by new ones. *Each old reference frame needs a transformation matrix of its own to change it to a new one, hence two different transformation matrices are used*, say $C^{\alpha}_{\alpha''}$ and $C^{\alpha}_{\alpha'''}$ (or $\mathbf{C}_1$ and $\mathbf{C}_2$), so that

$$\boxed{\mathbf{z}' = \mathbf{C}_{1t}\cdot\mathbf{z}\cdot\mathbf{C}_2} \qquad \Bigg| \qquad \boxed{z_{\alpha''\beta'''} = z_{\alpha\beta'}C^{\alpha}_{\alpha''}C^{\beta'}_{\beta'''}} \qquad 7.18$$

The first $\mathbf{C}_1 = C^{\alpha}_{\alpha''}$ changes the unprimed frame to the double-primed frame, and the second $\mathbf{C}_2 = C^{\alpha'}_{\alpha'''}$ changes the primed frame to a triple-primed one, so that after the transformation the geometric object still is expressed along two different reference frames. (Of course, it is possible to transform only *one* of the reference frames, thereby using only one transformation matrix.)

(*b*) *The transformation formula of the transformation tensor* $C^{\alpha}_{\alpha'}$ *itself becomes*

$$\boxed{\mathbf{C}' = \mathbf{C}_1^{-1} \cdot \mathbf{C} \cdot \mathbf{C}_2} \qquad \Big| \qquad \boxed{C^{\alpha''}_{\alpha'''} = C^{\alpha}_{\alpha'} C^{\alpha''}_{\alpha} C^{\alpha'}_{\alpha'''}} \tag{7.19}$$

This new transformation matrix $\mathbf{C}'$ can change indices from the double-primed to the triple-primed reference frame, while the old $\mathbf{C}$ changes indices from the unprimed to the primed axes (or vice versa).

In index notation the transformation formula of $\mathbf{C}$ follows automatically from the indices. In direct notation its derivation is as follows:

Given the equation

$$\mathbf{i} = \mathbf{C} \cdot \mathbf{i}' \tag{7.20}$$

Let both $\mathbf{i}$ and $\mathbf{i}'$ be changed by

$$\mathbf{i} = \mathbf{C}_1 \cdot \mathbf{i}'' \tag{7.21}$$

$$\mathbf{i}' = \mathbf{C}_2 \cdot \mathbf{i}''' \tag{7.22}$$

Substituting

$$\mathbf{C}_1 \cdot \mathbf{i}'' = \mathbf{C} \cdot \mathbf{C}_2 \cdot \mathbf{i}''' \tag{7.23}$$

or

$$\mathbf{i}'' = \mathbf{C}_1^{-1} \cdot \mathbf{C} \cdot \mathbf{C}_2 \cdot \mathbf{i}''' \tag{7.24}$$

Since

$$\mathbf{i}'' = \mathbf{C}' \cdot \mathbf{i}''' \tag{7.25}$$

therefore

$$\mathbf{C}' = \mathbf{C}_1^{-1} \cdot \mathbf{C} \cdot \mathbf{C}_2 \tag{7.26}$$

IX. INDUCED GEOMETRIC OBJECTS

(*a*) When the transformation tensor $C^{\alpha}_{\alpha'}$ is singular (not square) the new reference frame has a different number of axes, usually less than the old reference frame. Similarly the geometric objects found with the aid of $C^{\alpha}_{\alpha'}$ have fewer rows or columns than the original ones. *The geometric objects found with the aid of a singular* $C^{\alpha}_{\alpha'}$ *are called "induced geometric objects."* Practically all the geometric objects of mesh networks and junction networks (but not of orthogonal networks) are "induced" geometric objects.

Whenever a transformation tensor is singular *it is usually possible to establish another transformation tensor which is non-singular (square with non-zero determinant) by assuming additional reference frames (actual or hypothetical) so that the singular* $C^{\alpha}_{\alpha'}$ *forms a part of the non-singular one.* Similarly all induced geometric objects form a part of other objects that are found with the aid of the non-singular $C^{\alpha}_{\alpha'}$ and have components also along the additional reference axes.

It should be noted that the original geometric object cannot be reestablished from the induced ones.

(*b*) When the transformation tensor $C^{\alpha}_{\alpha'}$ is singular and its complementary part (augmenting it to a non-singular $C^{\alpha}_{\alpha'}$) is not available, then *geometric objects with contravariant* (upper) *indices cannot be transformed from the old to the new axes*, since the inverse of $C^{\alpha}_{\alpha'}$ cannot be found. Similarly, if the inverse of $C^{\alpha}_{\alpha'}$, namely $C^{\alpha'}_{\alpha}$, is known and is singular, then geometric objects with covariant (lower) indices cannot be transformed from the old to the new axes.

The situation is reversed when it is desired to transform from the new to the old axes. Then covariant indices like $y^{\alpha'\beta'}$ can be transformed as $y^{\alpha'\beta'}C^{\alpha}_{\alpha'}C^{\beta}_{\beta'}$, and contravariant ones cannot.

(*c*) *Most singular transformation tensors* $C^{\alpha}_{\alpha'}$ *occurring in this volume can be augmented to non-singular ones*, for instance, by considering both meshes and junction-pairs of a network, or by assuming both magnetizing and load currents. In the *synthesis* of networks many singular transformation tensors occur however, that cannot be so augmented.

X. INVARIANTS

(*a*) One of the functions of tensor analysis is to investigate what happens to a set of equations when new sets of variables are introduced. In particular, *tensor analysis* attempts to determine those characteristics of a set of equations that do not change when new sets of variables are introduced.

In such studies there are two main centers of interest:

1. The set of equations that is to be transformed.

2. The set of equations showing how the variables are to be transformed, that is, the " *group of transformations* " $x^{\alpha} = f(x^{\alpha'})$ to be used for changing the variables.

Now if a " set of equations " and a " group of transformations " are given, all properties of the set of equations that do not change while the variables are changed are called " invariants " of the set of equations under the group.

Of course the " invariants " of a set of equations are also " invariants " of the physical system described by the set of equations. Hence *tensor analysis investigates those properties of a physical system that do not change when the system is rearranged or is viewed from a different point of view.*

(*b*) In the equations hitherto considered the following types of " invariants " have been encountered among others:

1. *Invariant equations*, such as the equations of voltage, current, etc., whose form remains invariant under the group of transformations used.

2. *Invariant forms,* such as the stored magnetic energy $(1/2)l_{mn}i^m i^n$ or power $e_m i^m$, etc., whose form remains unchanged.

3. *Invariant geometric objects,* whose base letter remains unchanged in all reference axes.

4. *Invariant rules* for the addition and multiplication of geometric objects, etc.

5. *Invariant properties,* such as the sum of the number of meshes and junction-pairs is equal to the number of coils, etc.

6. *Invariant transformations,* that interconnect coils, etc.

(*c*) Depending on the *type of equations that are to be transformed,* the " invariants " may be classified as

1. Arithmetic invariants.
2. Algebraic invariants.
3. Differential invariants.
4. Integral invariants.

In other words, the invariants may contain, respectively, arithmetic, algebraic, differential, or integral expressions.

Tensor analysis is the study of " invariants." That is, it concerns those characteristics of a set of equations that remain invariant under a given group of transformations.

Other methods besides tensor analysis set up " invariants," *but in physical problems, where each entity can be determined only by measurements made along some reference frame, the ability of tensor analysis to give these values along any desired frame is a powerful analytical aid.*

Of course it should be understood that there is no sharp demarcation between the arithmetic, algebraic, differential, and integral invariants, and that their study overlaps.

(*d*) Considering the " *group of transformations* " *to be used,* the more restricted the group is, the larger is the number of invariants of the set of equations. For instance, if the transformation tensor is restricted to contain only constants (as in networks), the number of invariants of the set of equations is much larger than if the transformation tensor contains also functions of the variables.

(*e*) The setting up of invariants is chiefly a *mathematical* procedure. Once the invariants are set up, the next step is to interpret them geometrically or physically. *The invariants always represent actually existing geometrical forms* (lines, figures, etc.) *or relations* that are independent of the reference axes chosen.

Similarly, when a set of equations is interpreted physically by the physicist or the engineer, *the invariants represent only actually existing, measurable physical quantities or relations between them.* These physical quantities and relations are common to an infinite variety of physical

systems (differing only by a transformation of variables) and are not attributes of just one particular system or one set of reference axes.

To the engineer, who deals daily with the greatest variety of physically analogous problems, the knowledge of the *invariants* of the system must be of the utmost importance. Whenever an engineering problem (mechanical, thermal, electrical, etc.) is expressed in terms of an equation with several variables or in terms of a set of equations (arithmetic, algebraic, differential, or integral equations), the theory of invariants offers a powerful method of *visualization, organization, and simplification.*

XI. THE DEFINITION OF A "TENSOR"

(*a*) It has been shown that the components of the transformation tensor $C^{\alpha}_{\alpha'}$ may be:

1. Either all constants.
2. Or functions of the variable x^{α}.

Also it was shown that in transforming from one reference frame to another each geometric object is multiplied by $C^{\alpha}_{\alpha'}$ or by its inverse $C^{\alpha}_{\alpha'}$, as many times as it has indices.

This last statement requires amplification. *In the general case, when the transformation tensor $C^{\alpha}_{\alpha'}$ is a function of the variables x^{α}, the transformation formulas of many geometric objects are more complicated than indicated in equation 7.17. They contain also additional terms.*

Such a geometric object is, for instance, $z_{\alpha\beta}$ when the junctions of the coils are not fixed but move with a uniform velocity as in a rotating machine running at constant speed. The transformation formula of $z_{\alpha\beta}$ becomes in that case (given here without proof)

$$z_{\alpha'\beta'} = z_{\alpha\beta}C^{\alpha}_{\alpha'}C^{\beta}_{\beta'} + a_{\alpha\beta}C^{\alpha}_{\alpha'}\frac{\partial C^{\beta}_{\beta'}}{\partial x^{\gamma'}}px^{\gamma'} \tag{7.27}$$

That is, an additional term also occurs in the transformation formula of $z_{\alpha\beta}$. This additional term disappears, however, when the transformation tensor is not a function of the variables x^{α} (as in the networks of this volume) since $\partial C^{\beta}_{\beta'}/\partial x^{\gamma'}$ is zero.

As another example, the transformation formula of the differential of i^{α}, namely of di^{α}, is not $di^{\alpha} = di^{\alpha'}C^{\alpha}_{\alpha'}$. It can be derived as follows: Let

$$\mathbf{i} = \mathbf{C}\cdot\mathbf{i}' \qquad \Big| \qquad i^{\alpha} = C^{\alpha}_{\alpha'}i^{\alpha'} \tag{7.28}$$

Taking the differential of both sides

$$d\mathbf{i} = d(\mathbf{C}\cdot\mathbf{i}') \qquad \Big| \qquad di^{\alpha} = d(C^{\alpha}_{\alpha'}i^{\alpha'}) \tag{7.29}$$

Since a product is differentiated by differentiating each component separately

$$d\mathbf{i} = \mathbf{C} \cdot d\mathbf{i}' + d\mathbf{C} \cdot \mathbf{i}' \quad \Big| \quad di^{\alpha} = C^{\alpha}_{\alpha'} di^{\alpha'} + dC^{\alpha}_{\alpha'} i^{\alpha'} \qquad 7.30$$

Hence an extra term, $dC^{\alpha}_{\alpha'}\, i^{\alpha'}$, occurs in the transformation formula of di^{α}.

(*b*) *Now those geometric objects whose transformation formulas contain only as many* $\mathbf{C}$ *or* $\mathbf{C}^{-1}$ *as they have indices, but no additional term, are called "tensors." Those geometric objects whose transformation formulas contain additional terms besides those shown in equation* 7.17 *will retain the name "non-tensor geometric object." But the word "tensor" is limited to a definite class of geometric objects obeying the simple rules of transformation of equation* 7.17. The additional terms occurring in the transformation formulas usually contain derivatives or differentials of $C^{\alpha}_{\alpha'}$.

Whether a geometric object is a tensor or not does not depend on its characteristics in *one* given problem. It depends on how it behaves when new reference frames are introduced, that is it depends on its characteristics in a large number of problems.

(*c*) *When the components of the transformation tensor* $C^{\alpha}_{\alpha'}$ *are constants* (that is, when the transformation is linear but not functional), *all geometric objects transform as tensors. In the network studies of this volume all geometric objects transform as tensors, since the components of all transformation tensors are constant.* That, however, is not the case in rotating electrical machine studies where the components of the transformation tensor are functions of the variable.

(*d*) *A "tensor of valence one" is called a "vector," and a "tensor of valence zero" is called a "scalar."* Tensors of other valence have no specific names.

(*e*) It is emphasized that in the definition of a tensor nothing is said about the nature of its components. The components of a tensor may be numbers, functions, operators, etc. *The sole criterion of a tensor is its linear transformation formula under a group of transformations.* Its tensor character does not depend on the nature of its components or on the physical or geometrical interpretation given to it.

XII. THE IMPORTANCE OF TENSORS IN PHYSICAL PROBLEMS

One of the purposes of tensor analysis is to formulate physical problems so that in their equations all geometric objects should be tensors. That is, in all possible reference frames that may occur in nature all geometric objects introduced should have the simple transformation formulas of equation 7.17.

The requirement that in all equations of a physical phenomenon every geometric object should be a tensor is due to the fact that *only entities represented by tensors have actual physical existence in nature.* Non-tensor geometric objects are introduced into the equations only by the peculiarities of the reference frames and can be made to disappear from the equations by a judicious selection of the reference frames as in the case of apparent forces like centrifugal force. *The number of tensors necessary to describe natural phenomena is very limited, owing to the limited number of distinct entities existing in nature.*

The importance of the method of tensor analysis in physical problems is due to its ability to offer a definite criterion to decide whether a set of mathematical expression has any counterpart in the physical world or is just a figment of human imagination. *That criterion is the " law of transformation " of the set of quantities.*

In addition to the *law of transformation* of a set of quantities representing a physical entity it is also important to establish the *group of transformations* $C^{\alpha}_{\alpha'}$ under which the law of transformation is valid. Physical entities associated with *networks* are defined by a different $C^{\alpha}_{\alpha'}$ than those associated with *fields*. The physical entities of quantum dynamics are defined again by a different group of transformations than those of classical dynamics. Physical entities may have an existence under one group of $C^{\alpha}_{\alpha'}$ and may be non-existent under another group of $C^{\alpha}_{\alpha'}$.

XIII. THE NEED FOR A "THIRD GENERALIZATION POSTULATE"

The introduction of " *n-way matrices* " permits the setting up of the " first generalization postulate " establishing the same form of equation for *n* degrees of freedom that exists for one (or a minimum number of) degrees of freedom.

The introduction of " *geometric objects* " permits the setting up of the " second generalization postulate " establishing the same form of equation for an infinite number of *analogous* systems, each system having *n* degrees of freedom.

However, in physical problems numerous *types* of reference frames occur (each containing an infinite variety of axes) that radically differ from one another, and the physical phenomena are described by different equations when viewed from different types of frames. For instance, electrodynamic systems have different equations and offer different physical pictures depending whether: (1) the conducting material moves (2) the magnetic material moves; (3) the dielectric moves; (4) the electric charges move; (5) the magnetic charges move; (6) the

observers move; and so on. The concept of " geometric object " is insufficient to unify the equations of these apparently different types of systems. In their equation of performance occur derivatives and differentials such as di^α that do not transform as tensors (as shown in equation 7.30) since they depend on the type of motion of the component parts of the system.

In order to unify the equations of performance of electrodynamical systems irrespective of the relative motion of the material or electrical parts, *all derivatives and differentials (not being tensors) have to be replaced by " tensors " so that every symbol, every geometric object, in the equations becomes a tensor.* This replacement is accomplished by the " third generalization postulate," which, however, is not studied in this volume.

XIV. THE THREE STAGES OF ORGANIZATION

(*a*) In order to visualize better this growth of simple concepts through successive stages of organization, namely, from ordinary numbers through the stages of (1) n-way matrices, (2) geometric objects, (3) tensors, consider a *single stationary coil* with self-inductance L whose equation of performance is $e = zi$ or

$$e = L\frac{di}{dt} \qquad 7.31$$

containing four concepts e, i, L, and an operator d/dt.

(*b*) The *first* stage of organization is introduced by considering instead of a single coil a *set of n interconnected stationary coils* with self- and mutual inductances $L_{(\alpha)(\beta)}$. The equation of performance of the set is

$$e_{(\alpha)} = L_{(\alpha)(\beta)}\frac{di_{(\beta)}}{dt} \qquad 7.32$$

containing still four concepts $e_{(\alpha)}$, $i_{(\alpha)}$, $L_{(\alpha)(\beta)}$, and d/dt, all of them being n-dimensional matrices. The first generalization postulate covers this first stage of organization.

(*c*) The *second* stage of organization is introduced by considering *all the possible sets of n stationary coils* where the sets differ in manner of interconnections, etc. Their equation of performance is

$$e_\alpha = L_{\alpha\beta}\frac{di^\beta}{dt} \qquad 7.33$$

containing four concepts e_α, i^α, $L_{\alpha\beta}$, and d/dt. Although the first three

concepts are tensors, the combination di^α/dt is not a tensor. The second generalization postulate covers this second stage of organization by introducing the concept of transformation tensor $C^\alpha_{\alpha'}$ and the invariance of a form.

(*d*) The *third* stage of organization is introduced by considering *all the possible motions of all the possible sets of n coils* (as occurs, say, in the various types of rotating electrical machines used in industry). Their equation of performance is

$$e_\alpha = L_{\alpha\beta}\frac{\delta i^\beta}{dt} \tag{7.34}$$

containing four concepts e_α, i^α, $L_{\alpha\beta}$, and δ/dt, *each of these concepts being a tensor.* (δi^β is an " absolute " differential representing certain combinations of di^β and other geometric objects defined by certain routine rules of tensor analysis.)

The development of this third stage of organization is more involved than that of the first two, and its study forms the central part of tensor analysis. Of course the simple engineering structures introduced in this volume are to be augmented by more complex structures for the more advanced study. That is, the *zero-dimensional structures* (junctions) *and one-dimensional structures* (coils) are to be interconnected with *two-dimensional structures*, such as the *layers of windings of electrical machinery*, before motions of the junctions may be considered.

XV. IMPORTANT GEOMETRIC OBJECTS OF PHYSICS AND GEOMETRY

(*a*) The geometric objects arising in the study of networks, namely e_α, i^α, $r_{\alpha\beta}$, $l_{\alpha\beta}$, $z_{\alpha\beta}$, etc., are *special cases* of certain basic geometric objects that play a fundamental role in the analysis of general dynamical systems and in geometrical studies. These more general geometric objects will be introduced in a subsequent volume. For a bird's-eye view the following geometric objects may be mentioned that occur in general dynamical and geometrical studies:

(*b*) In physical problems the most important *scalar* (tensor of valence zero) is " *time*," denoted by t. In geometrical problems its place is taken by " displacement," denoted as s.

Other important scalars are the " kinetic energy " T, the " potential energy " V, the " power input " P, etc. In geometry their place is taken by an infinitesimal length ds, etc.

(*c*) In any general problem the *contravariant variables* $x, y, z \cdots$

are denoted by $x^1, x^2, x^3 \cdots$ or by $x^a, x^b, x^c \cdots$, and are represented as the components of the vector

$$\mathbf{x} = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\mathbf{a}} & \multicolumn{1}{c}{\mathbf{b}} & \multicolumn{1}{c}{\mathbf{c}} & \multicolumn{1}{c}{\mathbf{d}} \\ \hline x^a & x^b & x^c & x^d \\ \hline \end{array} \qquad \Bigg| \qquad x^\alpha = \overset{\diagdown\alpha}{} \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \hline x^a & x^b & x^c & x^d \\ \hline \end{array} \tag{7.35}$$

The variables may be known or unknown or partly known and partly unknown, etc.

In mechanical problems the contravariant variables x^α represent instantaneous " displacements "; in electrical problems they represent the " electric charges " that have passed through the windings. In geometrical problems x^α represents any one *point* of an n-dimensional space.

From the variables x^α additional vectors may be derived by successive differentiations. The first of these is the " *differential* " of the variables (an additional set of n variables)

$$d\mathbf{x} = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{\mathbf{a}} & \multicolumn{1}{c}{\mathbf{b}} & \multicolumn{1}{c}{\mathbf{c}} & \multicolumn{1}{c}{\mathbf{d}} \\ \hline dx^a & dx^b & dx^c & dx^d \\ \hline \end{array} \qquad \Bigg| \qquad dx^\alpha = \overset{\diagdown\alpha}{} \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \hline dx^a & dx^b & dx^c & dx^d \\ \hline \end{array} \tag{7.36}$$

representing an infinitesimal change of the displacement, or of the charges, etc. In geometry dx^α represents an infinitesimal vector drawn at point x^α.

If the vector dx^α is divided by the infinitesimal scalar dt, the " *velocity* " vector is

$$\frac{dx^\alpha}{dt} = v^\alpha = \overset{\diagdown\alpha}{} \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \hline \frac{dx^a}{dt} & \frac{dx^b}{dt} & \frac{dx^c}{dt} & \frac{dx^d}{dt} \\ \hline \end{array} = \overset{\diagdown\alpha}{} \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \hline v^a & v^b & v^c & v^d \\ \hline \end{array} \tag{7.37}$$

representing " velocity " or " current." In geometry dx^α/ds represents a unit tangent vector drawn to a line passing through the point x^α.

If the velocity vector dx^α/dt is differentiated again with respect to time, the *acceleration* is represented as

$$\frac{d^2x^\alpha}{dt^2} = \frac{dv^\alpha}{dt} = a^\alpha = \overset{\diagdown\alpha}{} \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \hline \frac{d^2x^a}{dt^2} & \frac{d^2x^b}{dt^2} & \frac{d^2x^c}{dt^2} & \frac{d^2x^d}{dt^2} \\ \hline \end{array} = \overset{\diagdown\alpha}{} \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \hline a^a & a^b & a^c & a^d \\ \hline \end{array} \tag{7.38}$$

Another important covariant vector, whose components are usually known, is the " force " vector

$$f_\alpha = \begin{array}{c} \\ \end{array}\begin{array}{cccc} a & b & c & d \\ \hline f_a & f_b & f_c & f_d \end{array} \qquad 7.39$$

representing the applied forces or impressed voltages.

(*d*) Among the geometric objects of valence two the most important is the so-called " *metric tensor* " or " fundamental tensor "

$$a_{\alpha\beta} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & a_{aa} & a_{ab} & a_{ac} & a_{ad} \\ b & a_{ab} & a_{bb} & a_{bc} & a_{bd} \\ c & a_{ac} & a_{bc} & a_{cc} & a_{cd} \\ d & a_{ad} & a_{bd} & a_{cd} & a_{dd} \end{array} \qquad 7.40$$

representing the various " moments of inertia " and " products of inertia " along the reference axes, or the self- and mutual inductances of the various windings. *In geometrical problems it plays a part in defining the concept of " magnitude " of a vector.* In most problems it is a function of x^α. This tensor will be studied in detail in Chapter XVIII.

Another important tensor of valence two is the " resistance tensor " $r_{\alpha\beta}$ representing the frictional forces. In geometrical problems it plays a part in defining the concept of " *direction* " of a vector.

(*e*) Among the geometric objects of valence three the most important is the so-called " *affine connection* " $\Gamma^\gamma_{\alpha\beta}$. In mechanical problems it appears in the calculation of the centrifugal and other types of *apparent* forces, in electrical problems in the calculation of the generated voltages due to rotation and also of the torques. In geometrical problems it occurs whenever curvilinear reference axes are used and *it plays a central part in defining the concept of " direction " of a vector.* It is not a tensor. In most problems it is a function of $a_{\alpha\beta}$, but in the general case it is not.

(*f*) Among the geometric objects of valence four the most important is the so-called " *Riemann-Christoffel curvature tensor* " $K_{\alpha\beta\gamma\delta}$. In dynamical problems it appears in the equations when small oscillations occur. It is always a function of $\Gamma^\gamma_{\alpha\beta}$.

(*g*) Around the above-mentioned geometric objects there has been built up a large number of other geometric objects of various valences.

The most important among these is a tensor of valence two, the so-called "*transformation tensor*" $C^{\alpha}_{\alpha'}$, whose study is the central feature of the present volume.

XVI. NON-INVARIANT TRANSFORMATIONS

(*a*) In analyzing engineering problems it is quite often necessary to introduce transformation matrices **C** to accomplish certain changes. However, these transformation matrices do not usually form a group, or they do not leave a "form" invariant, so that the concepts of covariance and contravariance cannot be introduced. Such transformations will be called "*non-invariant transformations.*"

An n-matrix like $e_{(\alpha)}$ *or* $i_{(\alpha)}$ or $z_{(\alpha)(\beta)}$ *is transformed in a different manner with different types of transformation matrices*, and their transformation formula has to be derived all over again each time a transformation matrix is introduced.

Such transformations occur usually in the *manipulation* of a set of equations to bring them to more suitable forms. For instance, a general impedance matrix $z_{(\alpha)(\beta)}$ is to be brought to a diagonal form in which all components are zero except the diagonal ones. Or an asymmetrical $z_{(\alpha)(\beta)}$ is to be brought to a symmetrical form in order to set up an equivalent circuit for it.

Examples of non-invariant transformations might be continued indefinitely.

(*b*) Let a set of linear equations

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i} \qquad \Big| \qquad e_{(\alpha)} = z_{(\alpha)(\beta)} i_{(\beta)} \qquad 7.41$$

be considered. They may represent, say, the equations of voltage of *some particular* mesh network. *Three examples of non-invariant transformations will be worked out.*

1. *Let the voltage* **e** *and the current* i *be transformed by two different transformation matrices* **A** *and* **B**, that is, let

$$\mathbf{e} = \mathbf{A} \cdot \mathbf{e}' \qquad \Big| \qquad e_{(\alpha)} = A_{(\alpha)(\alpha')} e_{(\alpha')} \qquad 7.42$$

$$\mathbf{i} = \mathbf{B} \cdot \mathbf{i}' \qquad \Big| \qquad i_{(\alpha)} = B_{(\alpha)(\alpha')} i_{(\alpha')} \qquad 7.43$$

The question is, what is the transformation formula of $z_{(\alpha)(\beta)}$ if it is desired to keep the *new* set of equations in the form $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$?

Substituting equations 7.42 and 7.43 into equation 7.41

$$\mathbf{A} \cdot \mathbf{e}' = \mathbf{z} \cdot \mathbf{B} \cdot \mathbf{i}' \qquad \Big| \qquad A_{(\alpha)(\alpha')} e_{(\alpha')} = z_{(\alpha)(\beta)} B_{(\beta)(\beta')} i_{(\beta')}$$

Multiplying both sides by $\mathbf{A}^{-1}$

$$\mathbf{e}' = \mathbf{A}^{-1} \cdot \mathbf{z} \cdot \mathbf{B} \cdot \mathbf{i}' \qquad \Big| \qquad e_{(\alpha')} = A^{-1}{}_{(\alpha)(\alpha')} z_{(\alpha)(\beta)} B_{(\beta)(\beta')} i_{(\beta')}$$

Since

$$\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}' \qquad \Big| \qquad e_{(\alpha')} = z_{(\alpha')(\beta')} i_{(\beta')}$$

the transformation formula of the impedance matrix $\mathbf{z}$ is

$$\boxed{\mathbf{z}' = \mathbf{A}^{-1} \cdot \mathbf{z} \cdot \mathbf{B}} \qquad \Big| \qquad \boxed{z_{(\alpha')(\beta')} = z_{(\alpha)(\beta)} \overset{-1}{A}_{(\alpha)(\alpha')} B_{(\beta)(\beta')}} \qquad 7.44$$

In many problems $\mathbf{A}$ or $\mathbf{B}$ is the unit matrix.

It should be expressly noted that the *power input* $\mathbf{e} \cdot \mathbf{i}$ *is not an invariant under this transformation* since

$$\mathbf{e} \cdot \mathbf{i} = (\mathbf{A} \cdot \mathbf{e}') \cdot (\mathbf{B} \cdot \mathbf{i}') = \mathbf{A} \cdot (\mathbf{e}' \cdot \mathbf{i}') \cdot \mathbf{B}_t$$

hence

$$\mathbf{e} \cdot \mathbf{i} \neq \mathbf{e}' \cdot \mathbf{i}'$$

2. *Let both* $\mathbf{e}$ *and* $\mathbf{i}$ *be transformed by the same transformation matrix* $\mathbf{A}$. That is, let $\mathbf{B} = \mathbf{A}$. Then the transformation formula of the impedance matrix is

$$\boxed{\mathbf{z}' = \mathbf{A}^{-1} \cdot \mathbf{z} \cdot \mathbf{A}} \qquad \Big| \qquad \boxed{z_{(\alpha')(\beta')} = z_{(\alpha)(\beta)} \overset{-1}{A}_{(\alpha)(\alpha')} B_{(\beta)(\beta')}} \qquad 7.45$$

This transformation formula occurs quite often in matrix algebra when a 2-matrix is to be brought to a diagonal form.

3. It is possible that, when $\mathbf{e}$ is transformed by $\mathbf{A}$ and $\mathbf{i}$ by $\mathbf{B}$, *the impedance matrix* $\mathbf{z}$ *is transformed quite independently from* $\mathbf{e}$ *and* $\mathbf{i}$ by two other transformation matrices $\mathbf{C}$ and $\mathbf{D}$ as

$$\boxed{\mathbf{z}' = \mathbf{C} \cdot \mathbf{z} \cdot \mathbf{D}} \qquad \Big| \qquad \boxed{z_{(\alpha')(\beta')} = z_{(\alpha)(\beta)} C_{(\alpha)(\alpha')} D_{(\beta)(\beta')}} \qquad 7.46$$

In such cases the physical set-up of the system usually changes, say from three-phase to single-phase, etc.

CHAPTER VIII

GEOMETRICAL INTERPRETATIONS

I. GEOMETRIC LANGUAGE

(*a*) Most sciences deal with phenomena that can be described in terms of a set of equations. It has long been noticed that when seemingly unrelated phenomena are described in terms of *basic* concepts the equations often have identical forms. For instance, many equations of hydrodynamics, elasticity, electrodynamics, and differential geometry have the same form, even though the various quantities represent different concepts in the different sciences.

During the last century the tendency has arisen to introduce a universal language, in which the analogous equations of various origins are interpreted and visualized on a common basis. *That universal language is geometry.* That is, instead of saying that a set of equations describes the performance of an electrical machine, or of a gyroscope, or of moving electron, it is said in the language of geometry that the set of equations describes a certain geometric curve in an n-dimensional space, or the motion of a particle along that curve. *The properties of the various types of curves, surfaces, and spaces always correspond to some property of the particular physical system under investigation.* For instance, a singularity on a surface corresponds to the presence of an electric charge; the curvature of space at a point is an indication of the dynamic stability or instability of an oscillating dynamical system.

Hence, if the equations of the various sciences are expressed in geometric language, all their results can be pooled into one fund and applied to other sciences, without developing the same theory all over again in a different language.

(*b*) Generally speaking, the situation in *engineering* is much less favorable for a unified point of view than in the basic sciences. Because of the pressure of competition and other special conditions, engineers had to develop an additional structure of elaborate reasonings to get a solution to their special problems. After the isolated growth of a few decades this additional engineering knowledge becomes so removed from its parents, the basic sciences, that the new store of knowledge

becomes a closed book to everybody but specialists who devote a lifetime study to it and who thereby lose contact with the basic sciences. The theory of networks and of rotating electrical machinery is a classical example of such an *isolated* growth of engineering knowledge, with all its attendant inconveniences, like having a different theory for each machine.

It is the purpose of this volume to reformulate the present knowledge of electrical networks in terms of *basic invariant concepts* used in modern electrodynamics and geometry, in order to simplify their understanding and to open up new channels of investigation for their analysis and synthesis, following as closely as possible the well-traveled channels of the basic sciences.

It will always be found that the geometrical formulation of engineering problems is far simpler, shorter, and more elegant than the manner in which electrical engineers are accustomed to express them; and that the geometrical formulation gives a better and clearer visualization. Also, it will be found that the most up-to-date mathematical tools like tensor analysis are simpler to understand, handier to use in complicated problems, and quicker to give a numerical answer than the so-called elementary engineering tools developed *haphazardly* under the stress of accidental need.

(*c*) In this chapter some of the physical concepts and analytical processes introduced previously will be illustrated in a geometrical language. The content of this chapter is not needed to understand the chapters to follow. However, anyone desiring to read other textbooks on tensor analysis must be familiar with the content of this chapter since practically all books on tensor analysis use the geometrical language.

Some parts of this chapter represent statements whose proofs are found in textbooks on tensor analysis. No attempt is made here to give *precise* definitions.

II. THE GEOMETRIZATION OF PHYSICAL PROBLEMS

(*a*) The transition from a *physical* problem to a *geometrical* problem is made by the following assumption:

" A set of n equations with n variables (with time as parameter) may represent either the performance of a dynamical system with n degrees of freedom or the motion of a point along a curve located in an n-dimensional hypothetical space and expressed along some particular reference frame."

(*b*) For instance, let two coils be given, in which the equations of the currents $e_\alpha = z_{\alpha\beta}i^\beta$ are

$$\left.\begin{aligned} e_1 &= R_{11}i^2 + L_{11}\frac{di^1}{dt} + M_{12}\frac{di^2}{dt} \\ e_2 &= R_{22}i^2 + L_{22}\frac{di^2}{dt} + M_{12}\frac{di^1}{dt} \end{aligned}\right\} \qquad 8.1$$

Let the values of these currents in the two coils at a certain instant be $i^1 = 1.5$ amp. and $i^2 = 1.765$ amp. An instant later let these currents be $i^1 = 0.87$, $i^2 = 1.49$. Still later $i^1 = 0.763$, $i^2 = 1.175$, and again $i^1 = 1.105$ and $i^2 = 0.235$, and so on.

To represent the variation of currents as a problem in geometry, let it be assumed first that the hypothetical space to be introduced is the familiar " linear " space in which the reference axes are straight lines and are assumed to lie at any arbitrary angle from one another. If it is found that the characteristics of this space are insufficient to explain the steps made in the study of networks, let this space be modified or superseded by another hypothetical space that more correctly corresponds to the actual steps.

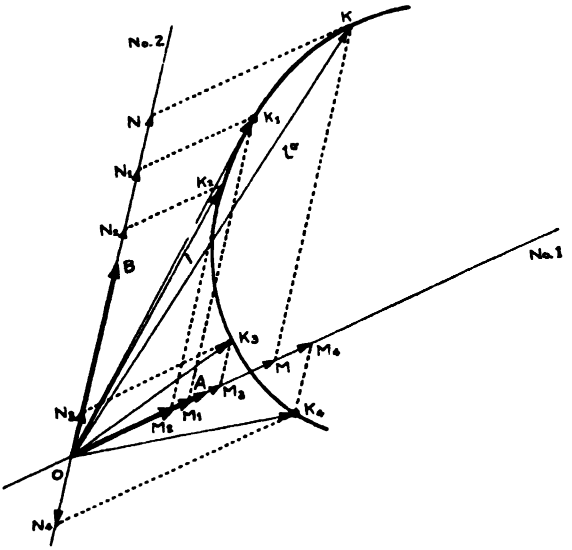


FIG. 8.1.—Representation of a Contravariant Vector i^α in a Rectilinear Reference Frame

Hence *as a first approximation* let two intersecting lines, called the "*reference axes*" or "*reference frame,*" be assumed at any *arbitrary* angle from each other as shown in Fig. 8.1, and along them let two *arbitrary* distances OA and OB be assumed, whose lengths will be denoted as " unity," representing 1 amp. Let the values of the current in coil 1 be plotted along line 1 in terms of OA as unity and the values of the current in coil 2 along line 2 in terms of OB as unity. Connect the two points that measure the current at each instant by drawing a parallelogram such as $OMKN$, where the projections of point

K represent the two currents $i^1 = OM = 1.5$ and $i^2 = ON = 1.765$ at a certain instant. *As time goes on, point K describes the curve $K\ K_1\ K_2\ K_3\ K_4 \cdots$, called a " trajectory." The components of point K along the reference axes give the instantaneous value of the currents flowing in the two coils.* The history of the currents in the two coils are incorporated in the trajectory described by point K.

The plane on which point K is assumed to move is called a " *linear* " space or " *affine* " space.

(c) When *three* coils exist, then three intersecting lines may be assumed as a reference frame located in space instead of in a plane and the motion of point K in a three-dimensional affine space would correspond to the history of the performance of the three coils.

When more than three coils exist then it is still customary to say *as a figure of speech* that a point K moves in an n-dimensional affine space and that the components of that point along n intersection lines represent at each instant the instantaneous values of the currents flowing in the n coils. However, *no attempt whatever should be made to visualize an n-dimensional space.*

III. THE REPRESENTATION OF A CONTRAVARIANT VECTOR

(a) It is customary to call the line connecting the moving point K with the origin O a " *contravariant vector* " and to say that in the n coils *only one current flows*, represented by the vector OK, called the " current vector," and that *the components of this vector along the axes represent the individual currents flowing in the coils.*

(b) The question now arises, if the *components* of the vector OK represent the actually existing currents flowing in the coils, what does *the vector OK itself* correspond to in the physical system?

The answer is that *the vector OK itself does not correspond to anything in particular in the physical system. The vector itself is a concept invented in order that something should exist that has components.* That is, the " vector " is a fiction; its " components," however, are measurable physical quantities.

It is at this very point where the vectors used in these pages differ from the usual vectors of the engineer. The vectors of the engineer, such as the " velocity " of a body, have a physical existence and their " components " are fictions, whereas the vectors used in these pages are fictions (as yet) and their components have a physical existence.

Expressing it differently, *the vectors that describe phenomena taking place in the actual three-dimensional space of our senses are different from those vectors that are invented to describe phenomena in which the actual physical space is an extraneous feature.*

(*c*) *Since the actual vector OK itself does not correspond to anything in particular in the physical system, it makes no sense to talk about the " magnitude " or " direction " of the current vector OK*, whose " components " do represent the currents in the coil, but whose " magnitude " and " direction " are not yet even defined geometrically.

In Chapter XVIII it will be shown that it is possible to assign to the vector OK *arbitrarily* a physical interpretation and thereby to assign to it arbitrarily certain numbers that are called " *magnitude* " and " *direction* " of the vector OK. However, these concepts have little to do with " magnitude " and " direction " as is ordinarily understood. They reduce to the usual meaning *only* in the special case when the intersecting lines of the reference frame are at right angles and the unit vectors are of equal length. In stationary networks this special case occurs only in the trivial case when the coils (1) are equal, (2) have unit self-inductance, (3) have no mutual inductance between them. Only then can it be said that the point K moves in an n-dimensional *ordinary* (Euclidean) space and that it has a " *magnitude* " $\sqrt{i_1^2 + i_2^2 + i_3^2 + \cdots}$ representing the square root of the double " *stored magnetic energy* " in the system.

Hence *it is emphasized that in general a vector has neither " magnitude " nor " direction." It has only " components," that is, an existence. It is defined as a set of quantities whose transformation formula contains* $C^{\alpha}_{\alpha'}$ *or its inverse* $C^{\alpha'}_{\alpha}$.

IV. THE INSUFFICIENCY OF ORDINARY SPACES

(*a*) The question may arise: Why select as a reference frame two intersecting lines at *any* angle, and why select as a unit ampere in each coil two distances of *unequal* length? It looks more logical to select two (or more) intersecting lines *at right angles* as shown in Fig. 8.2 and two (or more) *equal* vectors as unit vectors. In that case it is easier to picture and interpret the motion of a point since the space is an ordinary Euclidean space.

The reasons for introducing a rectilinear (and not a rectangular) reference frame and unequal unit vectors are two:

1. The first reason that ordinary space is insufficient to represent electromagnetic phenomena is that *the representation must be valid under all assumed reference frames*. Now if the current vector OK is represented only along rectangular frames with equal unit vectors, then the only transformations allowable are rotations (and translations) such that the determinant of the transformation tensor **C** would always be unity. The transformation tensors used in network studies, however,

are far more complex than that, and the reference frames and unit vectors must be allowed to have more variety.

2. The second reason is that in addition to the currents and voltages it is intended later on to find a geometrical representation of other electromagnetic quantities. For instance, when it is intended to interpret the *magnitude* of the current vector OK as being proportional at each instant to the instantaneous *stored magnetic energy* $l_{\alpha\beta}i^\alpha i^\beta$, then it is found that the various axes *must be* at different angles and the unit vectors *must be* unequal, the inequality of angles and magnitude being a function of the self- and mutual inductances. As another example, when it is intended to represent geometrically the *electric charges*, existing in a rotating machine, then the reference axes cannot be assumed to be located in a flat space like a plane; they must be located in an n-dimensional curved space like a spherical surface, the reference axes themselves being curved lines. *As more and more electromagnetic quantities are to be represented geometrically, the spaces and geometric configuration lying in them are correspondingly more complicated.*

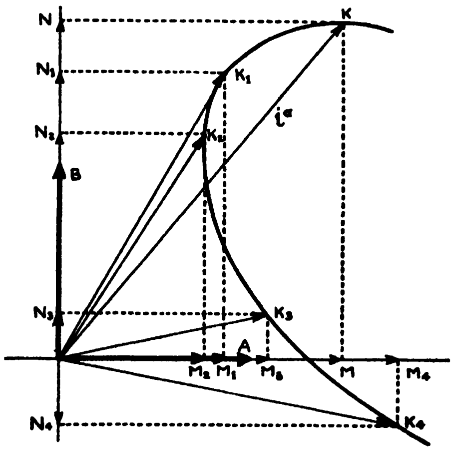


FIG. 8.2.—Rectangular Reference Frame

(*b*) Now if it is not intended to introduce a group of new reference frames to represent new systems or new points of view and if it is not intended to introduce later on the geometrical representation of other quantities besides the assumed variables, then *in any physical problem, no matter how complicated it is, the rectangular representation of the reference frame and the selection of equal unit vectors is perfectly correct.* Such a representation is for instance the pressure — volume — temperature (p — v — T) surface showing graphically the equations of state of thermodynamical systems. In such diagrams, however, the concepts of " distance between two points," or " angle between two lines " and so on have no equivalent in the physical system.

The geometrical representation of any physical problem is satisfactory only if:

1. *It is valid for all possible reference frames that are to be introduced.*
2. *It represents geometrically as many physical quantities occurring*

in the problem as possible, showing correctly their interrelations at all instants.

(*c*) The great variety of complicated reference frames and of non-Euclidean curved spaces have been introduced in modern physical studies in order to satisfy these two requirements of geometrical representation of physical phenomena. Especially electrodynamical phenomena which the electrical engineers deal with every day require quite complex types of space for their representation, and *much of modern tensor analysis and differential geometry owe their development to attempts to represent electrodynamical phenomena geometrically.* This representation is quite necessary in order to clarify the physical phenomenon and to suggest openings to attack still unsolved problems.

It is emphasized that it is not necessary to have an exact *quantitative* correspondence between the physical problem and its geometrical representation. It is sufficient to have a *qualitative* correspondence to offer a *rough but still workable* visualization.

(*d*) As the complexity of the electromagnetic phenomena and the generality of the reference frames to be employed increase, the complexity both of the geometrical configurations to be introduced and of the spaces in which these configurations are located increases proportionally. As traveling electromagnetic waves, moving conductors, accelerating magnetic fields appear, the geometrical representation of their performance becomes increasingly complex.

In fact, even the representation of Fig. 8.1 is insufficient to represent geometrically the performance of the system of coils under all conditions. For instance, when the currents in each coil are represented by a *complex number*, instead of a real number, then the values of the current cannot be measured off along axes 1 and 2 in terms of the unit vectors. In such cases the reference axes are to be pictured as being located, not in an ordinary space, but in a new type of space, called a " *unitary* " space, which can only be represented mathematically. Hence all diagrams, all geometrical expressions such as " reference axis," " unit vector," " space," " plane," and " intersection," should be used by the engineer only as an *aid* to visualize the mathematical steps, to keep their peculiarities in mind more easily. The geometrical language should be used only as a *figure of speech* and not as an exact reproduction of the equation of performance.

(*e*) It must be emphasized that *an n-matrix cannot be represented geometrically.* In order that a geometrical entity like a line, or a plane or an ellipsoid should be represented mathematically by an *n*-matrix as viewed from some reference frame, it is necessary that the same geometrical entity should be represented by *another n*-matrix when

viewed from a different reference frame. In other words, *a geometrical entity is to be represented by a series of n-matrices, not by a single n-matrix, each n-matrix representing the components of the geometrical entity along a particular reference frame.*

Such a series of n-matrices is what is called here a "*geometric object,*" namely an n-matrix with a permanent formula of transformation that allows the geometric object to assume a series of values, each separate value being representable by an n-matrix. That is, just as with a geometrical entity, so *with a "geometric object," a large, usually and infinite, number of n-matrices are associated, one for each reference frame.*

Hence a geometrical entity (line, plane, etc.) is represented by a "geometric object" and not merely by an n-matrix.

V. REPRESENTATION OF A COVARIANT VECTOR

It was shown in the previous sections that there are two types of vectors, contravariant and covariant vectors, each having a different transformation formula when the reference frame is changed. Because of this difference in the transformation formula it is found that a covariant vector cannot be represented geometrically by a point in space as a contravariant vector can.

Again let a reference frame be assumed with its two unit vectors, as shown in Fig. 8.3, and let the instantaneous voltages impressed on the two coils be 0.8 and 0.25 volt. That is, let

$$e_\alpha = \begin{array}{c|c|c} \alpha & 1 & 2 \\ \hline & 0.8 & 0.25 \end{array} \qquad 8.2$$

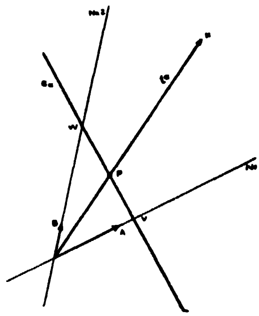


FIG. 8.3.—Representation of a Covariant Vector e_α

To represent these components measure along axis 1 the *reciprocal* of 0.8, namely 1.25 units to V, and along axis 2 the reciprocal of 0.25, namely 4 units to W. *The line passing through points V and W represents the covariant vector e_α.*

In general *in an n-dimensional space a "covariant vector" e_α, having n rather than two components, is represented by an $n - 1$ dimensional plane. The "components" of the vector e_α are the reciprocal of the intercepts cut off by the plane from the reference axes.* As the impressed voltages vary from instant to instant, the intercepts vary and the plane moves from one part of the space to another part.

If i^a is assumed as a variable then the equation of the plane (representing e_a) is

$$e_a i^a = 1 \qquad 8.3$$

or

$$0.8i^a + 0.25i^b = 1$$

For instance, if $i^b = 0$, then $i^a = 1/0.8 = 1.25 = OV$.

VI. TRANSFORMATION OF THE REFERENCE FRAMES

Let it be assumed that instead of lines 1 and 2 another set of lines is assumed as reference axes, 1′ and 2′ (Fig. 8.4). Along these new

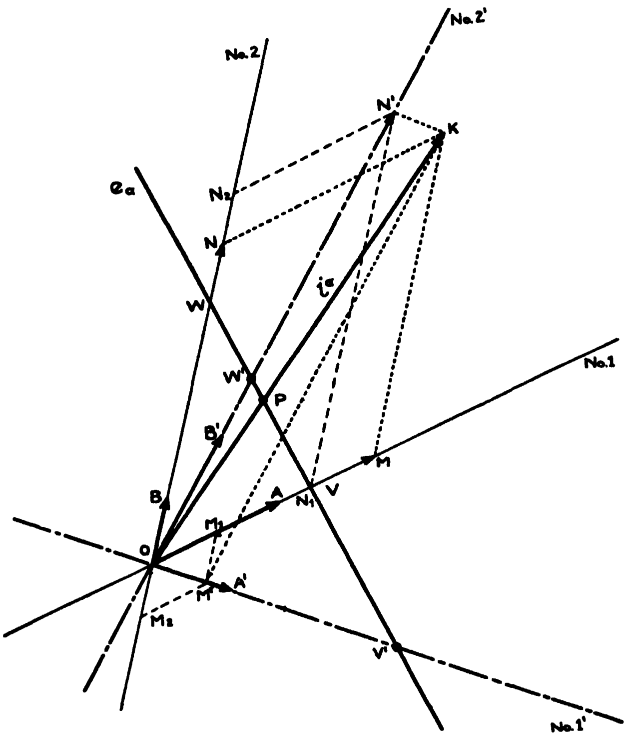


FIG. 8.4.—The Transformation of Reference Frame

axes let two *arbitary* vectors OA' and OB' be measured off again and called "unity." (*The fixed indices* 1, 2 *or* a, b *appearing along the n-matrices may be assumed to represent these unit vectors.*)

Along these new axes the components of the contravariant vector $i^a = OK$ are $OM' = 0.635$ and $ON' = 3.46$, both measured in terms

of the assigned unit vectors OA' and OB' respectively. That is, the components of i^α along the old and along the new reference axes are

$$i^\alpha = \begin{array}{cc} 1 & 2 \\ \hline 1.76 & 4.78 \end{array} \qquad i^{\alpha'} = \begin{array}{cc} 1' & 2' \\ \hline 0.635 & 3.46 \end{array} \tag{8.4}$$

In order to find the transformation tensor $\mathbf{C} = C_\alpha^{\alpha'}$ *a relation has to be set up between the old components* OM, ON *and the new components* OM', ON' *of* i^α as $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$. That is, both OM and ON should be expressed separately in terms of OM' and ON'.

Drawing parallels from points M' and N' to the *old* axes, both OM and ON can be expressed by the projections of OM' and ON' upon the *old* axes as

$$\begin{aligned} OM &= OM_1 + ON_1 = 0.494 + 1.266 = 1.76 \\ ON &= -OM_2 + ON_2 = -0.77 + 5.55 = 4.78 \end{aligned} \tag{8.5}$$

Expressing the projections in terms of OM' and ON' (where $OM' = 0.635$ and $ON' = 3.46$)

$$\begin{aligned} OM &= 0.777\, OM' + 0.366\, ON' \\ ON &= -1.21\, OM' + 1.6\, ON' \end{aligned} \tag{8.6}$$

or in terms of currents

$$\begin{aligned} i^1 &= 0.777 i^{1'} + 0.366 i^{2'} \\ i^2 &= -1.21 i^{1'} + 1.6 i^{2'} \end{aligned} \qquad C_{\alpha'}^{\alpha} = \begin{array}{c|cc} \alpha \backslash \alpha' & 1' & 2' \\ \hline 1 & 0.777 & 0.366 \\ 2 & -1.21 & 1\;6 \end{array} \tag{8.7}$$

the transformation tensor being the coefficients of the new current components.

Once the transformation tensor $\mathbf{C}$ is established, the components of any other vector or tensor along the new axes may be found by its transformation formula. The components of the covariant vector e_α along the new axes are $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$ or $e_{\alpha'} = C_{\alpha'}^{\alpha} e_\alpha$

$$e_{\alpha'} = \begin{array}{|c|c|} \hline 0.777 & -1.21 \\ \hline 0.366 & 1.6 \\ \hline \end{array} \; \begin{array}{|c|} \hline 0.8 \\ \hline 0.25 \\ \hline \end{array} = \begin{array}{cc} 1' & 2' \\ \hline 0.32 & 0.692 \end{array} \tag{8.8}$$

The *intercepts* OV' and OW' cut off by the plane e_α from the new axes as measured in Fig. 8.4 are 3.12 and 1.44. The *inverse of the*

intercepts (representing the components of $e_{\alpha'}$) is 0.32 and 0.692, *which checks the values of* $e_{\alpha'}$ found by the transformation formula.

It cannot be sufficiently emphasized that *the sole criterion that a set of quantities* (such as the various currents flowing in, or voltages impressed on, a set of coils) is arranged in a row and the quantities *are called " components of a vector "* (and are represented by a point or by an $n - 1$ dimensional plane) *is that in going over to a new reference frame these quantities are transformed by multiplication with* **C** *or its inverse* $\mathbf{C}^{-1}$ *only once.* There is no other criterion for calling a set of quantities the " components of a vector." *These vectors have no other attributes; they have neither " magnitude," nor "direction," nor "orientation."*

Because of the fact that their transformation formula contains **C** or $\mathbf{C}^{-1}$ *once*, these components are arranged in a *row*, and their base letter has *one index*. If a set of numbers needs **C** or $\mathbf{C}^{-1}$ *twice*, then and only then are these numbers arranged in a square and their base letter has two indices. *It is quite possible that the same set of numbers may be arranged in one problem in a row, and in another problem in a square.* In fact, even in the same problem it may happen that from one type of reference frame a set of numbers is considered as a vector, arranged in a row, but from a different type of reference frame this same set of numbers is considered arranged in a square. *The sole criterion of the method of arrangement of a set of quantities is their " equation of transformation."*

VII. TRANSFORMATION OF THE POINTS OF A SPACE

(*a*) Instead of considering geometrically the equation of transformation $i^\alpha = C^\alpha_{\alpha'} i^{\alpha'}$ or $x^\alpha = f(x^{\alpha'})$ as transforming the reference frames, another point of view is to assume *that the reference frame remains unchanged and in its place the point* i^α *is moved to another point by the equation of transformation* $\mathring{i}^\alpha = C^\alpha_{\alpha'} i^{\alpha'}$.

Considering, for instance, the example of Section VI, let i^α before the transformation be represented on Fig. 8.5 by point K with components $OM = 1.76$ and $ON = 4.78$. If after the transformation its components become $OM' = 0.635$ and $ON' = 3.46$, these numbers may be measured off along the *same* axes existing before the transformation giving another point K'. The vector OK' may now be assumed to represent $i^{\alpha'}$.

In other words, *the effect of the transformation* $i^\alpha = C^\alpha_{\alpha'} i^{\alpha'}$ *is to move point* K *to point* K'. Assuming different components for i^α, all the points of space may be shifted to new positions

Hence, there are two points of view regarding the geometrical representation of the transformation $i^{\alpha} = C^{\alpha}_{\alpha'} i^{\alpha'}$:

1. Either it transforms the reference frames and leaves the points of space unchanged.

2. Or it transforms the point of the space and leaves the reference frame unchanged.

The two points of view are equivalent.

(*b*) Just how much change is made on the points of the given space by the transformation $i^{\alpha} = C^{\alpha}_{\alpha'} i^{\alpha'}$ or $x^{\alpha} = f(x^{\alpha'})$ depends on the form of $C^{\alpha}_{\alpha'}$. *Each type of geometry has its own group of* $C^{\alpha}_{\alpha'}$. In general, it may be stated that:

1. In *Euclidean geometry* only such $C^{\alpha}_{\alpha'}$ are allowed as may *translate* or *rotate* the points of space without changing the shape of figures in any manner.

2. In *differential geometry* only such $C^{\alpha}_{\alpha'}$ are allowed as may *bend the space out of shape* without, however, stretching or tearing it.

3. In *topology* only such $C^{\alpha}_{\alpha'}$ are allowed as may *stretch and bend the space-structure* without tearing it.

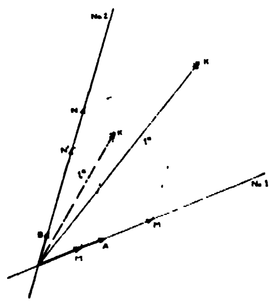


FIG. 8.5.—The Transformation of Point i^{α}

4. *However, it appears that the transformations* $C^{\alpha}_{\alpha'}$ *occurring in electrical engineering problems are more general than any used in these geometries, since they may tear the representative space-structure apart, transforming it to another type of space-structure.*

(*c*) Even though the spaces are moved, bent, stretched, or torn, certain properties associated with them remain "*invariant*" in spite of the changes made. For instance, while a surface is *bent*, the angle between the intersection of two lines drawn upon the surface is an invariant, and while a surface is stretched the property of two lines intersecting each other is an invariant property. Tensor analysis is a tool for discovering such invariants.

VIII. THE REPRESENTATION OF POWER

When a contravariant vector i^{α} and a covariant vector e_{α} are given, as in equations 8.2 and 8.4, their product, a linear form,

$$e_{\alpha} i^{\alpha} = (0.8)(1.76) + (0.25)(4.78) = 2.61 \qquad 8.9$$

is a scalar, a tensor of valence zero, representing the instantaneous

power input into all the coils. This scalar number is represented in Fig. 8.4 by the *ratio* of two lengths

$$P = e_\alpha i^\alpha = OK/OP = 2.61 \tag{8.10}$$

the lengths being measured in any unit, since they both lie along the *same direction.* The line OP is the line cut off by the plane e_α from the vector i^α.

That is *the product of a covariant vector* e_α *and a contravariant vector* i^α *is a scalar, represented by the ratio of:* (1) *the length of* i^α; (2) *the portion of* i^α *cut off by the plane* e_α.

Both lengths are measured in any unit.

When a new reference frame is introduced this ratio remains unchanged. (If the first point of view of **C** is assumed, this is obvious.) That is, *the linear form* $\dot{e}_\alpha i^\alpha$ *is an invariant under the transformation of the reference frames.*

IX. SPACES AND SUPERIMPOSED CONFIGURATIONS

(*a*) When a set of n coils is given and furthermore currents flow in the coils, the performance of the system may be represented by the motion of a point (and other geometrical configurations) in an n-dimensional space. (The geometrical picture presented is still a first approximation to the correct picture to be presented later on.)

The geometrical representation of the physical phenomena taking place in the coils can be grouped under two main headings:

1. *When n individual coils* interconnected in any arbitrary manner are given forming k meshes and $n - k$ junction-pairs but *having no voltages or currents impressed upon them,* geometrically the situation is represented by stating that *an n-dimensional space is given.*

With the space certain properties, called its "*structure,*" are associated that are defined with the aid of the design constants $r_{\alpha\beta}$, $l_{\alpha\beta}$, $C^{\alpha\beta}$, etc., of the coils. The relation between the design constants and the corresponding " structure " of the n-dimensional space is considered in Chapter XVIII.

2. *When instantaneous voltages and currents exist in the coils,* that is when the coils are excited, geometrically the situation is represented by assuming *geometrical configurations,* such as a point or an $n - 1$ dimensional plane, etc., *superimposed upon the given space.* As time goes on, the values of the electrical quantities in the coils *vary* and the configurations on the space *move,* describing certain curves called "*paths.*"

(*b*) The sharp distinction between an unexcited set of coils and

an excited set containing electrical quantities on one hand, and between an empty space and a space containing geometrical configurations on the other, must be emphasized. In the usual electrical engineering problems there is no difficulty in distinguishing between the design or test constants of the *material* configurations and the *electrical* quantities appearing on them. However, as the electrical phenomena become more elusive, as happens in electronics for example, difficulties arise and, culminating in the theory of relativity the distinction between the material quantities and the superimposed electrical quantities themselves entirely disappears and *all electrical phenomena are represented not as quantities superimposed upon our space, but as the "design constants," the "structure" of space itself.* If the electrical quantities (charges, field intensities, flux densities, etc.) are removed, space itself disappears.

X. ORTHOGONAL SUBSPACES

A plane lying in actual space is called a two-dimensional "*subspace*" of the three-dimensional space. If a set of three axes is assumed in space (Fig. 8.6), two of them lying in the plane and the third perpendicular to it, then any point moving in the plane has no component along the third reference axis.

1. *If a network of n coils is considered as a "mesh" network with only k meshes (and thereby with k variable currents), then the point is restricted to move only in a k-dimensional subspace of the n-dimensional space of the network.* The components of the moving point along the k reference axes lying in the subspace represent the k mesh currents. The components of the moving point along the remaining $n-k$ reference axes are zero, corresponding to the currents flowing into the leads at the junctions of the mesh network, that are also zero.

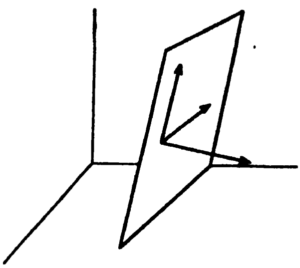


FIG. 8.6.—Two-dimensional Subspace in a Three-dimensional Space

2. *Similarly if the same network of n coils is considered as a junction network with only $n-k$ junction-pairs* (where the currents and voltages are known only at the junctions), *then the point is restricted to move only in an $n-k$ dimensional subspace of the n-dimensional space, whose points are not included in the previous k-dimensional subspace.* That is, the moving point has no projections along the previous k reference axes. It may be shown that *the two mutually exclusive subspaces are also orthogonal to each other.*

The points of the k and $n - k$ dimensional subspaces determine completely all the points of the n-dimensional space of the network.

3. Finally, *if the network of n coils is considered as an orthogonal network having both mesh and junction currents, then the point describes a trajectory in the n-dimensional space of the network and it has projections along all the n reference axes.*

That is, a network of n coils may be assumed to represent a hypothetical n-dimensional space irrespective of the number of meshes and junction-pairs of the network. The manner of interconnection of the coils inherently divides the n-dimensional space into two independent orthogonal subspaces whose points are mutually exclusive, so that the moving point *may* restrict its motion to one of the two subspaces, depending on the manner the network is excited. Hence *the currents flowing around the meshes and the currents flowing through the junction-pairs are orthogonal to each other.*

XI. RESTRICTIONS ON THE REPRESENTATION

(*a*) In the geometrical interpretation of the electrical phenomena appearing in networks the representative space in which the point moves was not assumed to have any boundaries. Around the origin of this n-dimensional space it is possible to assume an *infinite number of reference frames*, some of the reference frames corresponding to:

1. A definite n-coil network selected from all possible n-coil networks.
2. A definite set of n axes assumed in this n-coil network.
3. A definite linear function of the n currents flowing in the assumed n axes of the given network (say the "magnetizing" and "load" currents corresponding to n particular currents assumed).

(*b*) However, the assumption of an n-dimensional space of infinite extent in all directions, in which an infinite variety of reference frames can be assumed by simply assuming any linear transformation, *does not* exactly correspond to an n-coil network, since *in an n-coil network it is not possible to assume any linear transformation.* There is only a *limited* (though large) number of ways in which n coils can be interconnected. There are only a *limited* number of ways in which the n axes may be selected and a *limited* number of ways in which arbitrary linear relations (hypothetical currents) may replace the actual currents assumed.

In other words, in the n-dimensional space assumed, the variation of the currents that may occur in any n-coil network can fully be represented, *but not vice versa. In the space assumed, the moving point may*

follow paths that cannot be duplicated by the actual or hypothetical currents in any n-coil network, and consequently the assumed space is not an exact reproduction of all possible networks. There is no one-to-one relation between the space and the network, which, however, is necessary in order that the geometrical representation should be of more use. For instance, *there is nothing in the representative space to justify the existence of the dual concepts of " mesh " and " junction-pair "* that play such a fundamental part in the network analysis.

(c) In order to find a geometrical representation in which the *physical* concepts of " mesh," and " junction-pair," etc., find an analogous *geometrical* equivalent, *the character of the representative space so far considered has to be restricted by introducing boundaries, removing portions, adding projecting parts, etc., until the representative space has the same form as the network itself.*

(d) The study of n-dimensional space of *unbounded extent in all directions* is undertaken in " differential geometry"; while the study of n-dimensional spaces *bounded in several directions by spaces of smaller dimensions* (like a cube, or an icosahedron) is undertaken in " *topology* " or " *analysis situs*," where such bounded spaces are known as " *cells* " of various dimensions.

The tensorial method of approach (namely, the use of "reference frames," "transformation," "invariance," and "group") has been used in the study of differential geometry for fifty years and an extensive literature is available on the subject, but for the study of topology it has been used systematically *in tensor symbolism* only during the last year or two, and outside of scattered abstracts and discussions no literature is available on it as yet.

XII. INTERCONNECTED SPACES

(a) *A coil may be represented geometrically by a one-dimensional space (a line) which is bounded by two points, the two junctions of the coil. Also a junction point of two coils may be represented as a zero-dimensional space.*

Hence a network of n coils and k junctions may also be represented geometrically as a collection of n one-dimensional and k zero-dimensional spaces, the zero-dimensional spaces forming the boundaries of the one-dimensional spaces. (From a geometrical point of view an impedanceless branch is to be assumed as a coil with zero impedance. Hence it is represented by a one-dimensional space (1-cell) and its two ends by two zero-dimensional spaces (0-cells) instead of forming only one junction, as it is assumed in these pages for analytical purposes.)

When rotating machines with brushes and slip-rings, etc., are connected to the network, the whole system being as yet unexcited, the representative space contains also two- and higher dimensional spaces in addition to zero- and one-dimensional ones.

(*b*) The classification and the study of the properties of these complicated geometrical figures as such are too difficult without the aid of an auxiliary geometrical device. *This auxiliary device consists of drawing upon the given collection of spaces certain configurations of various dimensions.* The manner in which these additional nets can be drawn upon the given cells serves as a starting point for their study and classification. *These additional geometrical configurations drawn upon the given " cells " in order to study their properties are called " chains."*

For instance, a torus (an anchor-ring or a " doughnut " shown in Fig. 8.7) differs from a sphere in the respect that it is possible to draw on it *two* (and only two) *closed curves*, namely:

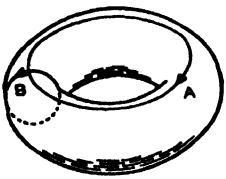


FIG. 8.7.—An Anchor Ring with Two Invariant Closed Curves

1. One parallel with the hole (curve A).
2. The other perpendicular to the hole (curve B).

The characteristics of these two closed curves is that they cannot be made to disappear by deforming them in any manner, that is, by stretching, bending, or moving them (without removing them, however, from the surface of the torus).

On the other hand, *no closed curves* can be drawn on a sphere that cannot be made to disappear by contracting them into a point.

Hence the difference between a torus and a sphere is established by means of configurations (chains) *superimposed* upon them. Because of the existence of the two invariant closed curves, the torus is called a " surface of genus 1" while the sphere, because of their absence, is called a " surface of genus 0."

(*c*) *Now, superimposing " chains " upon " cells " is analogous to superimposing " electromagnetic quantities " (voltages, currents, fluxes, m.m.f.'s, etc.) upon a " network."* Only by means of the superimposed electromagnetic quantities can a network be studied profitably by the engineer.

The process of classifying cells or networks by means of *superimposed* chains or electromagnetic quantities is analogous to studying *statical* problems as *dynamical* problems.

The reason that the process of superimposing chains upon cells and electromagnetic quantities upon networks is so fruitful as an analytical

tool is that the process of superimposing chains or electromagnetic quantities forms a " group," as will be shown in Chapter XI.

XIII. ANALOGOUS CONCEPTS IN TOPOLOGY

(*a*) The topological equivalence of some of the network concepts introduced here is shown in the following table, using the nomenclature of Veblen:

Network	One-dimensional complex
Sub-network	Sub-complex
Coil	1-cell
Junction	0-cell
Mesh	1-circuit
Junction-pair	Bounding 0-circuit

A network is also known as a " linear graph."

When *direction* is assigned to the left-hand terms, the adjective " *oriented* " is added to the right-hand terms, as for instance " *oriented* 1-*circuit*."

Other corresponding expressions are:

All-junction network	Tree
Number of meshes	Cyclomatic number

(*b*) When electromagnetic quantities are superimposed upon the network, topologically that step is equivalent to superimposing " chains " upon the " cells." Some of the analogous topological equivalents are given in the following table, using the nomenclature of Tucker:

Mesh current i^α	1-Chain
Current flowing into a junction	Boundary of 1-chain
Absolute potential at a junction	Dual 0-chain
Coil voltage e_α	Boundary of dual 0-chain
Power $e_\alpha i^\alpha$	Intersection index

The topological nomenclature in tensor parlance is not yet fully established.

XIV. THE TRANSFORMATION OF SPACE-STRUCTURES

(*a*) The various types of transformations $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ used in this volume may roughly be divided physically into two types:

1. The transformation of actual currents into *actual* currents.

2. The transformation of actual currents into *hypothetical* currents.

(*b*) The transformations using only actual physically existing currents may also be divided into two types:

1. The currents are changed from one set of branches of a network into another set of branches of the same network.

2. The currents are changed from one type of network into another type.

The first type of transformation is analogous to a " transformation of the reference frames." However, the second type of transformation involves more than that, since at the same time the interconnections of the coils are destroyed and the network is replaced by another network.

(*c*) From a geometrical point of view the concept of " transformation " $C_{\alpha}^{\alpha'}$ used in these pages for changing the interconnections of networks may be formulated as follows:

Given a set of n component spaces interconnected in various manner, forming thereby a large number of new spaces (or space-structures). Each of these new spaces has its own sets of reference frames. If on each of these spaces a certain geometrical configuration (say a set of trajectory or " path ") is drawn, it is possible to set up a set of equations (say $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$) for each space representing these configurations.

The problem under investigation consists of finding a group of transformation matrices $C_{\alpha}^{\alpha'}$ that transforms the different equations of all the configurations of the various spaces into each other. That is, the problem is to find the equations $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$ of any of the configurations if the equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ are known for one particular configuration. Or, briefly, *the problem is to set up a " correspondence " between the configurations drawn on different space-structures.*

(*d*) From a physical point of view the problem may be formulated as follows:

Given several different networks, each having a different number of sub-networks, of meshes and of junction-pairs (but the same number of coils). If currents and voltages are impressed on them, the performance of each network may be represented by a set of equations (say $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$). *The problem is to find a group of transformations $C_{\alpha}^{\alpha'}$ such that with their aid the equations of performance of each of the networks may be determined if those of one are already known.*

The set of equations $i^{\alpha} = C^{\alpha}_{\alpha'} i^{\alpha'}$ changes not only the reference frame α to α' but also the space on which the reference frames lie, since α and α' lie on two different types of spaces. Or the equations $i^{\alpha} = C^{\alpha}_{\alpha'} i^{\alpha'}$ transform a *portion* of one space into a portion of the other spaces (namely the points lying along the configuration).

(*e*) Summarizing the method of reasoning, one of the problems of the engineer is to calculate the performance of a great variety of dynamical systems under various conditions. The usual method of attack consists of analyzing each problem separately as it comes along and setting up its equation of performance from fundamentals. In their usual method of analysis it is possible to change the axes of the same system but *it is not possible to change the system itself into another system possessing an entirely different set of reference frames.*

The burden of this volume consists of suggesting another procedure, which involves the simultaneous analysis of a large variety of physically analogous but still different systems. It is suggested here that the first step in the analysis of any physical system should consist of finding a transformation matrix $C^{\alpha}_{\alpha'}$ *that sets up a " correspondence " between its equation of performance and that of another system (the primitive system) whose equation of performance already is known. Once* $C^{\alpha}_{\alpha'}$ *is determined, the remaining work consists of routine calculations. If the " group " of* $C^{\alpha}_{\alpha'}$ *is defined, the analysis of the large variety of physical systems becomes a routine procedure.*

(*f*) It should be remembered that in most engineering problems the concept of " transformation " $i^{\alpha} = C^{\alpha}_{\alpha'} i^{\alpha'}$ involves the " *transformation of paths from one space into another* " and not merely the "*transformation of the reference frame* " in the same space.

The correspondence between the various types of networks (spaces) is established by the recognition that in each all-mesh networks there exists one, and only one, reference frame along which the equations of performance of the different networks are identical. This reference frame consists of the individual coils of the networks.

That is, if in two n-coil, n-mesh networks (having different number of sub-networks) the n variables are assumed in the n coils, the n equations of performance of the two networks are identical. They become different, however, as soon as any other set of branches is assumed in each network.

(*g*) It will be shown in the chapter on synthesis that *it is possible to establish a transformation between paths drawn in space-structures in which each structure contains a different number and different types of component spaces,* provided the paths are subjected to certain conditions (say, all networks supply constant current).

CHAPTER IX

COMPOUND TENSORS

I. COMPOUND N-MATRICES, GEOMETRIC OBJECTS, AND TENSORS

(*a*) The most primitive type of manipulation of a set of n linear equations with n variables like $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ consists of treating all components of $\mathbf{e}$ and $\mathbf{i}$ and $\mathbf{z}$ as one unit and assuming that *all* the components of $\mathbf{e}$ and $\mathbf{z}$ are known and *all* the components of $\mathbf{i}$ are unknown. However, *in most problems not all the components of a geometric object are of equal significance.* For instance, part of the components of $\mathbf{e}$ and part of $\mathbf{i}$ may be known and the remaining components may be unknown; or some components of $\mathbf{e}$ or $\mathbf{i}$ may be missing, or knowledge of their value might be superfluous.

In order to treat separately the components of a geometric object that have different significance, *a single invariant equation, such as* $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$, *is replaced by several invariant equations, each invariant equation itself representing several ordinary equations* having the same general significance. Of course, not only linear but any other type of invariant equation may be replaced by several invariant equations.

First it will be shown how geometric objects may be subdivided into several parts; afterward *invariant equations* will be subdivided into several equations.

(*b*) *A geometric object in which each component is itself a geometric object of the same valence instead of an ordinary quantity will be called a "compound geometric object."*

Several special cases may be distinguished. The original n-dimensional quantity that is to be subdivided may be a tensor or a geometric object or an n-matrix. Also, the components of each of these into which they are divided may be tensors, or geometric objects, or n-matrices, depending on the transformation tensors available. For instance, each component of a tensor of valence three may be a 3-matrix.

In order to avoid lengthy circumlocutions it will be assumed in this chapter, unless otherwise stated, that all base letters represent tensors. However, in many cases they just as well can be considered as

n-matrices, by ignoring the reference frames shown, though with n-matrices many of the results arrived at assume simpler form than that shown.

II. THE MANNER OF SUBDIVISION OF TENSORS

(*a*) A vector **A** may be subdivided into the sum of two or more vectors. For instance, the vector

$$\mathbf{A} = \begin{array}{cccccccccccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} & \mathbf{g} & \mathbf{h} & \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} & \mathbf{m} & \mathbf{n} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{-8} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{-1} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{-3} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad 9.1$$

may be divided as $\mathbf{A} = \mathbf{B} + \mathbf{C} + \mathbf{D}$, where

$$\mathbf{B} = \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{-8} & \multicolumn{1}{c|}{9} \\ \hline \end{array} \quad \mathbf{C} = \begin{array}{cccccc} \mathbf{f} & \mathbf{g} & \mathbf{h} & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \hline \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{-1} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{-3} & \multicolumn{1}{c|}{6} \\ \hline \end{array} \quad \mathbf{D} = \begin{array}{ccc} \mathbf{l} & \mathbf{m} & \mathbf{n} \\ \hline \multicolumn{1}{|c|}{9} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad 9.2$$

so that the vector **A** may be expressed as a compound vector

$$\mathbf{A} = \begin{array}{ccc} \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \hline \multicolumn{1}{|c|}{\mathbf{B}} & \multicolumn{1}{c|}{\mathbf{C}} & \multicolumn{1}{c|}{\mathbf{D}} \\ \hline \end{array} \qquad 9.3$$

in which each component itself is a vector, instead of an ordinary number.

(*b*) *For compound tensors it is necessary to introduce new types of fixed indices*, each of them representing not one axis but several axes at the same time. For instance, the index **p** stands for the individual axes **a, b, c, d, e**; the index **q** for **f, g, h, i, j, k**; and the index **r** for the axes **l, m, n.** *Indices representing several indices will be called "compound indices"; the usual types will be called "individual indices."* Both fixed and variable indices may be individual or compound, that is, each fixed or variable compound index may represent several fixed or variable individual indices.

Just as the individual indices of the component tensors may be transformed by a transformation tensor $\mathbf{C}_1$, *similarly the compound indices may be transformed by another transformation tensor* $\mathbf{C}_2$. *In Chapter XX examples will be shown where the individual indices and the compound indices are transformed by different transformation tensors.*

(*c*) A tensor of valence two (briefly, a 2-tensor) may be divided into the sum of several 2-tensors in several different ways depending on the

problem at hand. It may be divided by one or more *horizontal* (or vertical) lines as $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$, where

$$\mathbf{A} = \begin{array}{c|c|c|c|c|c|c|c|} & a & b & c & d & e & f & g \\ \hline a & 2 & 0 & 0 & 4 & 0 & 3 & 0 \\ \hline b & 0 & 6 & 0 & 9 & 0 & -8 & 0 \\ \hline c & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline d & 0 & 5 & 0 & 5 & 0 & 2 & 0 \\ \hline e & 1 & 0 & 0 & 0 & -3 & 0 & 0 \\ \hline f & 0 & 0 & 5 & 0 & 0 & 7 & 0 \\ \hline g & 7 & -6 & 0 & 0 & 8 & 0 & 4 \\ \hline \end{array}$$

$$\mathbf{A}_1 = \begin{array}{c|c|c|c|c|c|c|c|} & a & b & c & d & e & f & g \\ \hline a & 2 & 0 & 0 & 4 & 0 & 3 & 0 \\ \hline b & 0 & 6 & 0 & 9 & 0 & -8 & 0 \\ \hline \end{array}$$

$$\mathbf{A}_2 = \begin{array}{c|c|c|c|c|c|c|c|} & a & b & c & d & e & f & g \\ \hline c & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline d & 0 & 5 & 0 & 5 & 0 & 2 & 0 \\ \hline e & 1 & 0 & 0 & 0 & -3 & 0 & 9 \\ \hline f & 0 & 0 & 5 & 0 & 0 & 7 & 0 \\ \hline g & 7 & -6 & 0 & 0 & 8 & 0 & 4 \\ \hline \end{array}$$

$$\mathbf{A} = \begin{array}{c|c|} & r \\ \hline p & \mathbf{A}_1 \\ \hline q & \mathbf{A}_2 \\ \hline \end{array} \qquad 9.4$$

Here the original tensor $\mathbf{A}$ is expressed as a *one-rowed* " compound 2-tensor " in which each component is a 2-tensor.

The 2-tensor $\mathbf{A}$ may be divided by *both* horizontal and vertical lines into four or more 2-tensors as $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4$, where

$$\mathbf{A} = \begin{array}{c|c|c|c|c|c|c|c|} & a & b & c & d & e & f & g \\ \hline a & 2 & 0 & 0 & 4 & 0 & 0 & 0 \\ \hline b & 0 & 6 & 0 & 9 & 0 & -8 & 0 \\ \hline c & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline d & 0 & 5 & 0 & 5 & 0 & 2 & 0 \\ \hline e & 1 & 0 & 0 & 0 & -3 & 0 & 9 \\ \hline f & 0 & 0 & 5 & 0 & 0 & 7 & 0 \\ \hline g & 7 & -6 & 0 & 0 & 8 & 0 & 4 \\ \hline \end{array}$$

$$\mathbf{A}_1 = \begin{array}{c|c|c|} & a & b \\ \hline a & 2 & 0 \\ \hline b & 0 & 6 \\ \hline c & 1 & 0 \\ \hline \end{array} \qquad \mathbf{A}_2 = \begin{array}{c|c|c|c|c|c|} & c & d & e & f & g \\ \hline a & 0 & 4 & 0 & 3 & 0 \\ \hline b & 0 & 9 & 0 & -8 & 0 \\ \hline c & -1 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

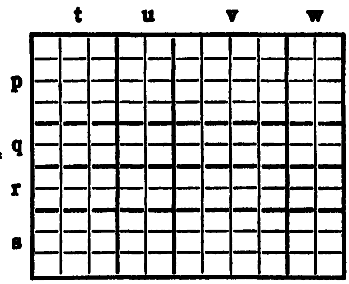
$$\mathbf{A}_3 = \begin{array}{c|c|c|} & a & b \\ \hline d & 0 & 5 \\ \hline e & 1 & 0 \\ \hline f & 0 & 0 \\ \hline g & 7 & -6 \\ \hline \end{array} \qquad \mathbf{A}_4 = \begin{array}{c|c|c|c|c|c|} & c & d & e & f & g \\ \hline d & 0 & 5 & 0 & 2 & 0 \\ \hline e & 0 & 0 & -3 & 0 & 9 \\ \hline f & 5 & 0 & 0 & 7 & 0 \\ \hline g & 0 & 0 & 8 & 0 & 4 \\ \hline \end{array} \qquad 9.5$$

so that $\mathbf{A}$ is expressed as

$$\mathbf{A} = \begin{array}{c|c|c|} & r & s \\ \hline p & \mathbf{A}_1 & \mathbf{A}_2 \\ \hline q & \mathbf{A}_3 & \mathbf{A}_4 \\ \hline \end{array} \qquad 9.6$$

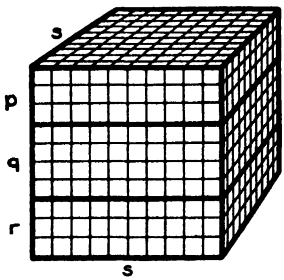
that is, the original 2-tensor is expressed as a *square* compound 2-tensor in which each component itself is a 2-tensor instead of a scalar.

As another example the following 2-tensor is divided into sixteen smaller 2-tensors as

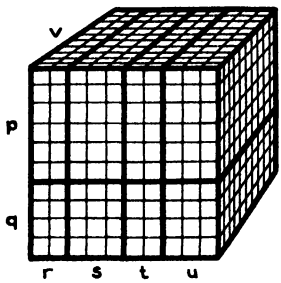
$z =$ 

$$= \begin{array}{c|cccc} & t & u & v & w \\ \hline p & Z_1 & Z_2 & Z_3 & Z_4 \\ q & Z_5 & Z_6 & Z_7 & Z_8 \\ r & Z_9 & Z_{10} & Z_{11} & Z_{12} \\ s & Z_{13} & Z_{14} & Z_{15} & Z_{16} \end{array} \tag{9.7}$$

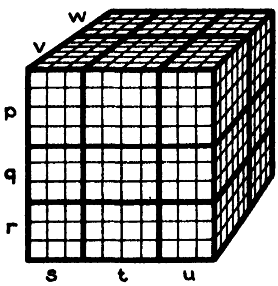
(*d*) Similarly a tensor of valence three (briefly, a 3-tensor) may be expressed as a compound 3-tensor in which each component is a 3-tensor, by dividing it with horizontal, or vertical, or both types of planes as shown in Fig. 9.1.



$$= \begin{array}{c|c} s\diagdown & s \\ \hline p & A_1 \\ q & A_2 \\ r & A_3 \end{array}$$



$$= \begin{array}{c|cccc} v\diagdown & r & s & t & u \\ \hline p & A_1 & A_2 & A_3 & A_4 \\ q & A_5 & A_6 & A_7 & A_8 \end{array}$$



$$= \begin{array}{c|ccc} v\diagdown & s & t & u \\ \hline p & A_1 & A_2 & A_3 \\ q & A_4 & A_5 & A_6 \\ r & A_7 & A_8 & A_9 \end{array} + \begin{array}{c|ccc} w\diagdown & s & t & u \\ \hline p & A_{10} & A_{11} & A_{12} \\ q & A_{13} & A_{14} & A_{15} \\ r & A_{16} & A_{17} & A_{18} \end{array}$$

FIG. 9.1.—Different Types of Subdivision of a Tensor of Valence Three

(*e*) In general, the ***components have the same valence as the original tensor that is subdivided,*** since a 2-tensor can be divided only into 2-tensors (with a one-rowed 2-tensor as a special case), a 3-tensor can be divided only into 3-tensors, etc. The manner of subdivision produces a ***compound*** tensor whose valence is also equal to that of the original tensor.

It will be shown in Chapter XXI that the " compound " tensors considered now (where the original tensor, the resultant tensor, and the component tensors all have the same valence) are special cases of " multiple tensors " with more complex structures.

III. DIFFERENCE BETWEEN TENSORS, GEOMETRIC OBJECTS, AND N-MATRICES

(*a*) In subdividing a tensor into several smaller tensors, care must often be taken to observe whether the component parts are tensors, geometric objects, or n-matrices.

It should be recalled that *with an n-matrix no reference axes are associated.* However, the fixed indices *always* have to be attached to every geometric object or tensor. Also it should be recalled that a geometric object has a different transformation formula in general from a tensor, while an n-matrix has no formula of transformation.

When each of the subdivisions is a geometric object or a tensor, then it is possible to change each partial group of reference axes into other types of reference axes without disturbing the remaining group of axes. Examples of such partial change of axes will be shown in Chapter XX.

(*b*) For instance, let the components of a compound 2-tensor under consideration along some particular reference frame be

$$
\begin{array}{c|c|c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
1 & A & B & E & F & S & T & U \\ \hline
2 & C & D & G & H & V & Z & W \\ \hline
3 & E & G & P & Q & I & J & K \\ \hline
z = 4 & F & H & R & S & L & M & N \\ \hline
5 & I & L & D & G & J & K & L \\ \hline
6 & J & M & E & H & M & U & P \\ \hline
7 & K & N & F & I & Q & R & S \\ \hline
\end{array}
\tag{9.8}
$$

Since some of the smaller components are identical, the question is

whether it is allowed to represent these components by the same base letters or by different letters, as

$$\mathbf{z} = \begin{array}{|c|c|c|}\hline \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \hline \mathbf{B}_t & \mathbf{E} & \mathbf{G}_t \\ \hline \mathbf{G} & \mathbf{H} & \mathbf{J} \\ \hline \end{array} \quad 9.9 \qquad \text{or } \mathbf{z} = \begin{array}{|c|c|c|}\hline \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \hline \mathbf{D} & \mathbf{E} & \mathbf{F} \\ \hline \mathbf{G} & \mathbf{H} & \mathbf{J} \\ \hline \end{array} \quad 9.10$$

(*c*) Now if each component is a ***tensor or a geometric object*** (that is, if it is intended to change its reference axes), then the two components **G** and **F**

$$\mathbf{G} = \begin{array}{c|c|c|} & 1 & 2 \\ \hline 5 & I & L \\ \hline 6 & J & M \\ \hline 7 & K & N \\ \hline \end{array} \qquad \mathbf{F} = \begin{array}{c|c|c|c|} & 5 & 6 & 7 \\ \hline 3 & I & J & K \\ \hline 4 & L & M & N \\ \hline \end{array} \quad 9.11$$

cannot both be represented by the same base letter **G, *since they are expressed along different reference axes, even though all their components are identical.*** In other words, the two tensors **G** and **F** happen in these particular reference frames to have the same components. However, in general in any other reference frame they have different components.

But if **G** and **F** are 2-matrices and the reference axes are absent then **F** may be written as $\mathbf{G}_t$.

(*d*) On the other hand, considering **B** and **D**

$$\mathbf{B} = \begin{array}{c|c|c|} & 3 & 4 \\ \hline 1 & E & F \\ \hline 2 & G & H \\ \hline \end{array} \qquad \mathbf{D} = \begin{array}{c|c|c|} & 1 & 2 \\ \hline 3 & E & G \\ \hline 4 & F & H \\ \hline \end{array}$$

they may be represented by the same base letter **B** and $\mathbf{B}_t$ if they are ***tensors,*** since they have the same components and are expressed along the same reference frames. In any other reference frames they still have the same components.

However, if **B** is a tensor and **D** a geometric object, that is, ***if* B *and* D *have different transformation formulas,*** then they cannot be represented by the same base letter, since in any other reference frame they may have different components. Hence:

1. ***Two n-matrices are identical*** (and are denoted by the same base letter) ***if they have the same components.***

2. ***Two tensors are identical if they have the same components and the same reference axes.***

3. *Two geometric objects are identical if they have the same components, the same reference axes, and the same formulas of transformation.*

In order to make the formulas of this chapter as general as possible, it will be assumed that each component of a compound tensor has a different base letter (the letters z_1, z_2, z_3 are considered as different base letters).

IV. MANIPULATION OF COMPOUND TENSORS

(*a*) *Compound tensors can be added, multiplied, differentiated, etc., just like ordinary tensors, with additional precautions* to avoid confusion. These precautions will be taken up as they are needed.

The sum of two compound vectors is a compound vector

$$X = \begin{array}{ccc} p & q & r \\ \hline \multicolumn{1}{|c|}{A_1} & \multicolumn{1}{c|}{A_2} & \multicolumn{1}{c|}{A_3} \\ \hline \end{array} \qquad Y = \begin{array}{ccc} p & q & r \\ \hline \multicolumn{1}{|c|}{B_1} & \multicolumn{1}{c|}{B_2} & \multicolumn{1}{c|}{B_3} \\ \hline \end{array} \tag{9.12}$$

$$X + Y = Z = \begin{array}{ccc} p & q & r \\ \hline \multicolumn{1}{|c|}{A_1 + B_1} & \multicolumn{1}{c|}{A_2 + B_2} & \multicolumn{1}{c|}{A_3 + B_3} \\ \hline \end{array}$$

The vectors to be added as $A_1 + B_1$, etc., must have the same number of components and the same individual indices.

The transpose of a compound 2-tensor in which each component is also a 2-tensor is taken by interchanging rows and columns as usual. In addition the transpose of each of the components is also taken. For instance

$$z = \begin{array}{|c|c|c|} \hline A & B & C \\ \hline D & E & F \\ \hline G & H & J \\ \hline \end{array} \qquad z_t = \begin{array}{|c|c|c|} \hline A_t & D_t & G_t \\ \hline B_t & E_t & H_t \\ \hline C_t & F_t & J_t \\ \hline \end{array} \tag{9.13}$$

(*b*) *Two compound tensors are multiplied together as if each component were a scalar. However, the order of the component tensors in a product cannot be disturbed.*

For instance, the product of a compound 2-tensor and a compound vector gives a compound vector

$$\begin{array}{c|c|c|c|c|} & p & q & r & s \\ \hline p & Z_1 & Z_2 & Z_3 & Z_4 \\ \hline q & Z_5 & Z_6 & Z_7 & Z_8 \\ \hline r & Z_9 & Z_{10} & Z_{11} & Z_{12} \\ \hline s & Z_{13} & Z_{14} & Z_{15} & Z_{16} \\ \hline \end{array} \cdot \begin{array}{c|c|} \hline p & i_1 \\ \hline q & i_2 \\ \hline r & i_3 \\ \hline s & i_4 \\ \hline \end{array} = \begin{array}{c|c|} \hline p & Z_1 \cdot i_1 + Z_2 \cdot i_2 + Z_3 \cdot i_3 + Z_4 \cdot i_4 \\ \hline q & Z_5 \cdot i_1 + Z_6 \cdot i_2 + Z_7 \cdot i_3 + Z_8 \cdot i_4 \\ \hline r & Z_9 \cdot i_1 + Z_{10} \cdot i_2 + Z_{11} \cdot i_3 + Z_{12} \cdot i_4 \\ \hline s & Z_{13} \cdot i_1 + Z_{14} \cdot i_2 + Z_{15} \cdot i_3 + Z_{16} \cdot i_4 \\ \hline \end{array} \tag{9.14}$$

The multiplications and additions indicated in each component are still to be performed.

The product of two compound 2-tensors is a compound 2-tensor

$$
\mathbf{X} = \begin{array}{c|c|c|c|}
 & p & q & r \\
\hline
s & A & B & C \\
\hline
t & D & E & F \\
\hline
u & G & H & J \\
\hline
\end{array}
\qquad
\mathbf{Y} = \begin{array}{c|c|c|c|}
 & v & w & z \\
\hline
p & K & L & M \\
\hline
q & N & P & Q \\
\hline
r & R & S & T \\
\hline
\end{array}
$$

$$
\mathbf{X}\cdot\mathbf{Y} = \begin{array}{c|c|c|c|}
 & v & w & z \\
\hline
s & A\cdot K + B\cdot N + C\cdot R & A\cdot L + B\cdot P + C\cdot S & A\cdot M + B\cdot Q + C\cdot T \\
\hline
t & D\cdot K + E\cdot N + F\cdot R & D\cdot L + E\cdot P + F\cdot S & D\cdot M + C\cdot Q + F\cdot T \\
\hline
u & G\cdot K + H\cdot N + J\cdot R & G\cdot L + H\cdot P + J\cdot S & G\cdot M + H\cdot Q + J\cdot T \\
\hline
\end{array} \tag{9.15}
$$

In the components the product, say **A** · **K**, cannot be written as **K** · **A**. It is important to note that:

1. Along the arrows the original tensors **X** and **Y** must have the same *individual fixed* indices.

2. Along the arrows the original tensors must be divided in a similar manner, that is, they must have also the same *compound fixed* indices.

(*c*) If the components in the compound tensors are identical, the final tensor contains several identical products, thereby *reducing greatly the necessary number of multiplications to be performed.* For instance,

$$
\mathbf{X} = \begin{array}{|c|c|c|}
\hline
A & B & B \\
\hline
B & A & B \\
\hline
B & B & A \\
\hline
\end{array}
\qquad
\mathbf{Y} = \begin{array}{|c|c|c|}
\hline
D & C & E \\
\hline
C & E & D \\
\hline
E & D & C \\
\hline
\end{array}
$$

$$
\mathbf{X}\cdot\mathbf{Y} = \begin{array}{|c|c|c|}
\hline
A\cdot D + B\cdot C + B\cdot E & A\cdot C + B\cdot E + B\cdot D & A\cdot E + B\cdot D + B\cdot C \\
\hline
B\cdot D + A\cdot C + B\cdot E & B\cdot C + A\cdot E + B\cdot D & B\cdot E + A\cdot D + B\cdot C \\
\hline
B\cdot D + B\cdot C + A\cdot E & B\cdot C + B\cdot E + A\cdot D & B\cdot E + B\cdot D + A\cdot C \\
\hline
\end{array} \tag{9.16}
$$

Only six products of the component tensors have to be further evaluated, namely, **A** · **C**, **A** · **D**, **A** · **E** and **B** · **C**, **B** · **D**, **B** · **E**.

If the original tensors **X** and **Y** are not replaced by compound tensors in the present case, the multiplication of **X** and **Y** involves an

amount of work equivalent to finding twenty-seven such products instead of six. If the original tensors involve a large number of rows, *the saving in the numerical calculations is considerable.*

(*d*) If the original tensors involve a large number of rows or columns or layers, etc., say twenty or more, then *even in the absence of any symmetry* it is worth while to divide them into component tensors and so to calculate first their products and then each product of the smaller tensors.

When a compound tensor is subdivided *arbitrarily* in order to reduce the amount of numerical calculation, its components are n-matrices instead of tensors, since no physical or geometrical interpretation can be attached to the components.

V. AN EXAMPLE OF MULTIPLICATION OF COMPOUND 2-TENSORS

(*a*) For instance, let the product $\mathbf{z}' = \mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ be found where the 2-tensors $\mathbf{z}$ and $\mathbf{C}$ have many components involving certain symmetry. Dividing $\mathbf{z}$ and $\mathbf{C}$ into component 2-matrices, they become

$$\mathbf{z} = \begin{array}{|ccc|ccccc|ccc|ccccc|}
\hline
a & v & c & k & l & m & n & p & a & b & c & k & l & m & n & p \\
d & e & f & q & r & s & t & u & d & e & f & q & r & s & t & u \\
g & h & j & v & w & x & y & z & g & h & j & v & w & x & y & z \\
\hline
k & q & v & m & a & n & b & p & k & q & v & m & a & n & b & p \\
l & r & w & c & q & d & r & e & l & r & w & c & q & d & r & e \\
m & s & x & s & f & t & g & u & m & s & x & s & f & t & g & u \\
n & t & y & h & v & i & w & j & n & t & y & h & v & i & w & j \\
p & u & z & x & k & y & l & z & p & u & z & x & k & y & l & z \\
\hline
a & b & c & k & l & m & n & p & a & b & c & k & l & m & n & p \\
d & e & f & q & r & s & t & u & d & e & f & q & r & s & t & u \\
g & h & j & v & w & x & y & z & g & h & j & v & w & x & y & z \\
\hline
k & q & v & m & a & n & b & p & k & q & v & m & a & n & b & p \\
l & r & w & c & q & d & r & e & l & r & w & c & q & d & r & e \\
m & s & x & s & f & t & g & u & m & s & x & s & f & t & g & u \\
n & t & y & h & v & i & w & j & n & t & y & h & v & i & w & j \\
p & u & z & x & k & y & l & z & p & u & z & x & k & y & l & z \\
\hline
\end{array}
\;\xrightarrow{\hspace{0pt}}\; = \begin{array}{|c|c|c|c|}
\hline
\mathbf{A} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\
\hline
\mathbf{B}_t & \mathbf{C} & \mathbf{B}_t & \mathbf{C} \\
\hline
\mathbf{A} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\
\hline
\mathbf{B}_t & \mathbf{C} & \mathbf{B}_t & \mathbf{C} \\
\hline
\end{array} \qquad 9.17$$

$$\mathbf{C} = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & 1 & & \\ \hline & & & & & 1 & \\ \hline & & & & & & 1 \\ \hline 1 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 1 & & & \\ \hline -1 & & & & & & \\ \hline & & & & 1 & & \\ \hline & & & & & 1 & \\ \hline & & & & & & 1 \\ \hline 1 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 1 & & & \\ \hline -1 & & & & & & \\ \hline \end{array} = \begin{array}{|c|c|} \hline \mathbf{0} & \mathbf{I} \\ \hline \mathbf{J} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \\ \hline \mathbf{J} & \mathbf{0} \\ \hline \end{array} \qquad \mathbf{C}_t = \begin{array}{|c|c|c|c|} \hline \mathbf{0} & \mathbf{J}_t & \mathbf{0} & \mathbf{J}_t \\ \hline \mathbf{I}_t & \mathbf{0} & \mathbf{I}_t & \mathbf{0} \\ \hline \end{array} \tag{9.18}$$

where the component 2-matrices are

$$\mathbf{A} = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & f \\ \hline g & h & j \\ \hline \end{array} \quad \mathbf{B} = \begin{array}{|c|c|c|c|c|} \hline k & l & m & n & p \\ \hline q & r & s & t & u \\ \hline v & w & x & y & z \\ \hline \end{array} \quad \mathbf{C} = \begin{array}{|c|c|c|c|c|} \hline m & a & n & b & p \\ \hline c & q & d & r & e \\ \hline s & f & t & g & u \\ \hline h & v & i & w & j \\ \hline x & k & y & l & z \\ \hline \end{array} \tag{9.19}$$

$$\mathbf{I} = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array} \qquad \mathbf{J} = \begin{array}{|c|c|c|c|} \hline 1 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline -1 & & & \\ \hline \end{array}$$

while the 2-matrices **0** contain all zero components. *It should be noted that no compound fixed indices are associated with the 2-matrices.*

It should also be noted that, *along the arrows* (along which the

multiplication $\mathbf{z} \cdot \mathbf{C}$ is to be performed), each component matrix has the same number of components in their proper order, namely, 3–5–3–5 along both arrows. *Along the other two directions the manner of division is arbitrary.*

(*b*) The product $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ is found just as if each component were an ordinary number. The first step is

$$\mathbf{z}\cdot\mathbf{C} = \begin{array}{|c|c|c|c|}\hline \mathbf{A} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \hline \mathbf{B}_t & \mathbf{C} & \mathbf{B}_t & \mathbf{C} \\ \hline \mathbf{A} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \hline \mathbf{B}_t & \mathbf{C} & \mathbf{B}_t & \mathbf{C} \\ \hline \end{array} \cdot \begin{array}{|c|c|}\hline & \mathbf{I} \\ \hline \mathbf{J} & \\ \hline & \mathbf{I} \\ \hline \mathbf{J} & \\ \hline \end{array} = \begin{array}{|c|c|}\hline 2\mathbf{B}\cdot\mathbf{J} & 2\mathbf{A}\cdot\mathbf{I} \\ \hline 2\mathbf{C}\cdot\mathbf{J} & 2\mathbf{B}_t\cdot\mathbf{I} \\ \hline 2\mathbf{B}\cdot\mathbf{J} & 2\mathbf{A}\cdot\mathbf{I} \\ \hline 2\mathbf{C}\cdot\mathbf{J} & 2\mathbf{B}_t\cdot\mathbf{I} \\ \hline \end{array}$$

The second step is

$$\mathbf{C}_t\cdot(\mathbf{z}\cdot\mathbf{C}) = \begin{array}{|c|c|c|c|}\hline & \mathbf{I}_t & & \mathbf{I}_t \\ \hline \mathbf{I}_t & & \mathbf{I}_t & \\ \hline \end{array} \cdot \begin{array}{|c|c|}\hline 2\mathbf{B}\cdot\mathbf{J} & 2\mathbf{A}\cdot\mathbf{I} \\ \hline 2\mathbf{C}\cdot\mathbf{J} & 2\mathbf{B}_t\cdot\mathbf{I} \\ \hline 2\mathbf{B}\cdot\mathbf{J} & 2\mathbf{A}\cdot\mathbf{I} \\ \hline 2\mathbf{C}\cdot\mathbf{J} & 2\mathbf{B}_t\cdot\mathbf{I} \\ \hline \end{array} =$$

$$= \mathbf{z}' = \begin{array}{|c|c|}\hline 4\mathbf{J}_t\cdot\mathbf{C}\cdot\mathbf{J} & 4\mathbf{J}_t\cdot\mathbf{B}_t\cdot\mathbf{I} \\ \hline 4\mathbf{I}_t\cdot\mathbf{B}\cdot\mathbf{J} & 4\mathbf{I}_t\cdot\mathbf{A}\cdot\mathbf{I} \\ \hline \end{array} \qquad 9.20$$

The final product $\mathbf{z}'$ comes out as a compound 2-tensor.

(*c*) To express $\mathbf{z}'$ as an ordinary 2-tensor with $7 \times 7 = 49$ components, the indicated four multiplications are to be performed, namely

$$\mathbf{C}\cdot\mathbf{J} = \begin{array}{|c|c|c|c|c|}\hline m & a & n & b & p \\ \hline c & q & d & r & e \\ \hline s & f & t & g & u \\ \hline h & v & i & w & j \\ \hline x & k & y & l & z \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline -1 & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline m-p & a & n & b \\ \hline c-e & q & d & r \\ \hline s-u & f & t & g \\ \hline h-j & v & i & w \\ \hline x-z & k & y & l \\ \hline \end{array}$$

$$\mathbf{J}_t\cdot\mathbf{C}\cdot\mathbf{J} = \begin{array}{|c|c|c|c|}\hline m-p-x+z & a-k & n-y & b-l \\ \hline c-e & q & d & r \\ \hline s-u & f & t & g \\ \hline h-j & v & i & w \\ \hline \end{array} \qquad 9.21$$

$\mathbf{J}_t \cdot \mathbf{B}_t \cdot \mathbf{I} =$

$k-p$	$q-u$	$v-z$
l	r	w
m	s	x
n	t	y

Also $\mathbf{I}_t \cdot \mathbf{A} \cdot \mathbf{I} = \mathbf{A}$ and $\mathbf{I}_t \cdot \mathbf{B} \cdot \mathbf{J} = (\mathbf{J}_t \cdot \mathbf{B}_t \cdot \mathbf{I})_t$.

Hence the final product 2-tensor $\mathbf{z}' = \mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ is

$\mathbf{z}' = 4 \times$ (9.22)

$m-p-x+z$	$a-k$	$n-y$	$b-l$	$k-p$	$q-u$	$v-z$
$c-e$	q	d	r	l	r	w
$s-u$	f	t	g	m	s	x
$h-j$	v	i	w	n	t	y
$k-p$	l	m	n	a	b	c
$q-u$	r	s	t	d	e	f
$v-z$	w	x	y	g	h	j

VI. MULTIPLY COMPOUND TENSORS

(a) *Just as ordinary tensors may be subdivided into several components, similarly each component of a compound tensor may be subdivided into several components in an entirely analogous manner, forming a "doubly compound tensor."*

For instance, the compound tensor $\mathbf{z}$ which contains actually, say, twenty-eight rows and columns becomes a doubly component tensor when again it is subdivided as

		u			v			w		t	
	$\mathbf{z}_1$		$\mathbf{z}_2$				$\mathbf{z}_3$			$\mathbf{z}_4$	
p		$\mathbf{z}_5$			$\mathbf{z}_6$		$\mathbf{z}_7$		$\mathbf{z}_8$		$\mathbf{z}_9$
	$\mathbf{z}_{10}$			$\mathbf{z}_{11}$				$\mathbf{z}_{12}$		$\mathbf{z}_{13}$	
q			$\mathbf{z}_{14}$		$\mathbf{z}_{15}$	$\mathbf{z}_{16}$		$\mathbf{z}_{17}$	$\mathbf{z}_{18}$		$\mathbf{z}_{19}$
	$\mathbf{z}_{20}$	$\mathbf{z}_{21}$	$\mathbf{z}_{22}$				$\mathbf{z}_{23}$			$\mathbf{z}_{24}$	
$\mathbf{z} =$		$\mathbf{z}_{25}$		$\mathbf{z}_{26}$				$\mathbf{z}_{27}$			$\mathbf{z}_{28}$
r		$\mathbf{z}_{29}$		$\mathbf{z}_{30}$			$\mathbf{z}_{31}$			$\mathbf{z}_{32}$	
			$\mathbf{z}_{33}$					$\mathbf{z}_{34}$			$\mathbf{z}_{35}$
	$\mathbf{z}_{36}$					$\mathbf{z}_{37}$					
s			$\mathbf{z}_{38}$					$\mathbf{z}_{39}$		$\mathbf{z}_{40}$	
	$\mathbf{z}_{41}$				$\mathbf{z}_{42}$			$\mathbf{z}_{43}$			$\mathbf{z}_{44}$

$=$

	u	v	w	t
p	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{A}_4$
q	$\mathbf{A}_5$	$\mathbf{A}_6$	$\mathbf{A}_7$	$\mathbf{A}_8$
r	$\mathbf{A}_9$	$\mathbf{A}_{10}$	$\mathbf{A}_{11}$	$\mathbf{A}_{12}$
s	$\mathbf{A}_{13}$	$\mathbf{A}_{14}$	$\mathbf{A}_{15}$	$\mathbf{A}_{16}$

9.23

where $\mathbf{A}_1$, $\mathbf{A}_2$, etc. are compound tensors. The indices $\mathbf{u}, \mathbf{v}, \mathbf{w}, \cdots$ are *doubly compound indices*, each representing several compound indices $\mathbf{m}, \mathbf{n} \cdots$.

(*b*) As another example a tensor of valence three may be expressed as a *doubly* compound 3-tensor, shown in Fig. 9.2, containing eight compound 3-tensors as its components (shown on Fig. 9.2 as separated), each compound 3-tensor containing two, four, or eight unequal 3-tensors (shown with heavy lines).

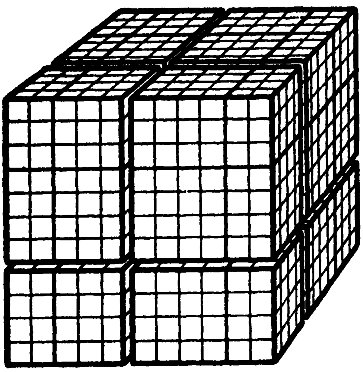


FIG. 9.2.—Doubly Compound Tensor of Valence Three

(*c*) Similarly each component of a "doubly compound tensor" itself may also be subdivided into component tensors. It may be called a "*triply compound tensor.*" For instance, the above original 2-tensor $\mathbf{z}$ may be expressed as a "triply compound 2-tensor" as

$$\mathbf{z} = \begin{array}{c|cc|cc|} & \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{t} \\ \hline \mathbf{p} & \mathbf{A} & \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{A}_4 \\ \mathbf{q} & \mathbf{A}_5 & \mathbf{A}_6 & \mathbf{A}_7 & \mathbf{A}_8 \\ \hline \mathbf{r} & \mathbf{A}_9 & \mathbf{A}_{10} & \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{s} & \mathbf{A}_{13} & \mathbf{A}_{14} & \mathbf{A}_{15} & \mathbf{A}_{16} \\ \hline \end{array} = \begin{array}{c|c|c|} & \mathbf{x} & \mathbf{y} \\ \hline \mathbf{x} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{y} & \mathbf{B}_3 & \mathbf{B}_4 \\ \hline \end{array} \tag{9.24}$$

so that the original 2-tensor $\mathbf{z}$ containing twenty-eight rows and columns has been subdivided as shown in Fig. 9.3.

This successive subdivision of a tensor may be continued indefinitely as the complexity of the problem increases, forming in general " multiply compound tensors " of any complexity, in which the various subdivisions may contain unequal number of component tensors and also may be tensors or geometric objects or *n*-matrices in any combination.

(*d*) The importance of multiply compound tensors is due to the fact that *any reasoning, any formula, or any equation developed for ordinary tensors is also valid without any change for " multiply compound tensors " of any complexity, and vice versa.*

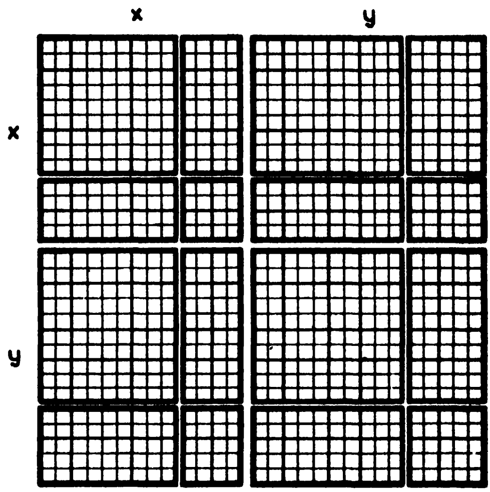


FIG. 9.3.—Triply Compound Tensor of Valence Two

VII. SUBDIVISION OF LINEAR EQUATIONS

(*a*) In the previous sections it was shown how a geometric object may be subdivided into several geometric objects. Now it will be shown how a single invariant equation may be subdivided into several invariant equations, each component equation representing several ordinary equations that behave in an analogous manner during the analysis. ***The subdivision of invariant equations represents a very flexible tool for the analysis and synthesis of systems in which the various reference axes have different physical significance.*** A more detailed treatment is given in Chapters XXI and XXII.

(*b*) Let a set of linear equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ be given, represented as

$$\begin{array}{c} \\ \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 7 \\ \hline 4 \\ \hline 3 \\ \hline 6 \\ \hline 5 \\ \hline \end{array} \end{array} \quad \begin{array}{c} \\ = \end{array} \quad \begin{array}{c} \begin{array}{|c|c|c|c|c|c|c|} \hline i^{\cdot 1} & i^{\cdot 2} & i^{\cdot 3} & i^{\cdot 4} & i^{\cdot 5} & i^{\cdot 6} & i^{\cdot 7} \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 0 & 2 & 0 & 3 & 4 & 4 \\ \hline 0 & 5 & 1 & 6 & 3 & 7 & 0 \\ \hline 8 & 0 & 9 & 0 & 2 & 0 & 1 \\ \hline 0 & 1 & 5 & 3 & 1 & 4 & 2 \\ \hline 5 & 0 & 6 & 6 & 7 & 5 & 8 \\ \hline 0 & 9 & 8 & 0 & 0 & 1 & 0 \\ \hline 2 & 9 & 3 & 0 & 4 & 7 & 5 \\ \hline \end{array} \end{array} \qquad 9.25$$

The single invariant equation can be divided into two invariant equations according to the requirement of the problem, by, say, a *horizontal* line, giving

$$\begin{array}{ccc} & & \boxed{\mathbf{i}} \\ & & \cdot \\ \begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline\end{array} & = & \begin{array}{|c|}\hline \mathbf{z}_1 \\ \hline \mathbf{z}_2 \\ \hline\end{array}\end{array} \qquad \text{or} \qquad \begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline\end{array} = \overrightarrow{\begin{array}{|c|}\hline \mathbf{z}_1 \\ \hline \mathbf{z}_2 \\ \hline\end{array}} \cdot \boxed{\mathbf{i}}\downarrow \tag{9.26}$$

If the multiplication is performed as indicated by the arrows, the result is

$$\begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline\end{array} = \begin{array}{|c|}\hline \mathbf{z}_1 \cdot \mathbf{i} \\ \hline \mathbf{z}_2 \cdot \mathbf{i} \\ \hline\end{array} \tag{9.27}$$

Equating each component of the two vectors separately, the single equation $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ *can be written as two equations.*

$$\left.\begin{array}{ccc} & & \begin{array}{|c|c|c|c|c|c|c|}\hline i^1 & i^2 & i^3 & i^4 & i^5 & i^6 & i^7 \\ \hline\end{array} \\ & & \cdot \\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 7 \\ \hline\end{array} & = & \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 0 & 2 & 0 & 3 & 4 & 4 \\ \hline 0 & 5 & 1 & 6 & 3 & 7 & 0 \\ \hline 8 & 0 & 9 & 0 & 2 & 0 & 1 \\ \hline\end{array}\end{array}\right\} \quad \mathbf{e}_1 = \mathbf{z}_1 \cdot \mathbf{i}$$

and

$$\left.\begin{array}{ccc} & & \begin{array}{|c|c|c|c|c|c|c|}\hline i^1 & i^2 & i^3 & i^4 & i^5 & i^6 & i^7 \\ \hline\end{array} \\ & & \cdot \\ \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 6 \\ \hline 5 \\ \hline\end{array} & = & \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 1 & 5 & 3 & 1 & 4 & 2 \\ \hline 5 & 0 & 6 & 6 & 7 & 5 & 8 \\ \hline 0 & 9 & 8 & 0 & 0 & 1 & 0 \\ \hline 2 & 9 & 3 & 0 & 4 & 7 & 5 \\ \hline\end{array}\end{array}\right\} \quad \mathbf{e}_2 = \mathbf{z}_2 \cdot \mathbf{i} \tag{9.28}$$

(*c*) The same set of equations can be divided by *a horizontal and a vertical* line into two invariant equations as

$$\begin{array}{ccc} & & \begin{array}{|c|c|}\hline \mathbf{i}^1 & \mathbf{i}^2 \\ \hline\end{array} \\ & & \cdot \\ \begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline\end{array} & = & \begin{array}{|c|c|}\hline \mathbf{z}_1 & \mathbf{z}_2 \\ \hline \mathbf{z}_3 & \mathbf{z}_4 \\ \hline\end{array}\end{array} \qquad \text{or} \qquad \begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline\end{array} = \underrightarrow{\begin{array}{|c|c|}\hline \mathbf{z}_1 & \mathbf{z}_2 \\ \hline \mathbf{z}_3 & \mathbf{z}_4 \\ \hline\end{array}} \cdot \begin{array}{|c|}\hline \mathbf{i}^1 \\ \hline \mathbf{i}^2 \\ \hline\end{array}\Bigg\downarrow \tag{9.29}$$

Performing the multiplication of compound tensors as indicated by the arrows yields the result

$$\begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline \mathbf{e}_1 \\ \hline \end{array} = \begin{array}{|c|}\hline \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 \\ \hline \mathbf{z}_3 \cdot \mathbf{i}^1 + \mathbf{z}_4 \cdot \mathbf{i}^2 \\ \hline \end{array} \qquad 9.30$$

Equating each components of the two vectors separately, the single invariant equation $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ *can be written as two invariant equations*:

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 \\ \mathbf{e}_2 &= \mathbf{z}_3 \cdot \mathbf{i}^1 + \mathbf{z}_4 \cdot \mathbf{i}^2 \end{aligned} \qquad 9.31$$

It should be noted that cross-products as $\mathbf{z}_1 \cdot \mathbf{i}^2$ do not enter.

(*d*) The same set of linear equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ may be divided *by two horizontal and two vertical lines* into *three* invariant equations

$$\begin{array}{ccc} & & \begin{array}{|c|c|c|}\hline \mathbf{i}^1 & \mathbf{i}^2 & \mathbf{i}^3 \\ \hline \end{array} \\ & & \cdot \\ \begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline \mathbf{e}_3 \\ \hline \end{array} & = & \begin{array}{|c|c|c|}\hline \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 \\ \hline \mathbf{z}_4 & \mathbf{z}_5 & \mathbf{z}_6 \\ \hline \mathbf{z}_7 & \mathbf{z}_8 & \mathbf{z}_9 \\ \hline \end{array} \end{array} \quad \text{or} \quad \begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline \mathbf{e}_3 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 \\ \hline \mathbf{z}_4 & \mathbf{z}_5 & \mathbf{z}_6 \\ \hline \mathbf{z}_7 & \mathbf{z}_8 & \mathbf{z}_9 \\ \hline \end{array} \cdot \begin{array}{|c|}\hline \mathbf{i}^1 \\ \hline \mathbf{i}^2 \\ \hline \mathbf{i}^3 \\ \hline \end{array} \qquad 9.32$$

Performing the multiplication as indicated by the arrows results in

$$\begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline \mathbf{e}_3 \\ \hline \end{array} = \begin{array}{|c|}\hline \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 + \mathbf{z}_3 \cdot \mathbf{i}^3 \\ \hline \mathbf{z}_4 \cdot \mathbf{i}^1 + \mathbf{z}_5 \cdot \mathbf{i}^2 + \mathbf{z}_6 \cdot \mathbf{i}^3 \\ \hline \mathbf{z}_7 \cdot \mathbf{i}^1 + \mathbf{z}_8 \cdot \mathbf{i}^2 + \mathbf{z}_9 \cdot \mathbf{i}^3 \\ \hline \end{array} \qquad 9.33$$

Equating each component of the two compound vectors separately, the single equation $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ may be written as

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 + \mathbf{z}_3 \cdot \mathbf{i}^3 \\ \mathbf{e}_2 &= \mathbf{z}_4 \cdot \mathbf{i}^1 + \mathbf{z}^5 \cdot \mathbf{i}^2 + \mathbf{z}_6 \cdot \mathbf{i}^3 \\ \mathbf{e}_3 &= \mathbf{z}_7 \cdot \mathbf{i}^1 + \mathbf{z}_8 \cdot \mathbf{i}^2 + \mathbf{z}_9 \cdot \mathbf{i}^3 \end{aligned} \qquad 9.34$$

each equation representing several scalar equations.

(*e*) Similarly $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ may be expressed as *four* equations by dividing $\mathbf{e}$ and $\mathbf{i}$ into four and $\mathbf{z}$ into sixteen components, giving

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 + \mathbf{z}_3 \cdot \mathbf{i}^3 + \mathbf{z}_4 \cdot \mathbf{i}^4 \\
\mathbf{e}_2 &= \mathbf{z}_5 \cdot \mathbf{i}^1 + \mathbf{z}_6 \cdot \mathbf{i}^2 + \mathbf{z}_7 \cdot \mathbf{i}^3 + \mathbf{z}_8 \cdot \mathbf{i}^4 \\
\mathbf{e}_3 &= \mathbf{z}_9 \cdot \mathbf{i}^1 + \mathbf{z}_{10} \cdot \mathbf{i}^2 + \mathbf{z}_{11} \cdot \mathbf{i}^3 + \mathbf{z}_{12} \cdot \mathbf{i}^4 \\
\mathbf{e}_4 &= \mathbf{z}_{13} \cdot \mathbf{i}^1 + \mathbf{z}_{14} \cdot \mathbf{i}^2 + \mathbf{z}_{15} \cdot \mathbf{i}^3 + \mathbf{z}_{16} \cdot \mathbf{i}^4
\end{aligned} \qquad 9.35$$

(*f*) Of course, in place of $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ any other set of linear equations $\mathbf{x} = \mathbf{A} \cdot \mathbf{y}$ can be subdivided in a similar manner. The equation $\mathbf{i} = \mathbf{y} \cdot \mathbf{e}$ is subdivided into three equations, as

$$\begin{aligned}
\mathbf{i}^1 &= \mathbf{y}^1 \cdot \mathbf{e}_1 + \mathbf{y}^2 \cdot \mathbf{e}_2 + \mathbf{y}^3 \cdot \mathbf{e}_3 \\
\mathbf{i}^2 &= \mathbf{y}^4 \cdot \mathbf{e}_1 + \mathbf{y}^5 \cdot \mathbf{e}_2 + \mathbf{y}^6 \cdot \mathbf{e}_3 \\
\mathbf{i}^3 &= \mathbf{y}^7 \cdot \mathbf{e}_1 + \mathbf{y}^8 \cdot \mathbf{e}_2 + \mathbf{y}^9 \cdot \mathbf{e}_3
\end{aligned} \qquad 9.36$$

Although $\mathbf{e}_1$ and $\mathbf{i}^1$ in general may have different number of components, similarly $\mathbf{e}_2$ and $\mathbf{i}^2$, etc., *in most problems* $\mathbf{e}_1$ *and* $\mathbf{i}^1$ *also* $\mathbf{e}_2$ *and* $\mathbf{i}^2$, *etc., will have the same number of components. This is equivalent to making the diagonal component 2-tensors, say* $\mathbf{z}_1$, $\mathbf{z}_6$, $\mathbf{z}_{11}$ *and* $\mathbf{z}_{16}$, *square* while the other components may assume any rectangular form. This assumption is made to allow the calculation of the inverse $\mathbf{z}_1^{-1}$, $\mathbf{z}_6^{-1}$, etc., of the diagonal tensors. The inverse of the non-diagonal 2-tensors can be found only if they are square.

(*g*) *The component equations such as* 9.35 *may be written down usually by inspection of the compound* $\mathbf{z}$, $\mathbf{e}$, *and* $\mathbf{i}$, *without performing such multiplications as indicated by equation* 9.33

VIII. COMPOUND TENSORS IN INDEX NOTATION *

(*a*) The manipulation of ordinary tensors is simpler if direct notation is used, as long as the manipulation is restricted to tensors of valence one and two, that is to vectors and 2-tensors, and as long as the manipulations are of elementary nature. As soon as tensors of valence three or more, or more complex manipulations, appear, the index notation must be resorted to.

A similar situation exists in respect to *compound* tensors of valence one and two, whose manipulation in general is simpler if direct nota-

* The remaining sections (except the last) of this chapter may be left out at the first reading.

tion is used since the indices are suppressed. Where the direct notation fails, the following index notation may be used.

(*b*) In case of ordinary tensors all the " fixed " indices $a, b, c, \cdots j$ are represented by each of the " variable " indices $\alpha, \beta, \gamma \cdots$, where α or β assume *all* the fixed indices in succession.

When the meshes are divided into *two* groups, then *the fixed indices are also divided into two groups*, say all meshes from a to e belong to the first group and those from f to j to the second group. *For each group of " fixed " indices a separate set of " variable " indices is assigned.* In the present case two sets of variable indices are used, say $k, l, m, n \cdots$ for the first group and $u, v, w \cdots$ for the second group, while $\alpha, \beta, \gamma \cdots$ are still retained to represent *all* the fixed indices, as shown in Table 9.1.

TABLE 9.1

Groups	Fixed Indices	Variable Indices
First Group	$a, b, c, \cdots$	$k, l, m, \cdots$
Second Group	$g, h, i, \cdots$	$u, v, w, \cdots$
Both Groups	$a, b, c \cdots g, h, i \cdots$	$\alpha, \beta, \gamma, \cdots$

The compound tensors are represented in both notations as

$$\begin{array}{cc} & \\ 1 & \boxed{\mathbf{e}_1} \\ 2 & \boxed{\mathbf{e}_2} \end{array} = \begin{array}{cc} 1 & 2 \\ \boxed{\mathbf{i}^1} & \boxed{\mathbf{i}^2} \\ & \cdot \\ \boxed{\mathbf{z}_1} & \boxed{\mathbf{z}_2} \\ \boxed{\mathbf{z}_3} & \boxed{\mathbf{z}_4} \end{array} \qquad \Bigg| \qquad \begin{array}{cc} & \\ k & \boxed{e_k} \\ u & \boxed{e_u} \end{array} = \begin{array}{cc} m & v \\ \boxed{i^m} & \boxed{i^v} \\ \boxed{z_{km}} & \boxed{z_{kv}} \\ \boxed{z_{um}} & \boxed{z_{uv}} \end{array} \qquad 9.37$$

That is, in direct notation one set of subscripts is used, $1, 2 \cdots$, *whereas in index notation two sets of subscripts are used* $k, m \cdots$, and $u, v \cdots$.

With the aid of the three sets of variable indices the equation of the *whole system* is written in both notations as

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i} \qquad \Big| \qquad e_\alpha = z_{\alpha\beta} i^\beta$$

and the two equations of the component systems as

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 \\ \mathbf{e}_1 &= \mathbf{z}^3 \cdot \mathbf{i}^1 + \mathbf{z}_4 \cdot \mathbf{i}^2 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} e_k &= z_{km} i^m + z_{kv} i^v \\ e_u &= z_{um} i^m + z_{uv} i^v \end{aligned} \qquad 9.38$$

IX. THE FLEXIBILITY OF THE INDEX NOTATION

(*a*) Although the direct notation uses fewer sets of variable indices, the index notation has far more flexibility. For instance, when the set of equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ is divided in a different manner, as

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{z}_5 \\ \mathbf{z}_6 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{i} \end{bmatrix} \qquad \Bigg| \qquad \begin{array}{c} k \\ u \end{array} \begin{bmatrix} e_k \\ e_u \end{bmatrix} = \begin{bmatrix} Z_{k\alpha} \\ Z_{u\alpha} \end{bmatrix} \overset{\alpha}{\begin{bmatrix} i^\alpha \end{bmatrix}} \qquad 9.39$$

the equations are

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_5 \cdot \mathbf{i} \\ \mathbf{e}_2 &= \mathbf{z}_6 \cdot \mathbf{i} \end{aligned} \qquad \Bigg| \qquad \begin{aligned} e_k &= Z_{k\alpha} i^\alpha \\ e_u &= Z_{u\alpha} i^\alpha \end{aligned} \qquad 9.40$$

In this case *the direct notation needs a new set of indices* 5, 6, *but the index notation does not need any.*

Or let $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ be divided as

$$\begin{bmatrix} \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_7 & \mathbf{z}_8 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{i}_1 & \mathbf{i}_2 \end{bmatrix} \qquad \Bigg| \qquad \alpha \begin{bmatrix} e_\alpha \end{bmatrix} = \begin{bmatrix} Z_{\alpha m} & Z_{\alpha v} \end{bmatrix} \begin{array}{c} \begin{array}{cc} m & v \end{array} \\ \begin{bmatrix} i^m & i^v \end{bmatrix} \end{array} \qquad 9.41$$

In this case the direct notation needs another set of new indices 7, 8, but not so the index notation.

(*b*) When a 3-tensor is associated with a system that is divided into

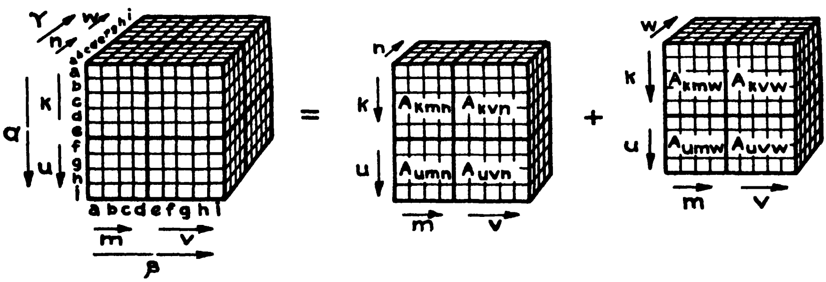


FIG. 9.4.—Subdivision of a Compound 3-Tensor

two groups, the same set of indices is used. That is, the 3-tensor $A_{\alpha\beta\gamma}$ is expressed as a compound 3-tensor in Fig. 9.4.

Other types of subdivision of the same 3-tensor may be expressed without introducing new variable indices as shown in Fig. 9.5.

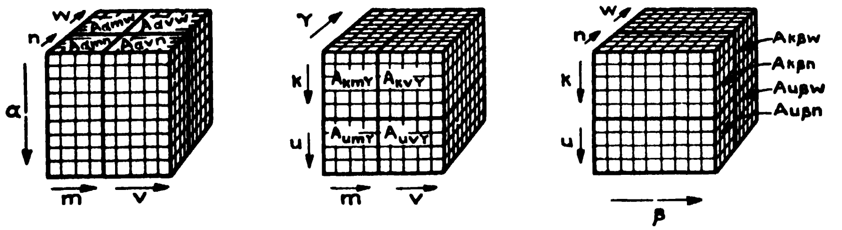


FIG. 9.5.—Subdivision of a Compound 3-Tensor

(*c*) *When particular rows or columns are to be picked out for special study, the index notation easily accomplishes it* as shown for a 3-tensor in Fig. 9.6.

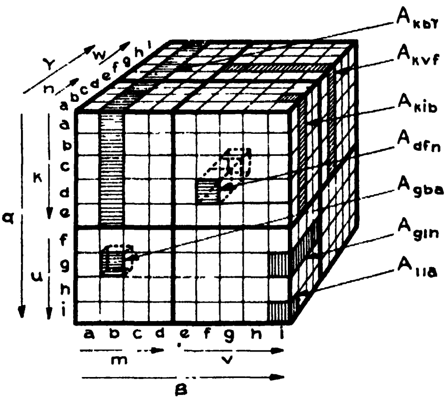


FIG. 9.6.—Various Subdivisions of a Compound 3-Tensor.

X. SEVERAL SETS OF VARIABLE INDICES

(*a*) The subdivision of a system of equations into more than two groups introduces *as many sets of variable indices as there are groups.*

For three groups a range of fixed and variable indices is shown in Table 9.2. Of course the indices may be selected differently from those shown. For instance, the fixed indices may be lower-case letters while the variable indices may be capital letters, or vice versa.

TABLE 9.2

Groups	Fixed Indices	Variable Indices
First	$a, b, c \cdots$	$k, l, m \cdots$
Second	$e, f, g \cdots$	$u, v, w \cdots$
Third	$h, i, j \cdots$	$x, y, z \cdots$
All	$a, b \cdots i, j$	$\alpha, \beta, \gamma \cdots$

The set of equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ or $e_\alpha = z_{\alpha\beta} i^\beta$ may be represented as

$$
\begin{array}{ccc}
 & & \begin{array}{ccc} \overset{m}{\longrightarrow} & \overset{v}{\longrightarrow} & \overset{y}{\longrightarrow} \end{array} \\
 & & \begin{array}{|c|c|c|} \hline i^m & i^v & i^y \\ \hline \end{array} \\
\begin{array}{c|c|} \cline{2-2} k \downarrow & e_k \\ \cline{2-2} u \downarrow & e_u \\ \cline{2-2} x \downarrow & e_x \\ \cline{2-2} \end{array} & = & \begin{array}{|c|c|c|} \hline z_{km} & z_{kv} & z_{ky} \\ \hline z_{um} & z_{uv} & z_{uy} \\ \hline z_{xm} & z_{xv} & z_{xy} \\ \hline \end{array} \Bigg\downarrow \alpha \\
 & & \xrightarrow[\beta]{\qquad\qquad}
\end{array}
\qquad
\begin{aligned}
e_k &= z_{km} i^m + z_{kv} i^v + z_{ky} i^y \\
e_u &= z_{um} i^m + z_{uv} i^v + z_{uy} i^y \\
e_x &= z_{xm} i^m + z_{xv} i^v + z_{xy} i^y
\end{aligned}
\qquad 9.42
$$

(*b*) When a system of equations is to be divided into four or more groups the single letters of the alphabet might be insufficient to supply enough different indices. In that case *each simple index may have a subscript* as $a_1, a_2, a_3 \cdots$ or $b_1, b_2, b_3 \cdots$, etc. For instance, a *five-group system* may have the set of indices shown in Table 9.3.

TABLE 9.3

Groups	Fixed Indices	Variable Indices
First	$1_1, 1_2, 1_3 \cdots$	$a_1, a_2, a_3 \cdots$
Second	$2_1, 2_2, 2_3 \cdots$	$b_1, b_2, b_3 \cdots$
Third	$3_1, 3_2, 3_3 \cdots$	$c_1, c_2, c_3 \cdots$
Fourth	$4_1, 4_2, 4_3, \cdots$	$d_1, d_2, d_3 \cdots$
Fifth	$5_1, 5_2, 5_3 \cdots$	$f_1, f_2, f_3 \cdots$
All	$1_1, 1_2, \cdots 2_1 \cdots 5_1 \cdots$	$\alpha, \beta, \gamma \cdots$

A 2-tensor having twenty fixed indices and five sets of variable indices and its various subdivisions are shown in Fig. 9.7.

When, say, a row belonging to *two* different groups is to be picked

out, then it is represented as the sum of two rows, each row belonging to its own group as $Z_{8a} + Z_{8b}$.

Similar notation is used for the subdivisions of a 3-tensor.

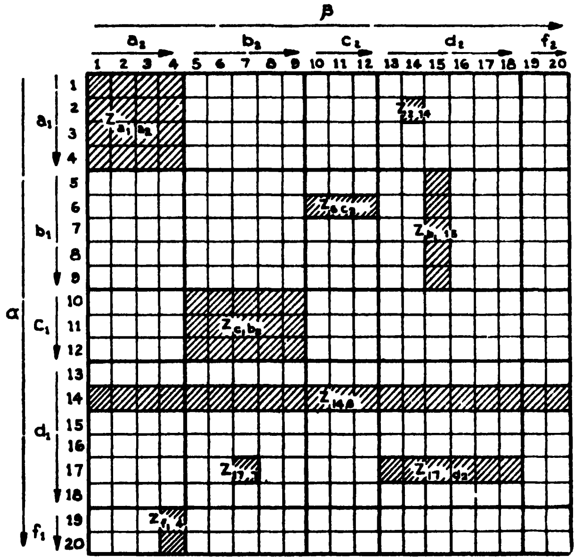


FIG. 9.7.—Subdivisions of a Compound 2-Tensor

(c) In case of *doubly compound tensors* two groups of variable indices are used as shown in Table 9.4, where the five groups of equations are divided into two larger groups.

TABLE 9.4

Groups	Fixed Indices	Variable Indices for Smaller Groups	Variable Indices for Larger Groups
First	$1_1, 1_2, 1_3 \cdots$	$a_1, a_2, a_3 \cdots$	
Second	$2_1, 2_2, 2_3 \cdots$	$b_1, b_2, b_3 \cdots$	$a, b, c \cdots$
Third	$3_1, 3_2, 3_3 \cdots$	$c_1, c_2, c_3 \cdots$	
Fourth	$4_1, 4_2, 4_3 \cdots$	$p_1, p_2, p_3 \cdots$	
Fifth	$5_1, 5_2, 5_3 \cdots$	$q_1, q_2, q_3 \cdots$	$p, q, r \cdots$
All	$1_1 \cdots, 2_1 \cdots, 5_1 \cdots$	$\alpha, \beta. \gamma \cdots$	

For a triply compound tensor three groups of variable indices are introduced. In general, there are as many groups of variable indices as there are sets of subdivisions.

XI. COMPOUND INDICES

Just as in direct notation, similarly in index notation, it is possible to introduce *compound indices*, both fixed and variable, to replace several individual indices. Denoting compound indices by capital letters, the compound 2-tensor of equation 9.42 is written as

$$\begin{array}{c|ccc} & \mathbf{D} & \mathbf{E} & \mathbf{F} \\ \hline \mathbf{A} & z_1 & z_2 & z_3 \\ \mathbf{z} = \mathbf{B} & z_4 & z_5 & z_6 \\ \mathbf{C} & z_6 & z_7 & z_8 \end{array} \qquad \begin{array}{c|ccc} P \diagdown Q & D & E & F \\ \hline A & Z_{km} & Z_{kv} & Z_{ky} \\ Z_{PQ} = B & Z_{um} & Z_{uv} & Z_{uy} \\ C & Z_{xm} & Z_{xv} & Z_{xy} \end{array} \qquad 9.43$$

where A, B, $C \cdots$ are *fixed* compound indices, each representing a particular set of fixed individual axes; A represents, say, axes a, b, c, while B represents d, e, f, C represents axes g, h, i, and so on. The corresponding *variable* compound indices are P, Q, $R \cdots$.

The *fixed* compound indices A, B, C should not be confused with the *variable* individual indices p, q, $\cdots u$, $v \cdots$ and x, $y \cdots$ of Table 9.2, since either p or q or r represents any *one* of the axes a, b, c, while A represents *all* the axes a, b, c and no other index.

The two symbols $z_{\alpha\beta}$ and z_{PQ} both represent the same tensor. However, $z_{\alpha\beta}$ *considers it as an ordinary tensor, whereas* z_{PQ} *considers it as a compound tensor.*

To represent multiply compound tensors in index notation, it is possible to introduce *multiply compound indices*, both fixed and variable.

XII. SUBDIVISION OF QUADRATIC EQUATIONS

(*a*) Let a set of quadratic equations be given in which the variable i^α occurs *twice* in the same term, as for instance in the explicit form of the equation of motion of Lagrange

$$e_\alpha = a_{\alpha\beta}\frac{di^\beta}{dt} + \Gamma_{\alpha\beta\gamma}i^\beta i^\gamma \qquad 9.44$$

This invariant equation contains three vectors, e_α, i^α, and di^α/dt; one 2-tensor, $a_{\alpha\beta}$ (the so-called " metric tensor "); and one geometric object of valence three, $\Gamma_{\alpha\beta\gamma}$ (the so-called " affine connection "). The invariant equation is represented in Fig. 9.8 along a particular frame.

(*b*) In an electrodynamical system, say in rotating electrical machinery or in a group of accelerated electrons, the axes are sub-

divided into two groups, the first group a, b, $c \cdots$ representing the *electrical* axes (say the meshes), the second group f, $g \cdots$ representing the geometrical axes (the axes of rotation).

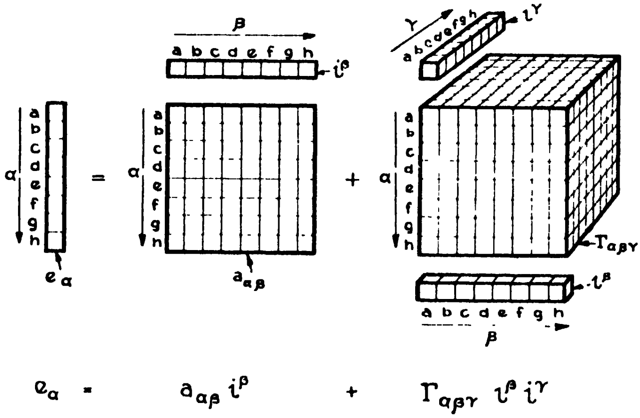


FIG. 9.8.—The Dynamical Equation of Lagrange in Invariant Form

Accordingly the single invariant equation of motion may be subdivided into *two* invariant equations by subdividing each vector into two parts, the 2-tensor into four parts, and the geometric object of valence three into eight parts as shown in Fig. 9.9.

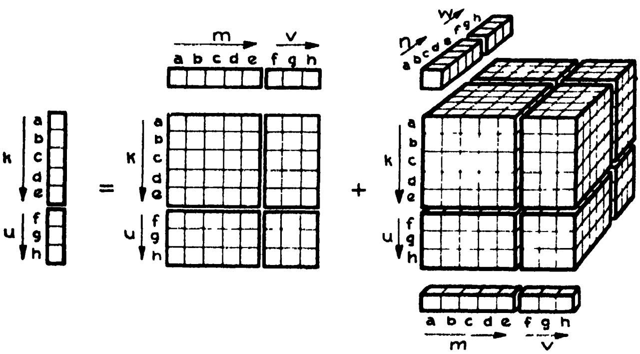


FIG. 9.9.—Subdivision of the Dynamical Equation of Lagrange along the Electrical and Geometrical Axes

If the variable indices of the first group are k, m, $n \cdots$, those of the second group u, v, $w \cdots$, then the two invariant equations are

$$e_k = a_{km}\frac{di^m}{dt} + a_{kv}\frac{di^v}{dt} + \Gamma_{kmn}i^m i^n + \Gamma_{kvn}i^v i^n + \Gamma_{kmw}i^m i^w + \Gamma_{kvw}i^v i^w \quad 9.45$$

$$e_u = a_{um}\frac{di^m}{dt} + a_{uv}\frac{di^v}{dt} + \Gamma_{umn}i^m i^n + \Gamma_{uvn}i^v i^n + \Gamma_{umw}i^m i^w + \Gamma_{uvw}i^v i^w \quad 9.46$$

The first equation is the "equation of voltage," the second is the "equation of torque."

In *classical* dynamical systems many of these component geometric objects are zero since there is no interchange between the electrical and material points of view. However, in *relativistic* dynamical systems a physical entity may appear as an electrical entity from one type of reference axes but it appears as a material entity from another type of axes, hence none of these component geometric objects is zero.

Each of these invariant equations can again be subdivided into several equations.

XIII. RECOMBINATION OF INVARIANT EQUATIONS

(*a*) When a single invariant equation is given and the various types of axes defined, it is a comparatively easy task to divide the single equation into several invariant equations. However, if several invariant equations are given, each expressed along different types of axes, *it is quite a difficult task to discover a single invariant equation that splits up smoothly into the given several equations.*

For instance, in rotating electrical machinery it is comparatively easy to set up the invariant equation of voltage and the invariant equation of torque *separately*. But it requires quite advanced mathematical concepts to be able to combine the two equations into the equation of motion 9.44. One of the reasons for the difficulty is that *the original equations do not have the form of equations* 9.45 *and* 9.46 because all 3-tensors are degenerated in them into 2-tensors (most of their components being zero) the vectors into scalars, etc. Of course, once the single equation is discovered, its establishment looks self-evident.

(*b*) Or as another example, the field equations of Maxwell were originally set up separately for the electric and for the magnetic fields. The theory of relativity owes much of its success to the discovery by Minkowsky of a means for combining the various tensors describing the two types of fields into compound tensors and combining the two sets of tensor equations into one set. *As a result of this combination the electric field and the magnetic field become part of one physical reality* and their separate electric and magnetic scalars, vectors, and 2-tensors used in the older physics become component parts of *compound* vectors and *compound* 2-tensors used in modern physics.

In combining several tensor equations into fewer tensor equations, the following successive developments occur:

1. The number of equations decreases.
2. The number of axes increases.
3. The valence of the tensors used increases.
4. The number of tensors introduced decreases.

The final goal of the simplification is to arrive at a *single* tensor equation that splits up into the various component equations.

(*c*) The so-called "*Unified Field Theory*" of modern physics represents attempts by physicists and mathematicians to discover a single tensor equation containing *compound tensors*, so that this single tensor equation should split up into several tensor equations representing respectively, say, the field equations of Maxwell and those of Einstein, the equations of motion of Lagrange, and the wave equation of Schrödinger. The discovery of such a tensor equation is important in unifying classical and quantum dynamics on one hand, and classical and relativistic dynamics on the other.

The final goal is to set up a ***single tensor equation*** that not only splits up into these various basic equations but also ***involves only one single tensor*** (of undetermined valence) ***so that the final single tensor equation that includes say the equations of Maxwell, Lagrange, Schrödinger, etc., has the form***

$$\boxed{T^{\alpha\beta\cdots}_{\cdots\cdots} = 0}$$

Several forms have been suggested for this single tensor in the absence of quantum phenomena.

(*d*) Once the manner of establishment of the compound invariant equation is known, in any complicated problem ***it is always easier to set up first the resultant compound equation and split it up afterward*** than to set up the component equations separately. By far the most important network problems require the establishment of several invariant equations, and in order to accomplish it, ***in all cases first the resultant compound equation*** $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ (or $\mathbf{I} = \mathbf{Y} \cdot \mathbf{E}$, etc.) ***is set up and then only is it subdivided into its component equations that are to be manipulated.***

CHAPTER X

REDUCTION FORMULAS

I. REDUCTION OF THE NUMBER OF EQUATIONS

(*a*) In many problems the total number of meshes may be divided into two groups. One group of meshes may have voltages impressed around it; the other group has no voltages, but is permanently short-circuited. If the knowledge of the currents in the permanently shorted meshes is not required for the physical analysis, the corresponding equations may be permanently eliminated.

In other problems some of the meshes may play no important part in the physical analysis even though voltages are impressed around them. In such cases both the meshes and their impressed voltages may be eliminated from view by permanently reducing the number of equations.

The reduction of the number of equations not only may be necessitated by a desire to ignore the presence of some of the meshes in simplifying the physical analysis, but also it may be due to an attempt to simplify the solution of a set of equations. The usual solution of a set of linear equations with the aid of determinants or by eliminating the variables one at a time is quite lengthy.

(*b*) A labor-saving device, the so-called " reduction formulas," is developed in this chapter to speed up the labor of reducing the number of linear equations or of solving them and to facilitate the physical analysis accompanying such a reduction.

II. THE ELIMINATION OF A SET OF VARIABLES

(*a*) *One of the important applications of compound n-tensors or compound n-matrices is to solve a set of n linear equations quickly and in an organized manner.* The method eliminates many of the intermediary and unnecessary steps needed in solving a set of linear equations either by determinants or by elimination of the variables, one at a time.

Two cases will be considered:

1. No voltages are impressed along those axes that are eliminated.
2. Voltages are impressed along the axes that are eliminated.

(*b*) Considering the first case, let the linear invariant equation $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ along a particular reference frame with six axes be

$$\begin{array}{|c|}\hline e_1 \\ \hline e_2 \\ \hline e_3 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|}\hline a & b & c & d & f & g \\ \hline v & w & x & y & z & a \\ \hline h & j & k & l & m & n \\ \hline b & c & d & f & g & h \\ \hline p & q & r & s & t & u \\ \hline j & k & l & m & n & p \\ \hline \end{array} \cdot \begin{array}{|c|}\hline i^1 \\ \hline i^2 \\ \hline i^3 \\ \hline i^4 \\ \hline i^5 \\ \hline i^6 \\ \hline \end{array} \qquad 10.1$$

Expressing each tensor as a compound tensor

$$\begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline 0 \\ \hline \end{array} = \begin{array}{|c|c|}\hline \mathbf{z}_1 & \mathbf{z}_2 \\ \hline \mathbf{z}_3 & \mathbf{z}_4 \\ \hline \end{array} \cdot \begin{array}{|c|}\hline \mathbf{i}^1 \\ \hline \mathbf{i}^2 \\ \hline \end{array} \qquad 10.2$$

$\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ can be written now as two invariant equations, analogously to the two equations of a two-winding transformer:

$$\begin{array}{ll|ll} e_1 &= z_1 i^1 + z_2 i^2 & \mathbf{e}_1 &= \mathbf{z}_1\cdot\mathbf{i}^1 + \mathbf{z}_2\cdot\mathbf{i}^2 \\ 0 &= z_3 i^1 + z_4 i^2 & 0 &= \mathbf{z}_3\cdot\mathbf{i}^1 + \mathbf{z}_4\cdot\mathbf{i}^2 \end{array} \qquad 10.3$$

Since such a subdivision of a set of equations occurs quite often in practice, their theory will be considered in greater detail.

If in the original equations $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ the equations containing the impressed voltages are not grouped together, they can always be rearranged to the desired order by a permutation with the aid of a transformation tensor **C** or in any other way.

(*c*) The six equations may represent, for instance, the performance of six coils. The existence or absence of an impressed voltage divides the coils *automatically* into two groups such that:

1. $\mathbf{z}_1$ represents the impedance tensor of those coils that have impressed voltages, assuming the other coils open-circuited.

2. $\mathbf{z}_4$ represents the impedance tensor of the other three coils with the first set of coils open-circuited.

3. $\mathbf{z}_2$ and $\mathbf{z}_3$ contain the mutual inductances between the two groups of coils. *The mutual inductances in the two directions are in general not the same,* that is, $\mathbf{z}_2$ is different from $\mathbf{z}_3$.

These impedance tensors of the groups of coils may be called "*open-circuit impedance tensors*" since they are measured while all other groups of coils that may exist are open-circuited.

(*d*) *The elimination of* $\mathbf{i}^2$ *follows closely the manipulation of ordinary*

equations. To eliminate $\mathbf{i}^2$ from the second equation the term containing $\mathbf{i}^2$ is brought to the left-hand side as

$$z_4 i^2 = -\,z_3 i^1 \qquad\Big|\qquad \mathbf{z}_4 \cdot \mathbf{i}^2 = -\,\mathbf{z}_3 \cdot \mathbf{i}^1$$

Multiplying both sides by $\mathbf{z}_4{}^{-1}$

$$z_4^{-1} z_4 i^2 = -\,z_4^{-1} z_3 i^1 \qquad\Big|\qquad \mathbf{z}_4^{-1} \cdot \mathbf{z}_4 \cdot \mathbf{i}^2 = -\,\mathbf{z}_4^{-1} \cdot \mathbf{z}_3 \cdot \mathbf{i}^1$$

$$i^2 = -\,z_4^{-1} z_3 i^1 \qquad\Big|\qquad \mathbf{i}^2 = -\,\mathbf{z}_4^{-1} \cdot \mathbf{z}_3 \cdot \mathbf{i}^1 \qquad 10.4$$

Substituting this value of $\mathbf{i}^2$ into the first equation

$$e_1 = z_1 i^1 - z_2 z_4^{-1} z_3 i^1 \qquad \mathbf{e}_1 = \mathbf{z}_1 \cdot \mathbf{i}^1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3 \cdot \mathbf{i}^1$$

Factoring out $\mathbf{i}^1$

$$\boxed{e_1 = (z_1 - z_2 z_4^{-1} z_3) i^1} \qquad\Big|\qquad \boxed{\mathbf{e}_1 = (\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3)^{-1} \cdot \mathbf{i}^1} \qquad 10.5$$

or

$$e_1 = z_1' i^1 \qquad\Big|\qquad \mathbf{e}_1 = \mathbf{z}_1' \cdot \mathbf{i}^1 \qquad 10.6$$

Hence the two invariant equations 10.3 have been reduced to one invariant equation, representing three ordinary equations and containing only the three scalar variables $\mathbf{i}^1$. The expression in parenthesis, $\mathbf{z}_1'$, is a 2-tensor having as many rows and columns as $\mathbf{z}_1$.

III. THE IMPEDANCE REDUCTION FORMULA

(*a*) For electrical circuits or machinery this last equation may be formulated as follows:

If no voltages are impressed along certain meshes, these meshes may be eliminated by assuming that the self-impedance tensor $\mathbf{z}_1$ *of the remaining meshes changes to* $\mathbf{z}_1'$, *called the " short-circuit impedance tensor," by the " reduction formula "*

$$\boxed{\mathbf{z}_1' = \mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3} \qquad 10.7$$

where $\mathbf{z}_4$ is the self-impedance tensor of the eliminated meshes, $\mathbf{z}_2$ and $\mathbf{z}_3$ are the mutual impedance tensors between the eliminated and retained meshes.

That is, $\mathbf{z}_1'$ *represents the impedance tensor of the first group of coils measured while the second group of coils is short-circuited.* Its definition, equation 10.7, is analogous to that of the " short-circuit impedance " of a coil

$$z_1' = z_1 - (z_m)^2/z_2 \qquad 10.8$$

measured while the other coil with self-impedance z_2 is short-circuited.

(*b*) Hence *the elimination of several variables at one step* involves:

1. The calculation of the inverse of $\mathbf{z}_4$ having as many rows and columns as the number of variables to be eliminated.

2. Two 2-tensor multiplications, $\mathbf{z}_4^{-1} \cdot \mathbf{z}_3$ and $\mathbf{z}_2 \cdot (\mathbf{z}_4^{-1} \cdot \mathbf{z}_3)$.

3. One subtraction.

(*c*) When $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ contains many equations, say twelve, with $\mathbf{e}$ having, say, three components, then *the equations of this section may be repeated several times in succession.*

That is, first the *twelve* equations are subdivided into two groups so that $\mathbf{z}_4$ has three rows and $\mathbf{z}_1$ has nine. Eliminating $\mathbf{z}_4$, the remaining *nine* equations again are subdivided into two groups so that $\mathbf{z}_4$ has three rows and $\mathbf{z}_1$ has six. Eliminating $\mathbf{z}_4$ the remaining *six* equations are again subdivided and the last three rows eliminated so that finally *three* equations are left.

The quickest procedure appears to be to eliminate *three* rows and columns at one step, since it is comparatively easy to calculate the determinant of a matrix with *three* rows and columns.

(*d*) When $\mathbf{z}_1'$ has been found, the currents in the first group of coils are

$$i^1 = z_1'^{-1} e_1 \qquad \Big| \qquad \mathbf{i}^1 = \mathbf{z}_1'^{-1} \cdot \mathbf{e}_1$$

Hence *the calculation of the inverse of the original* $\mathbf{z}$ containing *six* rows and columns has been reduced to the calculation of the inverse of two 2-tensors (or 2-matrices) $\mathbf{z}_4$ and $\mathbf{z}_1'$, each having *three* rows and columns instead of six. However, some additional multiplications have to be performed.

The eliminated current $\mathbf{i}^2$ in the second group of coils is found from $\mathbf{i}^1$ by equation 10.4, namely, by

$$i^2 = -\, z_4^{-1} z_3 i^1 \qquad \Big| \qquad \mathbf{i}^2 = -\, \mathbf{z}_4^{-1} \cdot \mathbf{z}_3 \cdot \mathbf{i}^1 \qquad 10.9$$

where $\mathbf{z}_4^{-1} \cdot \mathbf{z}_3$ has already been calculated as a step in finding $\mathbf{z}_1'$.

IV. THE USE OF DOUBLY COMPOUND TENSORS

(*a*) When the elimination of a set of variables is to be repeated several times in succession, the work is greatly speeded up by *expressing the original impedance tensor as a compound tensor. Then this compound tensor is reduced just as if it were an ordinary tensor by eliminating one row at a time.*

The advantage of this method of elimination is that it avoids the multiplication of matrices with many rows and columns. Also any symmetry that may exist among the smaller matrices may be utilized to *avoid the repetition of identical multiplications* which cannot be

avoided otherwise. *Even in the absence of any symmetry many duplicate multiplications are avoided.*

(*b*) For instance, let an impedance tensor **z** with eleven rows and columns be subdivided into 3^2 components, expressing it as a compound tensor with three rows and columns as

$$
\mathbf{z} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
m & & a & n & & b & p & & c & q & \\
\hline
 & d & r & & e & s & & f & t & u & g \\
\hline
v & & h & & z & i & & x & j & y & k \\
\hline
 & l & a & & & m & b & & n & c & \\
\hline\hline
d & & o & e & f & p & & g & q & h & r \\
\hline
s & i & & & j & & t & k & & u & l \\
\hline
 & & v & & & w & & & x & & \\
\hline\hline
y & m & y & & z & n & & & a & p & \\
\hline
 & b & & q & & c & r & s & d & & e \\
\hline
f & & t & g & u & & h & v & & h & \\
\hline
 & i & & & j & & & k & & & l \\
\hline
\end{array}
= \begin{array}{|c|c|c|}
\hline
\mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 \\
\hline
\mathbf{z}_4 & \mathbf{z}_5 & \mathbf{z}_6 \\
\hline
\mathbf{z}_7 & \mathbf{z}_8 & \mathbf{z}_9 \\
\hline
\end{array}
\qquad 10.10
$$

The equation $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ may represent the performance of a network with eleven meshes, the meshes being divided into *three* groups with voltages impressed only in the meshes of the first group. A special case of such a network is shown in Fig. 10.1, where the three groups of meshes are shown as independent sub-networks to keep the physical picture of the equations clearer.

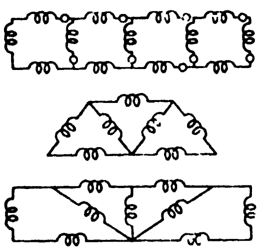


FIG. 10.1.—A Network with Three Groups of Meshes

The equation $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ may represent also the performance of any linear dynamical system, such as a group of uniformly rotating or oscillating electrical machines, etc.

(*c*) To eliminate the last row and column, **z** is subdivided into four smaller tensors, that is, it becomes a doubly compound tensor with two rows and columns

$$
\mathbf{z} = \begin{array}{|c|c|c|}
\hline
\mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 \\
\hline
\mathbf{z}_4 & \mathbf{z}_5 & \mathbf{z}_6 \\
\hline\hline
\mathbf{z}_7 & \mathbf{z}_8 & \mathbf{z}_9 \\
\hline
\end{array}
= \begin{array}{|c|c|}
\hline
\mathbf{z}_a & \mathbf{z}_b \\
\hline
\mathbf{z}_c & \mathbf{z}_d \\
\hline
\end{array}
\qquad 10.11
$$

From the last tensor the second row and column is eliminated by equation 10.7, that is, by

$$\mathbf{z}'_a = \mathbf{z}_a - \mathbf{z}_b \cdot \mathbf{z}_d^{-1} \cdot \mathbf{z}_c$$

where

$$\mathbf{z}_a = \begin{array}{|c|c|}\hline z_1 & z_2 \\ \hline z_4 & z_5 \\ \hline\end{array} \qquad \mathbf{z}_b = \begin{array}{|c|}\hline z_3 \\ \hline z_6 \\ \hline\end{array} \qquad \mathbf{z}_c = \begin{array}{|c|c|}\hline z_7 & z_8 \\ \hline\end{array} \qquad \mathbf{z}_d = \begin{array}{|c|}\hline z_9 \\ \hline\end{array}$$

Performing the indicated multiplications

$$\mathbf{z}_b \cdot \mathbf{z}_d^{-1} \cdot \mathbf{z}_c = \overset{\longrightarrow}{\begin{array}{|c|}\hline z_3 \\ \hline z_6 \\ \hline\end{array}} \cdot z_9^{-1} \cdot \begin{array}{|c|c|}\hline z_7 & z_8 \\ \hline\end{array}\Bigg\downarrow = \begin{array}{|c|c|}\hline z_3 \cdot z_9^{-1} \cdot z_7 & z_3 \cdot z_9^{-1} \cdot z_8 \\ \hline z_6 \cdot z_9^{-1} \cdot z_7 & z_6 \cdot z_9^{-1} \cdot z_8 \\ \hline\end{array}$$

Performing the subtraction $\mathbf{z}_a - \mathbf{z}_b \cdot \mathbf{z}_d^{-1} \cdot \mathbf{z}_c =$

$$= \begin{array}{|c|c|}\hline z_1 & z_2 \\ \hline z_4 & z_5 \\ \hline\end{array} - \begin{array}{|c|c|}\hline z_3 \cdot z_9^{-1} \cdot z_7 & z_3 \cdot z_9^{-1} \cdot z_8 \\ \hline z_6 \cdot z_9^{-1} \cdot z_7 & z_6 \cdot z_9^{-1} \cdot z_8 \\ \hline\end{array}$$

the reduced compound tensor becomes

$$\mathbf{z}'_a = \begin{array}{|c|c|}\hline z_1 - z_3 \cdot z_9^{-1} \cdot z_7 & z_2 - z_3 \cdot z_9^{-1} \cdot z_8 \\ \hline z_4 - z_6 \cdot z_9^{-1} \cdot z_7 & z_5 - z_6 \cdot z_9^{-1} \cdot z_8 \\ \hline\end{array} = \begin{array}{|c|c|}\hline z'_1 & z'_2 \\ \hline z'_4 & z'_5 \\ \hline\end{array} \qquad 10.12$$

having $4 + 3 = 7$ rows and columns.

Eliminating again the last row and column with equation 10.7 by finding the inverse of the right lower corner tensor $\mathbf{z}_5 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_8$ the final reduced tensor is

$$\boxed{z''_1 = (z_1 - z_3 \cdot z_9^{-1} \cdot z_7) - (z_2 - z_3 \cdot z_9^{-1} \cdot z_8) \cdot (z_5 - z_6 \cdot z_9^{-1} \cdot z_8)^{-1} \cdot (z_4 - z_6 \cdot z_9^{-1} \cdot z_7)} \qquad 10.13$$

This is the formula to be used immediately to reduce the eleven-rowed 2-tensor 10.10 *to a four-rowed 2-tensor.* The immediate use of this formula eliminates carrying along page after page the numerous sets of equations and writing them down on paper over and over again, as the variables are eliminated one at a time. In the absence of any symmetry the number of slide-rule operations that have to be performed is probably the same in any method of solution, but *with the aid of the invariant equation* 10.13 *the scalar equivalent of all the steps used in its derivation are completely eliminated.* When the components of the 2-tensors have some symmetry, many repetitions of slide-rule operations may also be eliminated.

(*d*) It should be noted that this formula for $\mathbf{z}_1''$ is quite similar to that of $\mathbf{z}_1'$ of equation 10.7, namely $\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3$, except that *each "open-circuit impedance tensor" of* $\mathbf{z}_1'$ *is replaced now by a "short-circuit impedance tensor." That is, the expressions in the parentheses of equation* 10.13 *represent the self- and mutual impedance tensors of the first and second group of coils of Fig.* 10.1 *measured with the third group of coils permanently short-circuited.*

The whole expression $\mathbf{z}_1''$ *itself represents the impedance tensor of the first group of coils measured with the other two groups of coils permanently short-circuited.* The inductance of *one* coil in the presence of two permanently short-circuited coils is, analogously to equation 10.13

$$L'' = (L_{11} - M_{13}M_{13}/L_{33}) - \frac{(L_{12} - M_{13}M_{23}/L_{33})(L_{12} - M_{13}M_{23}/L_{33})}{L_{22} - M_{23}M_{23}/L_{33}} \qquad 10.14$$

(*e*) The reduced equation is $\mathbf{e}_1 = \mathbf{z}_1'' \cdot \mathbf{i}^1$, where $\mathbf{z}_1''$ has as many rows and columns as $\mathbf{z}_1$, namely four. The equation is solved as $\mathbf{i}^1 = \mathbf{z}_1''^{-1} \cdot \mathbf{e}_1$.

Hence the solution of the eleven equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ *has been reduced to the calculation of the inverse of three 2-tensors, each having the same number of rows as the diagonal tensors* $\mathbf{z}_9$, $\mathbf{z}_5$, *and* $\mathbf{z}_1$, *namely four, three, and four respectively.* There are also several multiplications and subtractions to be performed.

It should be noted that, even in the absence of any symmetry, two products, $\mathbf{z}_3 \cdot \mathbf{z}_9^{-1}$ and $\mathbf{z}_6 \cdot \mathbf{z}_9^{-1}$, occur twice in the equation.

V. IMPRESSED VOLTAGES IN ELIMINATED AXES

(*a*) In the previous sections it was assumed that no voltages are impressed along the eliminated axes. It will be shown that the same formulas are valid even in the presence of applied voltages, except that *the presence of applied voltages in the eliminated axes changes the apparent value of the applied voltages in the retained axes.*

(*b*) Again let a set of linear equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ be given and let they be divided into two sets of equations as

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 \\ \mathbf{e}_2 &= \mathbf{z}_3 \cdot \mathbf{i}^1 + \mathbf{z}_4 \cdot \mathbf{i}^2 \end{aligned} \qquad 10.15$$

The subdivision of the system into two groups may be necessitated for various reasons. They may be subdivided just for the sake of easier manipulation. Or $\mathbf{e}_1$ may represent the voltages at the points of entry

and $\mathbf{e}_2$ may represent some internal voltages. (That is, the system is considered "active" instead of "passive.") Still another reason might be that $\mathbf{e}_1$ may be considered to remain constant and $\mathbf{e}_2$ variable. Or perhaps $\mathbf{z}_4$ may be considered as a variable load and the other impedances may remain constant, etc.

(*c*) Following closely the steps in Section II, let $\mathbf{i}^2$ be eliminated from the second equation

$$\mathbf{z}_4 \cdot \mathbf{i}^2 = \mathbf{e}_2 - \mathbf{z}_3 \cdot \mathbf{i}^1$$

$$\mathbf{i}^2 = \mathbf{z}_4^{-1} \cdot (\mathbf{e}_2 - \mathbf{z}_3 \cdot \mathbf{i}^1) \qquad 10.16$$

Substituting $\mathbf{i}^2$ into the first equation

$$\mathbf{e}_1 = \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot (\mathbf{e}_2 - \mathbf{z}_3 \cdot \mathbf{i}^1)$$

Factoring out $\mathbf{i}^1$

$$\mathbf{e}_1 = (\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3) \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2$$

$$\boxed{(\mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2) = (\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3) \cdot \mathbf{i}^1} \qquad 10.17$$

or

$$\boxed{\mathbf{e}_1' = \mathbf{z}_1' \cdot \mathbf{i}^1}$$

Comparing these equations with equations 10.5 and 10.6, it is found that *the reduced impedance tensor* $\mathbf{z}_1'$ *is found the same way* as in the previous case, namely by the "impedance reduction formula"

$$\boxed{\mathbf{z}_1' = \mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3} \qquad 10.18$$

(*d*) *However, the apparent value of the impressed voltage on the first group of coils changes from* $\mathbf{e}_1$ *to* $\mathbf{e}_1'$ *by the "voltage reduction formula"*

$$\boxed{\mathbf{e}_1' = \mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2} \qquad 10.19$$

where $\mathbf{e}_2$ *is the impressed voltage vector of the eliminated group.*

The multiplication $\mathbf{z}_2 \cdot \mathbf{z}_4^{-1}$ needed in the calculation of the new impressed voltage $\mathbf{e}'$ has already been performed in the calculation of $\mathbf{z}'$.

A physical interpretation of the new impressed voltage $\mathbf{e}'$ will be given in Section VIII with the aid of Thévenin's theorem.

(*e*) *The current* $\mathbf{i}^2$ *in the eliminated group is from equation* 10.16.

$$\boxed{\mathbf{i}^2 = \mathbf{z}_4^{-1}(\mathbf{e}_2 - \mathbf{z}_3 \cdot \mathbf{i}^1)} \qquad 10.20$$

VI. NUMERICAL EXAMPLE

(*a*) Given five equations with five unknowns,

$$10 = 1i^a + 2i^b - 3i^c + 4i^d + 5i^f$$
$$9 = 2i^a + 4i^b + 3i^c + 5i^d - 1i^f$$
$$8 = 3i^a + 4i^b + 5i^c + 2i^d + 3i^f$$
$$7 = 1i^a + 2i^b - 4i^c - 3i^d + 5i^f$$
$$6 = 5i^a + 1i^b - 3i^c + 3i^d + 2i^f$$

If the five equations are written as $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ then

$$\mathbf{e} = \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \\ \hline 10 & 9 & 8 & 7 & 6 \end{array}$$

$$\mathbf{i} = \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \\ \hline i^a & i^b & i^c & i^d & i^f \end{array}$$

$$\mathbf{z} = \begin{array}{c|ccccc} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \\ \hline \mathbf{a} & 1 & 2 & -3 & 4 & 5 \\ \mathbf{b} & 2 & 4 & 3 & 5 & -1 \\ \mathbf{c} & 3 & 4 & 5 & 2 & 3 \\ \mathbf{d} & 1 & 2 & -4 & -3 & 5 \\ \mathbf{f} & 5 & 1 & -3 & 3 & 2 \end{array}$$

The problem is to reduce the five equations to two equations. Three of the unknowns i^c, i^d, and i^f can be eliminated in one step, by separating the axes into two groups **a**, **b**, and **c**, **d**, **f** so that

$$\mathbf{e}_1 = \begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \hline 10 & 9 \end{array} \qquad \mathbf{e}_2 = \begin{array}{ccc} \mathbf{c} & \mathbf{d} & \mathbf{f} \\ \hline 8 & 7 & 6 \end{array}$$

$$\mathbf{i}^1 = \begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \hline i^a & i^b \end{array} \qquad \mathbf{i}^2 = \begin{array}{ccc} \mathbf{c} & \mathbf{d} & \mathbf{f} \\ \hline i^c & i^d & i^f \end{array}$$

$$\mathbf{z}_1 = \begin{array}{c|cc} & \mathbf{a} & \mathbf{b} \\ \hline \mathbf{a} & 1 & 2 \\ \mathbf{b} & 2 & 4 \end{array} \qquad \mathbf{z}_2 = \begin{array}{c|ccc} & \mathbf{c} & \mathbf{d} & \mathbf{f} \\ \hline \mathbf{a} & -3 & 4 & 5 \\ \mathbf{b} & 3 & 5 & -1 \end{array}$$

$$\mathbf{z}_3 = \begin{array}{c|cc} & \mathbf{a} & \mathbf{b} \\ \hline \mathbf{c} & 3 & 4 \\ \mathbf{d} & 1 & 2 \\ \mathbf{f} & 5 & 1 \end{array} \qquad \mathbf{z}_4 = \begin{array}{c|ccc} & \mathbf{c} & \mathbf{d} & \mathbf{f} \\ \hline \mathbf{c} & 5 & 2 & 3 \\ \mathbf{d} & -4 & -3 & 5 \\ \mathbf{f} & -3 & 3 & 2 \end{array}$$

(*b*) If the last three rows and columns are eliminated the remaining tensor (having two rows and columns) is, by equation 10.18,

$$\boxed{\mathbf{z}' = \mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3}$$

$$\mathbf{z}_4^{-1} = \frac{1}{-182} \times \begin{array}{|c|c|c|}\hline -21 & 5 & 19 \\ \hline -7 & 19 & -37 \\ \hline -21 & -21 & -7 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 0\ 115 & -0\ 0274 & -0\ 104 \\ \hline 0\ 0384 & -0\ 104 & 0\ 203 \\ \hline 0\ 115 & 0\ 115 & 0.0384 \\ \hline \end{array}$$

$$\mathbf{z}_2 \cdot \mathbf{z}_4^{-1} = \begin{array}{|c|c|c|}\hline 0\ 3835 & 0\ 2412 & 1\ 316 \\ \hline 0\ 422 & -0\ 717 & 0\ 665 \\ \hline \end{array} \qquad (\mathbf{z}_2 \cdot \mathbf{z}_4^{-1}) \cdot \mathbf{z}_3 = \begin{array}{|c|c|}\hline 7\ 97 & 3\ 332 \\ \hline 3\ 871 & 0\ 919 \\ \hline \end{array}$$

$$\mathbf{z}' = \mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3 = \begin{array}{c|c|c|} & a & b \\ \hline a & -6\ 97 & -1\ 332 \\ \hline b & -1\ 871 & 3\ 081 \\ \hline \end{array}$$

(*c*) The new applied voltages are, by equation 10.19,

$$\boxed{\mathbf{e}' = \mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2}$$

Since $\mathbf{z}_2 \cdot \mathbf{z}_4^{-1}$ has already been calculated

$$\mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2 = \begin{array}{|c|c|} a & b \\ \hline 12\ 64 & 2.35 \\ \hline \end{array}$$

$$\mathbf{e}' = \mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2 = \begin{array}{|c|c|} a & b \\ \hline -2\ 64 & 6\ 65 \\ \hline \end{array}$$

(*d*) Hence the remaining two equations with two unknowns $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$ are

$$-2.64 = -\ 6.97 i^a - 1.332 i^b$$

$$6.65 = -\ 1.87 i^a + 3.081 i^b$$

VII. REPLACING AN ACTIVE BY A PASSIVE NETWORK

Since the effect of any impressed voltage $\mathbf{e}_2$ in the eliminated meshes is to change the impressed voltage $\mathbf{e}_1$ of the retained group to $\mathbf{e}_1'$, therefore *the presence of* $\mathbf{e}_2$ may be ignored and in its place $\mathbf{e}_1$ may be changed to $\mathbf{e}_1'$ so that the set of equations 10.15 may also be written as

$$\begin{aligned} \mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2 &= \mathbf{z}_1 \cdot \mathbf{i}^1 - \mathbf{z}_2 \cdot \mathbf{i}^2 \\ 0 &= \mathbf{z}_3 \cdot \mathbf{i}^1 - \mathbf{z}_4 \cdot \mathbf{i}^2 \end{aligned} \qquad 10.21$$

That is, *in any active, asymmetrical, linear system the internal forces* $\mathbf{e}_2$ *(voltages) may be eliminated (and the system made passive) by replacing the terminal forces* $\mathbf{e}_1$ *by a new set of forces*

$$\mathbf{e}_1' = \mathbf{e}_1 - \mathbf{Z}_2 \cdot \mathbf{Z}_4^{-1} \cdot \mathbf{e}_2$$

but leaving the whole system arrangement unchanged.

The currents $\mathbf{i}^1$ flowing through the terminals are the same as in the presence of the internal voltages $\mathbf{e}_2$. However, the internal currents $\mathbf{i}^2$ are different in the two cases, since in the presence of the internal voltages $\mathbf{i}^2$ is found in terms of $\mathbf{i}^1$ by equation 10.20 whereas in the absence of internal voltages $\mathbf{i}^2$ is found in terms of $\mathbf{i}^1$ from the second equation of 10.21 by equation 10.9. That is, *the equivalence of the active and passive networks*, each having different terminal voltages, *is valid only when viewed from the terminals* (that is, from the meshes of the first group).

VIII. A GENERALIZATION OF THÉVENIN'S THEOREM

(*a*) In most engineering problems the steps of eliminating certain variables occur while the equations still contain design or test constants instead of pure numbers. For instance, in synchronous machines the equations of the field and amortisseur windings are usually eliminated, in induction motors those of the rotor windings, etc. The variables in these cases are eliminated in order to simplify the *physical analysis* of the problem.

Since there is hardly any engineering problem where the reduction formulas are not used to simplify the physical or analytical set-up of the problem (whether recognized as so or not), it is worth while to point out that *these formulas may be considered as a generalization of the so-called "Thévenin's theorem"* sometimes used to simplify the analysis of electrical engineering problems. It may be stated (among other ways) as follows: (Fig. 10.2)

Let a network Z_4 containing impressed voltages be open-circuited at point A and let the instantaneous *open-circuit voltage* at that point be e_0. If a second and passive network Z_1 is connected at point A, the instantaneous current flowing through point A is

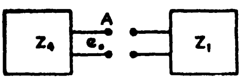


FIG. 10.2.—Thévenin's Theorem

$$i^A = \frac{-e_0}{Z_1 + Z_4} \qquad 10.22$$

where $Z_1 + Z_4$ is the combined impedance of the two networks looking in from point A.

It will be shown that this last equation 10.22 is a special case of equation 10.17.

(*b*) The reduced equations are, from equation 10.17,

$$\mathbf{i}^1 = (\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3)^{-1} \cdot (\mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2) \qquad 10.23$$

The terminals from which the network is viewed are the meshes of the first group. The impedance of the whole network viewed from the first group is $\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3$, the current through it is $\mathbf{i}^1$, and if no impressed voltage $\mathbf{e}_1$ exists on it the current is due solely to the addition voltage $-\mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2$ appearing upon the first group.

Now the additional voltage $-\mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2$ *appearing around the meshes of the first group represents the open-circuit voltage appearing across their terminals when the meshes of the first group are open-circuited and when currents flow in the eliminated second group owing to their impressed voltage vector* $\mathbf{e}_2$.

(*c*) That is, if the second group of four coils $\mathbf{z}_4$ alone is excited (the first group being open-circuited) the current in them is

$$\mathbf{i}^0 = \mathbf{z}_4^{-1} \cdot \mathbf{e}_2 \qquad 10.24$$

The voltage induced in the first group of three coils due to this current is

$$\mathbf{e}_0 = \mathbf{z}_2 \cdot \mathbf{i}^0 = \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2 \qquad 10.25$$

where $\mathbf{z}_2$ represents the mutual inductance between the two groups.

If no voltage $\mathbf{e}_1$ is impressed across the first group of coils, the current flowing through them is due entirely to the open-circuit voltage $-\mathbf{e}_0$ appearing across them.

(*d*) *Hence the " reduction formulas " represent the generalization of " Thévenin's theorem " by*:

1. Replacing *scalars* (impedance operators and voltages) with *tensors* of various valence.

2. Assuming *asymmetrical mutual impedances* between the two systems to be interconnected, hence making the theorem valid for moving or oscillating systems also, such as rotating machines.

3. Assuming impressed voltages on both systems.

4. Following the theorem in a manner that makes it independent of electrical phenomena and applicable to any physical phenomena that can be described by a set of *linear* invariant equations.

IX. IMPRESSED VOLTAGES IN THREE GROUPS OF COILS

(*a*) The complete set of invariant equations of the three groups of coils are

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 + \mathbf{z}_3 \cdot \mathbf{i}^3 \\ \mathbf{e}_2 &= \mathbf{z}_4 \cdot \mathbf{i}^1 + \mathbf{z}_5 \cdot \mathbf{i}^2 + \mathbf{z}_6 \cdot \mathbf{i}^3 \\ \mathbf{e}_3 &= \mathbf{z}_7 \cdot \mathbf{i}^1 + \mathbf{z}_8 \cdot \mathbf{i}^2 + \mathbf{z}_9 \cdot \mathbf{i}^3 \end{aligned} \qquad 10.26$$

(*b*) The subdivision of a system of equations into three groups may be necessitated for numerous reasons. They may be subdivided just to simplify their manipulation. Or $\mathbf{e}_1$ may represent the accessible terminal voltages, $\mathbf{e}_2$ the internal voltages of the second group, and $\mathbf{e}_3$ of the third group may be zero. As another case, in the first group the voltages $\mathbf{e}_1$ may be kept constant, in the second group the currents $\mathbf{i}^2$ may be kept constant, and in the third group both $\mathbf{e}_3$ and $\mathbf{i}^3$ may assume any values. The enumeration of the various reasons for subdividing a system into two, three, four, or more groups can be continued indefinitely. Each of the reasons requires a different manipulation and a different formula or criterion to be sought. Various cases are considered in detail in Chapter XXI.

The use of the invariant equations of these sections eliminates the necessity of repeating all the steps of the analysis each time a new set of equations is established or a new type of answer is sought. The equations derived here are independent of the method of interconnection of the networks or the arrangement of the linear dynamical systems and their reference axes. The systems may be active or passive, symmetrical or asymmetrical. Each symbol in the equations represents a definite physical *entity* associated with the system that influences the performance, *while all disturbing details are absent.*

(*c*) The problem at hand is to eliminate the equations of the second and the third group of meshes and to express the performance of the whole system in terms of the first group of meshes and its impressed voltages. Expressed in another way, the problem at hand is to solve the set of equations 10.26 for $\mathbf{i}^1$.

By analogy with the two-group case, the final formulas will be the same as before except that the self- and mutual impedance tensors of the coils of the first and second group are measured while the coils of the third group are *short-circuited.*

(*d*) Eliminating $\mathbf{i}^3$ from the third equation

$$\mathbf{i}^3 = \mathbf{z}_9^{-1} \cdot (\mathbf{e}_3 - \mathbf{z}_7 \cdot \mathbf{i}^1 - \mathbf{z}_8 \cdot \mathbf{i}^2) \qquad 10.27$$

Substituting into the first and second equations

$$\mathbf{e}_1 = \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 + \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot (\mathbf{e}_3 - \mathbf{z}_7 \cdot \mathbf{i}^1 - \mathbf{z}_8 \cdot \mathbf{i}^2)$$
$$\mathbf{e}_2 = \mathbf{z}_4 \cdot \mathbf{i}^1 + \mathbf{z}_5 \cdot \mathbf{i}^2 + \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot (\mathbf{e}_3 - \mathbf{z}_7 \cdot \mathbf{i}^1 - \mathbf{z}_8 \cdot \mathbf{i}^2)$$

Factoring out $\mathbf{i}^1$ and $\mathbf{i}^2$

$$(\mathbf{e}_1 - \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{e}_3) = (\mathbf{z}_1 - \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_7) \cdot \mathbf{i}^1 + (\mathbf{z}_2 - \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_8) \cdot \mathbf{i}^2$$
$$(\mathbf{e}_2 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{e}_3) = (\mathbf{z}_4 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_7) \cdot \mathbf{i}^1 + (\mathbf{z}_5 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_8) \cdot \mathbf{i}^2 \qquad 10.28$$

The three sets of equations 10.26 have now been reduced to the two sets of equations with two variables, equation 10.28, that can be written shortly as

$$\begin{aligned} \mathbf{e}'_1 &= \mathbf{z}'_1 \cdot \mathbf{i}^1 + \mathbf{z}'_2 \cdot \mathbf{i}^2 \\ \mathbf{e}'_2 &= \mathbf{z}'_3 \cdot \mathbf{i}^1 + \mathbf{z}'_4 \cdot \mathbf{i}^2 \end{aligned} \qquad 10.29$$

These equations represent the first two groups of coils, with the third group eliminated. The effect of the elimination of a group is: (1) to change the *open-circuit* self- and mutual impedance tensors of the remaining groups to the *short-circuit* self- and mutual impedance tensors; (2) to change the impressed voltages of the remaining groups to a new value.

(*e*) The manipulation from now on may follow two paths. Either the results of Section V are substituted into equation 10.29 or the manipulation of equation 10.28 or 10.29 is continued. Using the results of Section V, *the reduced equation, containing only the meshes of the first group, is*

$$\boxed{\mathbf{e}''_1 = \mathbf{z}''_1 \cdot \mathbf{i}^1} \qquad 10.30$$

where

$$\mathbf{z}''_1 = \mathbf{z}'_1 - \mathbf{z}'_2 \cdot \mathbf{z}'^{-1}_4 \cdot \mathbf{z}'_3 \qquad 10.31$$

$$\mathbf{e}''_1 = \mathbf{e}'_1 - \mathbf{z}'_2 \cdot \mathbf{z}'^{-1}_4 \cdot \mathbf{e}'_2 \qquad 10.32$$

Substituting the values of the primed quantities from equation 10.28, the reduced tensors are

$$\boxed{\mathbf{z}''_1 = (\mathbf{z}_1 - \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_7) - (\mathbf{z}_2 - \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_8) \cdot (\mathbf{z}_5 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_8)^{-1} \cdot (\mathbf{z}_4 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_7)} \qquad 10.33$$

$$\boxed{\mathbf{e}''_1 = (\mathbf{e}_1 - \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{e}_3) - (\mathbf{z}_2 - \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_8)(\mathbf{z}_5 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_8)^{-1} \cdot (\mathbf{e}_2 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{e}_3)} \qquad 10.34$$

The form of the short-circuit impedance tensor $\mathbf{z}''_1$ of the first group (already given in equation 10.13) and that of the new reduced voltage

vector $\mathbf{e}_1''$ to be impressed on the first group are analogous to equations 10.18 and 10.19 except that *each open-circuit impedance tensor is replaced by a short-circuit impedance tensor, and each actually impressed voltage vector* $\mathbf{e}$ *is replaced by its equivalent voltage vector* $\mathbf{e}'$.

(*f*) The three sets of currents flowing in the three groups of meshes are found from equations 10.30, 10.28, and 10.27

$$\mathbf{i}^1 = \mathbf{z}_1''^{-1} \cdot \mathbf{e}_1''$$

$$\mathbf{i}^2 = (\mathbf{z}_5 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_8)^{-1} \cdot [\mathbf{e}_2 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{e}_3 - (\mathbf{z}_4 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_7) \cdot \mathbf{i}^1]$$

$$\mathbf{i}^3 = \mathbf{z}_9^{-1} \cdot (\mathbf{e}_3 - \mathbf{z}_7 \cdot \mathbf{i}^1 - \mathbf{z}_8 \cdot \mathbf{i}^2) \qquad 10.35$$

To calculate these currents *three inverse 2-tensors have to be calculated*, namely

$$\mathbf{y} = \mathbf{z}_9^{-1}$$

$$\mathbf{y}' = (\mathbf{z}_5 - \mathbf{z}_6 \cdot \mathbf{y} \cdot \mathbf{z}_8)^{-1} \qquad 10.36$$

$$\mathbf{y}'' = (\mathbf{z}_1 - \mathbf{z}_3 \cdot \mathbf{y} \cdot \mathbf{z}_7) - (\mathbf{z}_2 - \mathbf{z}_3 \cdot \mathbf{y} \cdot \mathbf{z}_8) \cdot \mathbf{y}' \cdot (\mathbf{z}_4 - \mathbf{z}_6 \cdot \mathbf{y} \cdot \mathbf{z}_7)$$

In terms of these inverse 2-tensors the three current equations are

$$\mathbf{i}^1 = \mathbf{y}'' \cdot [\mathbf{e}_1 - \mathbf{z}_3 \cdot \mathbf{y} \cdot \mathbf{e}_3 - (\mathbf{z}_2 - \mathbf{z}_3 \cdot \mathbf{y} \cdot \mathbf{z}_8) \cdot \mathbf{y}' \cdot (\mathbf{e}_2 - \mathbf{z}_6 \cdot \mathbf{y} \cdot \mathbf{e}_3)]$$

$$\mathbf{i}^2 = \mathbf{y}' \cdot [\mathbf{e}_2 - \mathbf{z}_6 \cdot \mathbf{y} \cdot \mathbf{e}_3 - (\mathbf{z}_4 - \mathbf{z}_6 \cdot \mathbf{y} \cdot \mathbf{z}_7) \cdot \mathbf{i}^1] \qquad 10.37$$

$$\mathbf{i}^3 = \mathbf{y} \cdot (\mathbf{e}_3 - \mathbf{z}_7 \cdot \mathbf{i}^1 - \mathbf{z}_8 \cdot \mathbf{i}^2)$$

(*g*) *Each formula in these sections is valid if each tensor is a compound tensor of any complexity.* That is, each of the three groups of circuits themselves may be built up from several smaller groups of circuits.

X. ARBITRARY SUBDIVISION OF COMPOUND TENSORS

(*a*) When a set of ordinary equations is replaced by a set of n invariant equations in which each component is a 2-tensor (or a 1-tensor) they are simplified by: (1) eliminating *one* row and column at a time with the equations of Section V by considering it a doubly compound tensor with $2^2 = 4$ components; (2) eliminating *two* rows and columns at a time with the equations of Section IX by considering it a doubly compound tensor with $3^2 = 9$ components.

That is, when the previous equations are to be used in connection with a compound tensor, its subdivisions, except the first one, include only *one* row and column as

$$\mathbf{z} = \begin{array}{|ccccc|} \hline z_1 & z_2 & z_3 & z_4 & z_5 \\ z_6 & z_7 & z_8 & z_9 & z_{10} \\ z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{16} & z_{17} & z_{18} & z_{19} & z_{20} \\ \hline z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ \hline \end{array} \qquad \mathbf{z} = \begin{array}{|ccccc|} \hline z_1 & z_2 & z_3 & z_4 & z_5 \\ z_6 & z_7 & z_8 & z_9 & z_{10} \\ z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ \hline z_{16} & z_{17} & z_{18} & z_{19} & z_{20} \\ \hline z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ \hline \end{array}$$

This type of subdivision was necessary, since then only the *inverse* of a diagonal 2-tensor, $\mathbf{z}_{25}$ or $\mathbf{z}'_{19}$, had to be calculated.

(*b*) However, *the doubly compound tensor may be subdivided by including two or three rows and columns in each subdivision, such as*

$$\mathbf{z} = \begin{array}{|cc|ccc|cc|} \hline z_1 & & z_2 & z_3 & & z_4 & \\ z_5 & z_6 & & & z_7 & & z_8 \\ \hline & z_9 & & & z_{10} & & z_{11} \\ z_{12} & & z_{13} & & & z_{14} & \\ z_{15} & & & z_{16} & & z_{17} & z_{18} \\ \hline & z_{19} & & z_{20} & & & z_{21} \\ z_{22} & & z_{23} & & & z_{24} & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline z_a & z_b & z_c \\ \hline z_d & z_f & z_g \\ \hline z_h & z_j & z_k \\ \hline \end{array} \qquad 10.38$$

$$\mathbf{e} = \begin{array}{|cc|ccc|cc|} \hline e_1 & e_2 & & e_4 & e_5 & & e_7 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline e_a & e_b & e_c \\ \hline \end{array} \qquad 10.39$$

$$\mathbf{i} = \begin{array}{|cc|ccc|cc|} \hline i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline i_a & i_b & i_c \\ \hline \end{array} \qquad 10.40$$

In this case also the previously developed equations may be used to eliminate several sets of variables or to solve the equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$. However, they require the calculation of the inverse of compound 2-tensors with two or three rows and columns. These formulas are developed presently.

XI. THE INVERSE OF A TWO-ROWED COMPOUND 2-TENSOR

(*a*) One way to establish the inverse of $\mathbf{z}$ where $\mathbf{z}$ is a compound 2-tensor is actually to solve the several sets of invariant equations and establish $\mathbf{i} = \mathbf{z}^{-1} \cdot \mathbf{e}$. In Sections V and IX two and three sets of linear tensor equations were already solved, but the solutions were not completed to the presently desired form, since $\mathbf{i}^2$ and $\mathbf{i}^3$ were expressed in terms of $\mathbf{i}^1$ while for the inverse it is required to express $\mathbf{i}^2$ and $\mathbf{i}^3$ also in terms of the impressed voltages $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

(*b*) Considering first *two* equations given in Section V, the currents are expressed in equations 10.17 and 10.20 as

$$\left.\begin{aligned} \mathbf{i}^1 &= (\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3)^{-1} \cdot (\mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2) \\ \mathbf{i}^2 &= \mathbf{z}_4^{-1} \cdot (\mathbf{e}_2 - \mathbf{z}_3 \cdot \mathbf{i}^1) \end{aligned}\right\} \qquad 10.41$$

The currents $\mathbf{i}^1$ and $\mathbf{i}^2$ have to be expressed as linear functions of $\mathbf{e}_1$ and $\mathbf{e}_2$, hence substituting the value of $\mathbf{i}^1$ into $\mathbf{i}^2$

$$\mathbf{i}^1 = (\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3)^{-1} \cdot \mathbf{e}_1 - (\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3)^{-1} \cdot \mathbf{z}_2\mathbf{z}_4^{-1} \cdot \mathbf{e}_2$$

$$\begin{aligned} \mathbf{i}^2 = \mathbf{z}_4{}^{-1} \cdot \{\mathbf{e}_2 - \mathbf{z}_3 \cdot [(\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4{}^{-1} \cdot \mathbf{z}_3)^{-1} \cdot \mathbf{e}_1 \\ -(\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4{}^{-1} \cdot \mathbf{z}_3)^{-1} \cdot \mathbf{z}_2 \cdot \mathbf{z}_4{}^{-1} \cdot \mathbf{e}_2]\} \end{aligned}$$

$$\begin{aligned} \mathbf{i}^2 = -\, \mathbf{z}_4^{-1} \cdot \mathbf{z}_3 \cdot (\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3)^{-1} \cdot \mathbf{e}_1 \\ + [\mathbf{z}_4^{-1} + \mathbf{z}_4^{-1} \cdot \mathbf{z}_3(\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3)^{-1} \cdot \mathbf{z}_2 \cdot \mathbf{z}_4^{-1}] \cdot \mathbf{e}_2. \end{aligned}$$

If $(\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3)^{-1} = \mathbf{y}'$ and $\mathbf{z}_4^{-1} = \mathbf{y}$, then the inverse set of equations $\mathbf{i} = \mathbf{y} \cdot \mathbf{e}$ is

$$\left.\begin{aligned} \mathbf{i}^1 &= \mathbf{y}' \cdot \mathbf{e}_1 - \mathbf{y}' \cdot \mathbf{z}_2 \cdot \mathbf{y} \cdot \mathbf{e}_2 \\ \mathbf{i}^2 &= -\, \mathbf{y} \cdot \mathbf{z}_3 \cdot \mathbf{y}' \cdot \mathbf{e}_1 + (\mathbf{y} + \mathbf{y} \cdot \mathbf{z}_3 \cdot \mathbf{y}' \cdot \mathbf{z}_2 \cdot \mathbf{y}) \cdot \mathbf{e}_2 \end{aligned}\right\} \qquad 10.42$$

(*c*) That is, *the inverse of a two-rowed compound tensor* $\mathbf{z}$ *is*

$$\mathbf{z} = \boxed{\begin{array}{c|c} \mathbf{z}_1 & \mathbf{z}_2 \\ \hline \mathbf{z}_3 & \mathbf{z}_4 \end{array}} \qquad \mathbf{z}^{-1} = \boxed{\begin{array}{c|c} \mathbf{y}' & -\,\mathbf{y}'\cdot\mathbf{z}_2\cdot\mathbf{y} \\ \hline -\,\mathbf{y}\cdot\mathbf{z}_3\cdot\mathbf{y}' & \mathbf{y} + \mathbf{y}\cdot\mathbf{z}_3\cdot\mathbf{y}'\cdot\mathbf{z}_2\cdot\mathbf{y} \end{array}} \qquad 10.43$$

where the two inverse 2-tensors to be calculated are

$$\boxed{\mathbf{y} = \mathbf{z}_4^{-1}} \quad 10.44 \qquad \boxed{\mathbf{y}' = (\mathbf{z}_1 - \mathbf{z}_2\cdot\mathbf{y}\cdot\mathbf{z}_3)^{-1}} \quad 10.45$$

The steps in the calculation of $\mathbf{z}^{-1}$ are:

1. $\mathbf{z}_4^{-1} = \mathbf{y}$
2. $\mathbf{y} \cdot \mathbf{z}_3$
3. $[\mathbf{z}_1 - \mathbf{z}_2 \cdot (\mathbf{y} \cdot \mathbf{z}_3)]^{-1} = \mathbf{y}'$
4. $(\mathbf{y} \cdot \mathbf{z}_3) \cdot \mathbf{y}'$
5. $\mathbf{y}' \cdot \mathbf{z}_2 \cdot \mathbf{y}$
6. $\mathbf{y} + (\mathbf{y} \cdot \mathbf{z}_3) \cdot (\mathbf{y}' \cdot \mathbf{z}_2 \cdot \mathbf{y})$

That is, only the same two inverse 2-tensors have to be calculated as in the previous section. The first three steps include these inverse calculations, the steps being the same as in the previous sections. The remaining work consists of three multiplications and one subtraction.

The formulas are equally valid if $\mathbf{z}$ is a multiply compound 2-tensor of any multiplicity.

(*d*) If the compound 2-tensor is a *diagonal* tensor its inverse is found by calculating the inverse of each diagonal component separately, as for ordinary 2-matrices. That is

$$\mathbf{z} = \begin{array}{|c|c|c|c|}\hline \mathbf{z}_1 & & & \\ \hline & \mathbf{z}_2 & & \\ \hline & & \mathbf{z}_3 & \\ \hline & & & \mathbf{z}_4 \\ \hline\end{array} \qquad \mathbf{z}^{-1} = \begin{array}{|c|c|c|c|}\hline \mathbf{z}_1^{-1} & & & \\ \hline & \mathbf{z}_2^{-1} & & \\ \hline & & \mathbf{z}_3^{-1} & \\ \hline & & & \mathbf{z}_4^{-1} \\ \hline\end{array} \qquad 10.46$$

These formulas are valid also if each component of the compound tensor is not a 2-tensor or 2-matrix but a linear operator such as $d(\cos \omega t)/dt$.

XII. THE INVERSE OF A THREE-ROWED COMPOUND 2-TENSOR

(*a*) The inverse of a three-rowed compound 2-tensor may be derived by considering it a two-rowed doubly compound tensor as

$$\mathbf{z} = \begin{array}{|c|c|c|}\hline \mathbf{z}_a & \mathbf{z}_b & \mathbf{z}_c \\ \hline \mathbf{z}_d & \mathbf{z}_e & \mathbf{z}_f \\ \hline \mathbf{z}_g & \mathbf{z}_h & \mathbf{z}_i \\ \hline\end{array} = \begin{array}{|c|c|}\hline \mathbf{z}_1 & \mathbf{z}_2 \\ \hline \mathbf{z}_3 & \mathbf{z}_4 \\ \hline\end{array} \qquad \begin{array}{ll} \mathbf{z}_1 = \begin{array}{|c|c|}\hline \mathbf{z}_a & \mathbf{z}_b \\ \hline \mathbf{z}_d & \mathbf{z}_e \\ \hline\end{array} & \mathbf{z}_2 = \begin{array}{|c|}\hline \mathbf{z}_c \\ \hline \mathbf{z}_f \\ \hline\end{array} \\ \mathbf{z}_3 = \begin{array}{|c|c|}\hline \mathbf{z}_g & \mathbf{z}_h \\ \hline\end{array} & \mathbf{z}_4 = \begin{array}{|c|}\hline \mathbf{z}_i \\ \hline\end{array} \end{array} \qquad 10.47$$

The inverse of the doubly compound tensor has been given in equation 10.43, *where each tensor is a doubly compound tensor*, that is $\mathbf{y}'$ has two rows and columns, etc. Four compound tensors have to be found altogether, namely $\mathbf{z}_2$, $\mathbf{z}_3$, $\mathbf{y}$, and $\mathbf{y}'$. Now $\mathbf{z}_2$ and $\mathbf{z}_3$ are already known, $\mathbf{y}$ is easily calculated as $\mathbf{z}_1^{-1}$, so that only $\mathbf{y}'$ is left to be calculated.

The inverse of $\mathbf{y}'$ is calculated first as $\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3$, giving

$$\mathbf{y}'^{-1} = \begin{array}{|c|c|}\hline \mathbf{z}_a - \mathbf{z}_c \cdot \mathbf{z}_i^{-1} \cdot \mathbf{z}_g & \mathbf{z}_b - \mathbf{z}_c \cdot \mathbf{z}_i^{-1} \cdot \mathbf{z}_h \\ \hline \mathbf{z}_d - \mathbf{z}_f \cdot \mathbf{z}_i^{-1} \cdot \mathbf{z}_g & \mathbf{z}_e - \mathbf{z}_f \cdot \mathbf{z}_i^{-1} \cdot \mathbf{z}_h \\ \hline\end{array} = \begin{array}{|c|c|}\hline \mathbf{z}_1' & \mathbf{z}_2' \\ \hline \mathbf{z}_3' & \mathbf{z}_4' \\ \hline\end{array}$$

Its inverse is found by equation 10.43.

$$\mathbf{y}' = \begin{array}{|c|c|}\hline \mathbf{y}''' & -\mathbf{y}''' \cdot \mathbf{z}_2' \cdot \mathbf{y}'' \\ \hline -\mathbf{y}'' \cdot \mathbf{z}_3' \cdot \mathbf{y}''' & \mathbf{y}'' + \mathbf{y}'' \cdot \mathbf{z}_3' \cdot \mathbf{y}''' \cdot \mathbf{z}_2' \cdot \mathbf{y}'' \\ \hline\end{array} \qquad 10.48$$

where $\mathbf{y}'' = \mathbf{z}_4'^{-1}$ and $\mathbf{y}''' = (\mathbf{z}_1' - \mathbf{z}_2' \cdot \mathbf{y}'' \cdot \mathbf{z}_3')^{-1}$. 10.49

(*b*) To substitute $\mathbf{y}$, $\mathbf{y}'$, $\mathbf{z}_2$, and $\mathbf{z}_3$ into equation 10.43 calculate

$$\mathbf{y}'\cdot\mathbf{z}_2 = \boxed{\begin{array}{c} \mathbf{y}'''\cdot(\mathbf{z}_c - \mathbf{z}_2'\cdot\mathbf{y}''\cdot\mathbf{z}_f) \\ \hline \mathbf{y}''\cdot[\mathbf{z}_f - \mathbf{z}_3'\cdot\mathbf{y}'''\cdot(\mathbf{z}_c - \mathbf{z}_2'\cdot\mathbf{y}''\cdot\mathbf{z}_f)] \end{array}}$$

$$\mathbf{z}_3\cdot\mathbf{y}' = \boxed{\begin{array}{c|c} (\mathbf{z}_g - \mathbf{z}_h\cdot\mathbf{y}''\cdot\mathbf{z}_3')\cdot\mathbf{y}''' & [\mathbf{z}_h - (\mathbf{z}_g - \mathbf{z}_h\cdot\mathbf{y}''\cdot\mathbf{z}_3')\cdot\mathbf{y}'''\cdot\mathbf{z}_2']\cdot\mathbf{y}'' \end{array}}$$

$$\begin{aligned} \mathbf{z}_3\cdot\mathbf{y}'\cdot\mathbf{z}_2 = {} & (\mathbf{z}_g - \mathbf{z}_h\cdot\mathbf{y}''\cdot\mathbf{z}_3')\cdot\mathbf{y}'''\cdot\mathbf{z}_c \\ & + [\mathbf{z}_h - (\mathbf{z}_g - \mathbf{z}_h\cdot\mathbf{y}''\cdot\mathbf{z}_3')\cdot\mathbf{y}'''\cdot\mathbf{z}_2']\cdot\mathbf{y}''\cdot\mathbf{z}_f \end{aligned}$$

Substituting into equation 10.43 *the inverse of the three-rowed 2-tensor* of equation 10.47 is $\mathbf{z}^{-1}$ =

$$\begin{array}{|c|c|c|} \hline \mathbf{y}''' & -\mathbf{y}'''\cdot\mathbf{z}_2'\cdot\mathbf{y}'' & -\mathbf{y}'''\cdot(\mathbf{z}_c-\mathbf{z}_2'\cdot\mathbf{y}''\cdot\mathbf{z}_f)\cdot\mathbf{y} \\ \hline -\mathbf{y}''\cdot\mathbf{z}_3'\cdot\mathbf{y}''' & \mathbf{y}''+\mathbf{y}''\cdot\mathbf{z}_3'\cdot\mathbf{y}'''\cdot\mathbf{z}_2'\cdot\mathbf{y}'' & -\mathbf{y}''\cdot[\mathbf{z}_f-\mathbf{z}_3'\cdot\mathbf{y}'''\cdot(\mathbf{z}_c-\mathbf{z}_2'\cdot\mathbf{y}''\cdot\mathbf{z}_f)]\cdot\mathbf{y} \\ \hline -\mathbf{y}\cdot(\mathbf{z}_g-\mathbf{z}_h\cdot\mathbf{y}''\cdot\mathbf{z}_3')\cdot\mathbf{y}''' & \begin{array}{r} -\mathbf{y}\cdot[\mathbf{z}_h-(\mathbf{z}_g+\mathbf{z}_h\cdot\mathbf{y}''\cdot\mathbf{z}_3') \\ \cdot\mathbf{y}'''\cdot\mathbf{z}_2']\cdot\mathbf{y}'' \end{array} & \begin{array}{r} \mathbf{y}+\mathbf{y}\cdot(\mathbf{z}_g-\mathbf{z}_h\cdot\mathbf{y}''\cdot\mathbf{z}_3')\cdot\mathbf{y}'''\cdot\mathbf{z}_c\cdot\mathbf{y}+ \\ \mathbf{y}\cdot[\mathbf{z}_h-(\mathbf{z}_g-\mathbf{z}_h\cdot\mathbf{y}''\cdot\mathbf{z}_3')\cdot\mathbf{y}'''\cdot\mathbf{z}_2']\cdot \\ \mathbf{y}''\cdot\mathbf{z}_f\cdot\mathbf{y} \end{array} \\ \hline \end{array} \qquad 10.50$$

Several of the products occur many times, hence duplication of work is avoided even if there is no symmetry in the original 2-tensor.

XIII. SUMMARY OF THE REDUCTIONS OF LINEAR EQUATIONS

The set of n linear equations $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ may be reduced to less than n equations or may be completely solved by:

1. Eliminating three, four, or more rows at a time with equations 10.18 and 10.19. This requires the calculation of the inverse of a matrix with three, four, or more rows *during each step* of the elimination.

2. Eliminating all rows at one step by dividing $\mathbf{z}$ into 2^2 or 3^2 components and calculating the reduced $\mathbf{z}$ by *one* formula with equations 10.18 or 10.33. This requires again the calculation of the inverse of two or three smaller matrices. Instead of 2^2 or 3^2 components $\mathbf{z}$ may be divided into n^2 components and formulas may be developed for the immediate calculation of the reduced $\mathbf{z}$ by following the reasoning of Section IX.

3. Instead of developing new formulas for the elimination of $n - 1$ groups, the n^2 matrices into which $\mathbf{z}$ is subdivided may again be

subdivided into 2^2 or 3^2 compound 2-matrices ($\mathbf{z}$ being then a doubly compound 2-matrix), and the procedure of the previous paragraph is followed by assuming that each n-matrix in the equation stands for a compound n-matrix.

4. *The calculation of the inverse of matrices can be entirely avoided by eliminating one row and column at a time, instead of three or four with equations* 10.18 *and* 10.19. In that case the matrix $\mathbf{z}_4$ becomes a scalar Z_4 and its inverse is simply $1/Z_4$. However, $\mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3$ is a matrix having one less rows and columns than the original matrix.

It should be noted that when the matrix $\mathbf{z}$ is divided *arbitrarily* into several smaller matrices to eliminate the variables, without considering the *physical* meaning of the subdivisions, then *only n-matrices enter into the picture during the elimination and not tensors.*

XIV. STAR-MESH TRANSFORMATIONS

(*a*) When $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ represents the equations of a mesh network then *the elimination of one row and column at a time is the equivalent to eliminating one mesh at a time from the network by a star-mesh transformation.* The mesh that is eliminated may have mutual inductances with *any* of the other meshes and may have voltages impressed around it. In the usual methods of star-mesh transformations the mesh to be eliminated may have a mutual inductance only with its neighboring meshes and may have no impressed voltages.

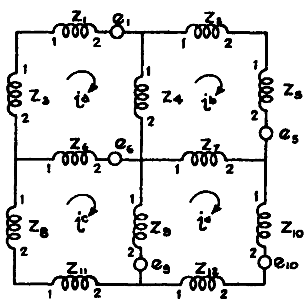


FIG. 10.3.—Network to be Reduced

(*b*) Let for instance twelve coils be interconnected into four meshes as shown in Fig. 10.3.

Before interconnection the components of the impressed voltage vector $\mathbf{e}$ and impedance tensor $\mathbf{z}$ are

$$
\mathbf{e} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}
\begin{array}{|c|} \hline e_1 \\ \hline \\ \hline \\ \hline \\ \hline e_5 \\ \hline e_6 \\ \hline \\ \hline \\ \hline e_9 \\ \hline e_{10} \\ \hline \\ \hline \\ \hline \end{array}
\qquad 10.51
$$

$\mathbf{z} =$ 10.52

	1	2	3	4	5	6	7	8	9	10	11	12
1	Z_1											
2		Z_2										
3			Z_3									
4				Z_4								
5					Z_5							
6						Z_6						
7							Z_7					
8								Z_8				
9									Z_9			
10										Z_{10}		
11											Z_{11}	
12												Z_{12}

The transformation tensor is

$$
\begin{aligned}
i^1 &= i^a \\
i^2 &= i^b \\
i^3 &= -\,i^a \\
i^4 &= i^a - i^b \\
i^5 &= i^b \\
i^6 &= i^c - i^a \\
i^7 &= i^d - i^b \\
i^8 &= -\,i^c \\
i^9 &= i^c - i^d \\
i^{10} &= i^d \\
i^{11} &= -\,i^c \\
i^{12} &= -\,i^d
\end{aligned}
$$

$\mathbf{C} =$ 10.53

	a	b	c	d
1	1			
2		1		
3	−1			
4	1	−1		
5		1		
6	−1		1	
7		−1		1
8			−1	
9			1	−1
10				1
11			−1	
12				−1

After interconnection the impressed voltage vector $\mathbf{e}'$ and the impedance tensor $\mathbf{z}'$ are found by $\mathbf{C}_t \cdot \mathbf{e}$ and by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ as

$$\mathbf{e}' = \begin{array}{c|c|} & \\ \mathbf{a} & e_1 - e_6 \\ \mathbf{b} & e_5 \\ \mathbf{c} & e_6 + e_9 \\ \mathbf{d} & -e_9 + e_{10} \end{array} \qquad 10.54$$

$$\mathbf{z}' = \begin{array}{c|c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \mathbf{a} & Z_1+Z_3+Z_4+Z_6 & -Z_4 & -Z_6 & 0 \\ \mathbf{b} & -Z_4 & Z_2+Z_4+Z_5+Z_7 & 0 & -Z_7 \\ \mathbf{c} & -Z_6 & 0 & Z_6+Z_8+Z_9+Z_{11} & -Z_9 \\ \mathbf{d} & 0 & -Z_7 & -Z_9 & Z_7+Z_9+Z_{10}+Z_{12} \end{array} \qquad 10.55$$

(c) Eliminating the last row and column by equation 10.18, namely by $\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4{}^{-1} \cdot \mathbf{z}_3$, then

$$\mathbf{z}_2 \cdot \mathbf{z}_4^{-1} = \begin{array}{c|c|} & \mathbf{d} \\ \mathbf{a} & 0 \\ \mathbf{b} & -Z_7/(Z_7+Z_9+Z_{10}+Z_{12}) \\ \mathbf{c} & -Z_9/(Z_7+Z_9+Z_{10}+Z_{12}) \end{array}$$

$$\mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3 = \begin{array}{|c|c|c|} 0 & 0 & 0 \\ 0 & (Z_7)^2/(Z_7+Z_9+Z_{10}+Z_{12}) & Z_7Z_9/(Z_7+Z_9+Z_{10}+Z_{12}) \\ 0 & Z_7Z_9/(Z_7+Z_9+Z_{10}+Z_{12}) & (Z_9)^2/(Z_7+Z_9+Z_{10}+Z_{12}) \end{array}$$

Hence the reduced impedance tensor is

$$\begin{array}{c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} & Z_1+Z_3+Z_4+Z_6 & -Z_4 & -Z_6 \\ \mathbf{b} & -Z_4 & Z_2+Z_4+Z_5+Z_7 - \dfrac{(Z_7)^2}{Z_7+Z_9+Z_{10}+Z_{12}} & -\dfrac{Z_7Z_9}{Z_7+Z_9+Z_{10}+Z_{12}} \\ \mathbf{c} & -Z_6 & -\dfrac{Z_7Z_9}{Z_7+Z_9+Z_{10}+Z_{12}} & Z_6+Z_8+Z_9+Z_{11} - \dfrac{(Z_9)^2}{Z_7+Z_9+Z_{10}+Z_{12}} \end{array} \qquad 10.56$$

The reduced impressed voltage vector, by equation 10.19, namely by $\mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2$, is

$$
\mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2 = \begin{array}{c|c|}
\hline
\mathrm{a} & 0 \\
\hline
\mathrm{b} & [-Z_7/(Z_7 + Z_9 + Z_{10} + Z_{12})]\,(e_{10} - e_9) \\
\hline
\mathrm{c} & [-Z_9/(Z_7 + Z_9 + Z_{10} + Z_{12})]\,(e_{10} - e_9) \\
\hline
\end{array}
$$

$$
\mathbf{e}' = \begin{array}{c|c|}
\hline
\mathrm{a} & e_1 - e_6 \\
\hline
\mathrm{b} & e_5 + [Z_7/(Z_7 + Z_9 + Z_{10} + Z_{12})](e_{10} - e_9) \\
\hline
\mathrm{c} & e_6 + e_9 + [Z_9/(Z_7 + Z_9 + Z_{10} + Z_{12})](e_{10} - e_9) \\
\hline
\end{array} = \begin{array}{|c|}
\hline
e_1 - e_6 \\
\hline
e_5 + e_b \\
\hline
e_6 + e_9 + e_c \\
\hline
\end{array} \qquad 10.57
$$

The reduced network corresponding to the reduced $\mathbf{z}'$ and $\mathbf{e}'$ is shown in Fig. 10.4, where $Z = Z_7 + Z_9 + Z_{10} + Z_{12}$.

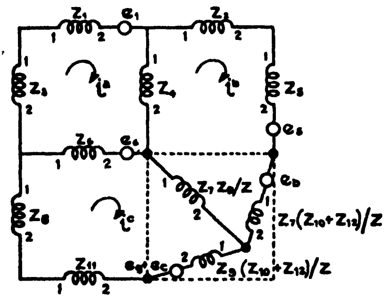


FIG. 10.4.—Star-mesh Transformation

(*d*) *Now when* $\mathbf{z}'$ *is considered as a compound 2-tensor, the elimination of one row and column from the compound tensor is the equivalent of eliminating several meshes at a time by a multiplicity of star-mesh transformations.* The meshes that are simultaneously eliminated may have mutual inductances with one another and with the other meshes; also they may have voltages impressed around them.

XV. THE IMPEDANCE BETWEEN TWO POINTS

(*a*) Quite often the problem is to find the impedance of an n-mesh network between two points. In such problems *it is advantageous to introduce an additional mesh* by impressing a voltage e_1 between the two points (or junctions) and by assuming the current i in this impedanceless branch as the first variable.

As a result the equation $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ contains $n + 1$ rows and columns. *If the last n rows and columns of* $\mathbf{z}$ *are eliminated* (either in one or in several steps), *the remaining scalar is the impedance of the network between the two points.*

(*b*) For instance, let the impedance of the five-mesh network of Fig. 10.5 be found between junctions A and B. (There are 12 coils, 8 junctions, $8 - 1 = 7$ junction-pairs; hence there are $12 - 7 = 5$ meshes.) The junctions A and B are interconnected to form a sixth mesh.

Assuming six variables (the first one i^1 flowing in the branch connecting A and B), the currents flowing through each coil are shown

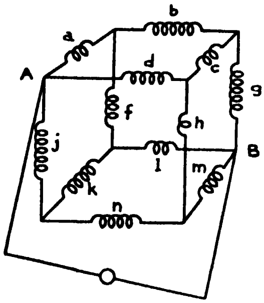


FIG. 10.5.—Impedance between Two Points A–B

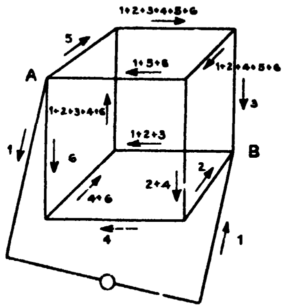


FIG. 10.6.—Currents in Individual Coils

in Fig. 10.6. Equating the old and the new currents flowing in each coil, the transformation tensor is

$$
\begin{aligned}
i^a &= i^5 \\
i^b &= i^1 + i^2 + i^3 + i^4 + i^5 + i^6 \\
i^c &= i^1 + i^2 + i^4 + i^5 + i^6 \\
i^d &= i^1 + i^5 + i^6 \\
i^f &= i^1 + i^2 + i^3 + i^4 + i^6 \\
i^g &= i^3 \\
i^h &= i^2 + i^4 \\
i^j &= i^6 \\
i^k &= i^4 + i^6 \\
i^l &= i^1 + i^2 + i^3 \\
i^m &= i^2 \\
i^n &= i^4 \\
i^p &= i^1
\end{aligned}
$$

$$
\mathbf{C} = \begin{array}{c|cccccc|}
 & 1' & 2' & 3' & 4' & 5' & 6' \\
\hline
\mathbf{a} & & & & & 1 & \\
\mathbf{b} & 1 & 1 & 1 & 1 & 1 & 1 \\
\mathbf{c} & 1 & 1 & & 1 & 1 & 1 \\
\mathbf{d} & 1 & & & & 1 & 1 \\
\mathbf{f} & 1 & 1 & 1 & 1 & & 1 \\
\mathbf{g} & & & 1 & & & \\
\mathbf{h} & & 1 & & 1 & & \\
\mathbf{j} & & & & & & 1 \\
\mathbf{k} & & & & 1 & & 1 \\
\mathbf{l} & 1 & 1 & 1 & & & \\
\mathbf{m} & & 1 & & & & \\
\mathbf{n} & & & & 1 & & \\
\mathbf{p} & 1 & & & & & \\
\hline
\end{array}
\qquad 10.58
$$

The impedance tensor $\mathbf{z}$ of the primitive network contains thirteen rows and columns, the last row and column $\mathbf{p}$ having all zero compo-

nents. *Assuming all equal impedances Z without any mutual impedances*, the $\mathbf{z}'$ of the network is, by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$

$$\mathbf{z}' = Z \times \begin{array}{c|cccccc|} & 1' & 2' & 3' & 4' & 5' & 6' \\ \hline 1' & 5 & 4 & 3 & 3 & 3 & 4 \\ 2' & 4 & 6 & 3 & 4 & 2 & 3 \\ 3' & 3 & 3 & 4 & 2 & 1 & 2 \\ \hline 4' & 3 & 4 & 2 & 6 & 2 & 4 \\ 5' & 3 & 2 & 1 & 2 & 4 & 3 \\ 6' & 4 & 3 & 2 & 4 & 3 & 6 \\ \hline \end{array} = \begin{array}{|c|c|} \hline \mathbf{z}_1 & \mathbf{z}_2 \\ \hline \mathbf{z}_3 & \mathbf{z}_4 \\ \hline \end{array} \qquad 10.59$$

Eliminating the last *three* rows and columns by $\mathbf{z}_1' = \mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3$,

$$\mathbf{z}_1' = \begin{array}{|c|c|c|} \hline 5 & 4 & 3 \\ \hline 4 & 6 & 3 \\ \hline 3 & 3 & 4 \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline 3 & 3 & 4 \\ \hline 4 & 2 & 3 \\ \hline 2 & 1 & 2 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline .3 & 0 & .2 \\ \hline 0 & .4 & .2 \\ \hline -\ 2 & -.2 & .4 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline 3 & 4 & 2 \\ \hline 3 & 2 & 1 \\ \hline 4 & 3 & 2 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1.9 & 1\ 6 & 1.6 \\ \hline 1\ 6 & 3\ 2 & 1.6 \\ \hline 1\ 6 & 1\ 6 & 3.2 \\ \hline \end{array}$$

Eliminating again the last *two* rows and columns by the same formula

$$\mathbf{z}_1' = 1.9 - \begin{array}{|c|c|} \hline 1\ 6 & 1.6 \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline 0.4165 & -0.2083 \\ \hline -0.2083 & 0\ 4165 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 1\ 6 \\ \hline 1\ 6 \\ \hline \end{array} = 0.833\, Z = \tfrac{5}{6} Z \qquad 10.60$$

Hence, if the impedance of each border of the cube is Z, then the impedance of the whole cube measured across the diagonal axis is $\frac{5}{6} Z$. Of course because of its symmetry, this particular example can be solved much more quickly by simple inspection.

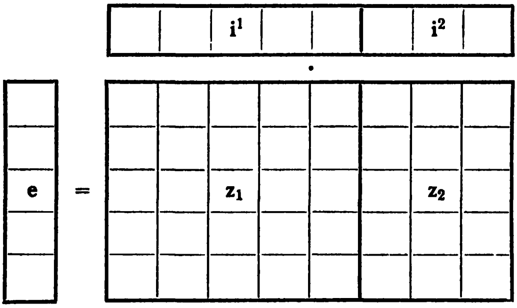
(*c*) If the impedance of a network as viewed from *several sets of points of entry* is wanted, then each set of points of entry is short-circuited and is considered as an additional mesh. From the calculated impedance tensor $\mathbf{z}'$ all rows and columns are eliminated except those of the additional axes. The remaining $\mathbf{z}''$ represents the self- and mutual impedances of the network as viewed from the points of entry.

XVI. *n* SIMULTANEOUS EQUATIONS WITH *k* VARIABLES

(*a*) In the equations hitherto considered it was assumed that: (1) $\mathbf{z}$ is square; (2) the inverse of the diagonal matrices $\mathbf{z}_1$, $\mathbf{z}_4$, etc., can be calculated.

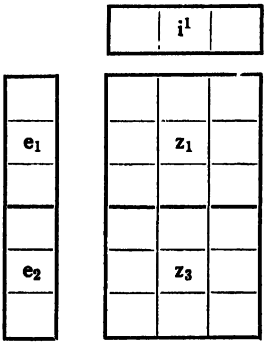
However, quite often linear equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ occur where there are:

1. More variables than there are equations, such as



$$\mathbf{e} = \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 \quad 10.61$$

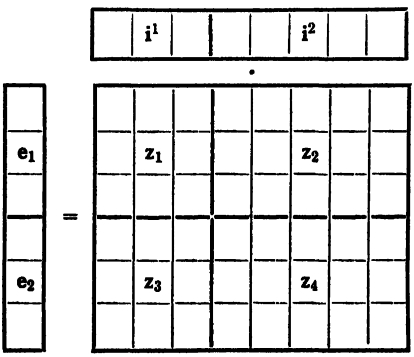
2. Fewer variables than equations



$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_1 \cdot \mathbf{i}^1 \\ \mathbf{e}_2 &= \mathbf{z}_3 \cdot \mathbf{i}^1 \end{aligned} \quad 10.62$$

In these cases $\mathbf{z}$ is a rectangle instead of a square.

3. Also it happens with square or with rectangular $\mathbf{z}$ that only a component matrix $\mathbf{z}_1$ with a small number of rows and columns has an inverse, as in



$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 \\ \mathbf{e}_2 &= \mathbf{z}_3 \cdot \mathbf{i}^1 + \mathbf{z}_4 \cdot \mathbf{i}^2 \end{aligned} \quad 10.63$$

(*b*) In all these cases the set of tensor equations can be solved only for $\mathbf{i}^1$ (assuming that only $\mathbf{z}_1$ has an inverse). Even then it is expressed as a function of $\mathbf{i}^2$, so that the components of $\mathbf{i}^2$ may assume arbitrary values.

If $\mathbf{z}_1$ contains no more than the maximum number of rows and columns which will permit the calculation of its inverse (its determinant being different from zero), this maximum number is called the " rank " of the matrix $\mathbf{z}$. The equations should be rearranged so that the matrix $\mathbf{z}_1$ with the maximum rank is in the upper left-hand corner of $\mathbf{z}$.

(*c*) The solution of the above three cases is

(1).

$$\mathbf{e} = \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2$$

$$\boxed{\mathbf{i}^1 = \mathbf{z}_1^{-1} \cdot (\mathbf{e} - \mathbf{z}_2 \cdot \mathbf{i}^2)} \qquad 10.64$$

(2).

$$\mathbf{e}_1 = \mathbf{z}_1 \cdot \mathbf{i}^1$$

$$\mathbf{e}_2 = \mathbf{z}_3 \cdot \mathbf{i}^1$$

$$\boxed{\mathbf{i}^1 = \mathbf{z}_1^{-1} \cdot \mathbf{e}_1} \qquad 10.65$$

(3).

$$\mathbf{e}_1 = \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2$$

$$\mathbf{e}_2 = \mathbf{z}_3 \cdot \mathbf{i}^1 + \mathbf{z}_4 \cdot \mathbf{i}^2$$

$$\boxed{\mathbf{i}^1 = \mathbf{z}_1^{-1} \cdot (\mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{i}^2)} \qquad 10.66$$

If this value of $\mathbf{i}^1$ is substituted in the second equation to solve for $\mathbf{i}^2$ as

$$\mathbf{i}^2 = (\mathbf{z}_4 - \mathbf{z}_3 \cdot \mathbf{z}_1^{-1} \cdot \mathbf{z}_2)^{-1}(\mathbf{e}_2 - \mathbf{z}_3 \cdot \mathbf{z}_1^{-1} \cdot \mathbf{e}_1) = \mathbf{z}_4'^{-1} \cdot \mathbf{e}_2'$$

in the numerical calculations $\mathbf{z}_4'$ comes out to be zero.

Of course, other cases are possible.

(*d*) As an example let $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ be

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 1 \\ \hline 2 \\ \hline\end{array} = \begin{array}{|c|c|c|c|}\hline x & y & z & w \\ \hline\end{array} \cdot \begin{array}{|c|c|c|c|}\hline 3 & 4 & -1 & -6 \\ \hline 4 & 8 & -2 & -8 \\ \hline 5 & 4 & -1 & -10 \\ \hline 3 & 8 & -2 & -6 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline\end{array} = \begin{array}{|c|c|}\hline \mathbf{i}^1 & \mathbf{i}^2 \\ \hline\end{array} \cdot \begin{array}{|c|c|}\hline \mathbf{z}_1 & \mathbf{z}_2 \\ \hline \mathbf{z}_3 & \mathbf{z}_4 \\ \hline\end{array} \qquad 10.67$$

The determinant of $\mathbf{z}$ is zero, as is any three-rowed determinant of $\mathbf{z}$. However, the determinant of the upper left-hand corner matrix is not zero; hence *the rank of* $\mathbf{z}$ *is two.* Therefore, by equation 10.66

$$\mathbf{i}^1 = \mathbf{z}_1^{-1} \cdot (\mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{i}^2)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 3/8 \end{bmatrix} \cdot \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 & -6 \\ -2 & -8 \end{bmatrix} \cdot \begin{bmatrix} z \\ w \end{bmatrix} \right\}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 3/8 \end{bmatrix} \cdot \begin{bmatrix} 1 + z + 6w \\ 2 + 2z + 8w \end{bmatrix} = \begin{bmatrix} 2w \\ 1/4 + z/4 \end{bmatrix} \qquad 10.68$$

Hence the solution is $x = 2w$ and $y = (1 + z)/4$, where w and z may assume any arbitrary values.

CHAPTER XI

THE THEORY OF "GROUPS"

I. PROPERTIES OF THE TRANSFORMATION TENSOR

(*a*) The transformation tensor $C^{\alpha}_{\alpha'}$ differs in physical interpretation from all other tensors that occur in the analysis of physical systems, such as e_{α}, $z_{\alpha\beta}$, i^{β}, etc. Whereas other tensors represent some *definite property* of the physical system itself, the transformation tensor C represents certain *operations* performed on the system, certain *changes* made in the point of view. *The transformation tensor* $C^{\alpha}_{\alpha'}$ is not a part of the system as $z_{\alpha\beta}$ or e_{α} is, and it *does not occur explicitly in its equation of performance.* Certain transformation tensors may be represented physically by *the pieces of conductors* that interconnect the various impedances but by themselves have no impedance and contribute nothing to the power or energy of the system. They represent only the *reference frame* along which the currents are made to flow.

(*b*) Because of their different physical significance, *the various types of transformation matrices form a separate class in themselves. They belong to a new type of aggregate, called a " group."* That is, each transformation matrix is one element of a " *group* " of transformation matrices, in addition to being an element of a " transformation tensor."

By virtue of the fact that transformation matrices belong to a " group " they are endowed with certain properties that other set of n-matrices (that do not belong to a " group ") *do not possess.*

The study of *one* of these special and additional properties is the subject matter of this chapter. There are, of course, other properties in which the transformation tensor differs from other tensors.

II. THE THEORY OF GROUPS

(*a*) *The " theory of groups " is a new type of algebra in which the entities A, B, C $\cdots$ of a " group " are subjected to one operation only.* The result of this operation is an entity which also belongs to the same group. The operation will be denoted in general by a dot as $A \cdot B = C$.

In ordinary algebra the entities a, b, c are subjected to *several* operations, such as addition, multiplication, division, etc. Because of the large variety of manipulations existing in ordinary algebra, the entities a, b, c that are manipulated are restricted to a rather narrow field. They are usually *single* numbers like 5, 7, $2 + 3j$, $x_1y^2{}_1$, $ax + y$, etc. However, in the theory of groups the number of operations is restricted to *one*, hence the entities A, B, C are of a far greater variety than in ordinary algebra. They may be, of course, single numbers, like 5, 7, etc., but they may also be n-dimensional matrices or still more abstract entities like " rotations " or " permutations " or " transformations," etc.

(b) It may be mentioned that the " algebra of n-dimensional matrices " as developed in the first two chapters of this volume may be looked upon as still another type of algebra in which the entities are of a greater variety than in ordinary algebra (being a " set " of numbers arranged in a row, or a square, or a cube, etc.), but the operations on the entities are more restricted than in ordinary algebra (division is excluded, also in a product $\mathbf{A} \cdot \mathbf{B}$ the factors cannot be interchanged, etc.). The "algebra of n-dimensional matrices" lies between " ordinary algebra " and " the theory of groups." There are, of course, numerous other types of algebras.

It may be shown that, as the number of operations allowed on the entities is successively restricted, the field of concepts from which the entities are selected successively widens. If no operation at all is allowed on the entities, their variety becomes still greater.

When the *entities* of a group are all the possible transformation matrices $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3 \cdots$, the single *operation* that is allowed on them is *multiplication* $\mathbf{C}_1 \cdot \mathbf{C}_2$ in the sense of Section XII, Chapter I. *Other n-matrices besides the transformation matrices may also form a group.*

III. DEFINITION OF A "GROUP"

(a) Not all aggregates of entities that are subjected to a single operation can form a " group." *The entities and the operation they are subjected to must satisfy four conditions in order that they should form a " group."* These four rules, which are satisfied by all transformation matrices $\mathbf{C}$, are as follows:

1. The product entity C of any two entities $A \cdot B$ must belong to the same group. This is called the " *group property.*" That is, C_3 formed by $C_1 \cdot C_2$ belongs to the same group.

2. In the product of several elements $A \cdot B \cdot C$ the multiplication may be performed in any order as $A \cdot (B \cdot C)$ or as $(A \cdot B) \cdot C$. This is the " associative law."

3. One of the elements of the group is the "*unit element*" I, so that multiplication with it leaves any element unchanged. (See equation 1.5 for the definition of the unit matrix **I**.)

4. Each element A has an "*inverse*" element $A^{-1} = B$ also belonging to the group, so that the product of an element A and its inverse A^{-1} is the unit element, I. (See Section XVI, Chapter I, for the calculation of the inverse transformation matrix.)

When the elements do not have an inverse and so the last condition is not satisfied, *the group is called a "semi-group."* Many theorems of the theory of groups apply also to semi-groups.

(*b*) The last three of these conditions are satisfied by square matrices in general, hence also by square transformation matrices. The associative law applies to their multiplication, the unit matrix **I** exists and most square matrix **A** has an inverse $\mathbf{A}^{-1}$, so that $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}$. However, the first rule does not apply in general to matrices, since for instance *two impedance matrices* $\mathbf{Z}_1$ *and* $\mathbf{Z}_2$ *cannot be multiplied together in general.* It is chiefly the first rule which differentiates entities that belong to a group from other entities that do not belong, and for this reason *entities that obey the first rule are said to have the "group property." Such entities are the transformation matrices* **C**, *whether singular or non-singular.* A detailed study of the first rule, the "group property," is the subject matter of this chapter; the last three of these rules have already been studied in connection with the transformation tensor **C**.

(*c*) There is an extensive mathematical literature on the theory of groups. In the last decade it has found wide applications in crystallography, in atomic structure, in quantum dynamics, and in many other fields. The importance of the concept of "groups" may be seen, for instance, from the fact that *the transformation matrix* **C** *of a network alone contains a surprising amount of information about the properties and behavior of the network, without even establishing any other tensors, such as the* **e** *vector or the* **z** *tensor of the actual network.* In a subsequent chapter on the synthesis of networks it will be shown that by simply knowing the transformation matrix **C** of a network (and the impedances of its *individual* coils) it can be foretold that the network has, for instance, a *desired response at all frequencies*, or that the network supplies *constant currents* at all loads, or that it supplies various currents *at constant potential*, without actually establishing $\mathbf{z}'$ of the network and examining it for its performance characteristics.

That is, *the basic characteristics, the skeleton of the performance of a dynamical system, is completely incorporated in the transformation tensor* **C**. *The other tensors simply supply the flesh and blood, but do not*

change the basic qualities of the performance. The exclusiveness of the transformation tensor **C** as an aristocratic member of the society of tensors is clearly shown by the fact that *the transformation tensor* **C** *does not even appear in any equation of performance of a dynamical system.* It keeps itself aloof from the plebeian pick and shovel work of the various types of performance calculations.

IV. EXAMPLES OF FINITE GROUPS

(*a*) The *number* of elements that form a group may vary from a few to infinity. According to the number of elements in a group, they divide into two classes: (1) "*finite groups*," and (2) "*infinite groups*."

An example of a "*finite group*" is the collection of the following four entities:

1. Rotation through 90° $= A$.
2. Rotation through 180° $= B$.
3. Rotation through 270° $= C$.
4. Rotation through 360° $= D$.

The four entities A, B, C, and D form a group, the so-called "*rotation group*," since they satisfy the four rules:

1. The product of any two rotations as $B \cdot C$ is the rotation A (rotation through 90°). Or $B \cdot C = A$.

2. The rotations may be performed in any order, the final angles of rotation being the same.

3. There is a unit element D that leaves all rotations unchanged as $B \cdot D = B$.

4. Each rotation has an inverse. For instance, the inverse of C is A, since $C \cdot A = D =$ unit element.

(*b*) Another example of a "*finite group*" is the group of all transformation matrices $\mathbf{C}_1$, $\mathbf{C}_2$ $\cdots$ that leave a particular n-mesh network unchanged but change the variables to different n branches (such as shown in Section IV, Chapter VI). There are only a finite number of ways in which the n variables may be selected in an n-mesh network.

Another example of a "finite group" is the group of all transformation matrices $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ $\cdots$ *that change a particular n-mesh stationary network into any other n-mesh network.* There are a large (though not infinite) number of ways to change an n-mesh network into other n-mesh networks.

(*c*) An example of a "*semi-group*" is the group of all transformation matrices that changes a network with n coils into other networks with ***different number of coils.*** None of these transformation matrices

has an inverse. Examples of semi-groups will occur in Chapter XXIII in connection with the synthesis of networks.

V. EXAMPLES OF INFINITE GROUPS

(*a*) Of the many important examples of infinite groups are: the *procedure of impressing electromagnetic quantities on a network*, the superposition of geometrical configurations upon a space (or chains upon cells). There is an infinite variety of these procedures, but they all satisfy the four group conditions, if the operation performed upon them is *addition*. That is, using impressed voltages as examples:

1. Impressing two voltages in succession is also equivalent of impressing a voltage, $\mathbf{e}_1 + \mathbf{e}_2 = \mathbf{e}_3$.

2. The resultant voltage is independent of the manner of grouping $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = (\mathbf{e}_1 + \mathbf{e}_2) + \mathbf{e}_3$, etc.

3. There is a *unit* element $\mathbf{e}_0$ that leaves the network unchanged. In this case $\mathbf{e}_0$ is zero.

4. Each impressed voltage $\mathbf{e}$ has an *inverse*, $-\mathbf{e}$, so that the product of the two (which in the present example is an addition), $\mathbf{e} - \mathbf{e}$, gives the unit element, $\mathbf{e}_0$.

(*b*) Another example of an "*infinite group*" is the group of all transformation matrices $\mathbf{C}_1, \mathbf{C}_2 \cdots$ that introduce *hypothetical currents* as variables instead of the actual branch currents. Such hypothetical currents are the "load" and "magnetizing currents." Although it is possible to introduce an infinite variety of hypothetical currents, still the interconnection of the coils puts a certain limitation upon the relation between them.

VI. SUBGROUPS

(*a*) *When all the elements of a group belong also to another group, then the first group is said to be a "subgroup" of the second.* For instance, the elements of the finite group changing the variables in the branches of a network belong to the finite group changing also the interconnections of the network, being only a special case of the latter. That is, the group of transformation matrices *changing the branch currents* is a "subgroup" of the group of transformation matrices *changing also the interconnections* of networks.

The group of transformation matrices changing the interconnection of networks is itself a subgroup of the group of transformation matrices that change the interconnections of rotating machinery with stationary reference axes. In turn the latter are a subgroup of the so-called "*group of linear transformations*," and so on.

(*b*) The various types of groups are usually denoted by the letter G with a subscript as G_1, $G_2 \cdots$, etc., the subscript denoting what particular group is meant. *In this volume the following " groups " of transformation matrices have hitherto been introduced*:

1. G_i = All **C** that change the *interconnections* of the various coils; that is that change an n-coil network into another n-coil network. This **C** may be called the " *connection tensor*."

This group of transformation matrices forms the basis of all network studies, since it represents the codification of Kirchhoff's laws.

2. G_b = All **C** that leave the interconnections unchanged but change the currents from one set of *branches* or meshes to another set.

The components of all transformation matrices belonging to these two groups are always integers. These groups are related to the so-called " Betti-group " used in Topology.

3. G_p = All **C** that represent *permutations* of the axes.

4. G_t = All **C** that change the number of *turns* of the coils. Their components are rational numbers.

5. G_m = All **C** that change actual currents to *magnetizing* and *load* currents.

All these groups are " subgroups " of the following groups:

6. G_n = All **C** that may occur in *network* studies, having real components only.

The group of real transformations that occur in network studies are of a limited type. They form a subgroup of a larger group called

7. G_l = The group of " *linear transformations* " or " *affine transformations*." The components of **C** belonging to this group are any real constants (integers, fractions, irrational numbers).

The latter group in turn is a subgroup of

8. G_f = The group of all *functional transformations* in which the components of **C** are functions of the variables.

The groups G_i and G_m also include the " *semi-groups* " of those **C** that have no inverse. In G_i the number of meshes after interconnection may be less than before interconnection, in G_m the magnetizing current may be neglected. The semigroups will be denoted the same way as the corresponding groups. *Each group of transformation matrices forms a tensor.*

It is emphasized that each time a group of *real* transformations are introduced they form only a *subgroup* of the group of real linear or " affine " transformations. That is *there are many affine transformations that have no meaning in network studies.*

In addition to the groups of real transformations there occur groups of transformations in network studies whose **C** contains complex

components instead of real components. Such transformations are shown in Chapter XIII.

VII. TRANSFORMATION, INVARIANCE, GROUP

(*a*) Due to the large variety of "groups" of transformation matrices that may exist, *whenever an invariant equation is set up as being "invariant" under a "group" of "transformation matrices," it is extremely important to define the "group" of transformation matrices that are understood, since an invariant equation that is invariant under one group, is not invariant under another group.* For instance, the equation $e_\alpha = z_{\alpha\beta} i^\beta$ is valid for a network only if the interconnections of the coils do not vary in time. If they do vary, the equation does not hold, and the equation is not invariant under the latter group G_f.

Hence the three words: (1) "*transformation,*" (2) "*invariance*" *and* (3) "*group*" *always occur together in one phrase,* as for instance: "The power input $\mathbf{e} \cdot \mathbf{i} = P$ is an *invariant* under the *group* of *transformations* G_n.

The concept "transformation" need not necessarily be expressed with the aid of a "transformation matrix $\mathbf{C}$" as $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$. It is a much wider conception than the $\mathbf{C}$. A "transformation" that may be expressed with the aid of a transformation matrix $\mathbf{C}$ is a highly specialized one. Hence the accepted phraseology uses the "group of transformations G" instead of "group of transformation matrices G" in order not to restrict the form of transformation. In this volume "transformation" will mean "transformation of the variables" (i^α or E_α) with the aid of $C^\alpha_{\alpha'}$ or $C^{\alpha'}_\alpha$.

(*b*) It is possible to introduce a set of *transformation* matrices $\mathbf{C}$ which leave the form of an equation invariant, but the $\mathbf{C}$'s do not form a "group." It is also possible to introduce a set of $\mathbf{C}$'s that form a group, but these $\mathbf{C}$'s do not leave a linear or quadratic form "invariant." In these cases *one of the three concepts is missing and the study of such transformations is not tensor analysis*, since among other requirements the concepts of "covariance" and "contravariance" cannot be introduced.

VIII. THE "GROUP PROPERTY"

(*a*) In Chapter VI several types of transformation tensor $\mathbf{C}$ were introduced, each of which changes the system set up or the point of view employed. For each change $\mathbf{C}$ the various tensors have been separately calculated with the aid of the transformation formulas.

However, in many engineering problems not one but several types of

changes are assumed simultaneously. For instance, in a multiwinding transformer system the windings may be subjected to four changes *simultaneously*:

1. The number of turns are changed.
2. The windings are interconnected.
3. The magnetizing currents are neglected.
4. Symmetrical components are introduced.

Or, as another example, in a direct-current machine the conductors in the slots are subjected to several different types of transformations *simultaneously*, in particular:

1. The conductors are connected into coils.
2. The coils are connected to the commutator bars.
3. The commutator bars are interconnected by the brushes.
4. The brushes are interconnected into circuits.
5. The equalizer rings introduce additional current paths.

(*b*) In the usual analysis of engineering problems, these simultaneous changes rarely, if ever, are separated and handled individually. *All non-tensor methods of analysis attempt to take care of all changes simultaneously, without separating them into successive steps.*

Now, *it is possible to set up a transformation matrix* $\mathbf{C}_1$, $\mathbf{C}_2 \cdots \mathbf{C}_n$ *for each of these component changes, and the transformation matrix* $\mathbf{C}$ *of the resultant change is found by taking the product of the individual changes*, as

$$\boxed{\mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2 \cdot \mathbf{C}_3 \cdots \mathbf{C}_n} \qquad \Big| \qquad \boxed{C^{\alpha}_{\alpha^{n\prime}} = C^{\alpha}_{\alpha'} C^{\alpha'}_{\alpha''} C^{\alpha''}_{\alpha'''} \cdots C^{\alpha^n}_{\alpha^{n\prime}}} \qquad 11.1$$

This formula is valid since each transformation matrix is an element of the group of linear transformation G_l and their product is a transformation matrix which also belongs to the group G_l. It may be shown that *this formula is valid even if several or all of the transformation matrices are singular* (rectangular, not square) since this formula represents the so-called "*group property*" which is satisfied by the elements of a semi-group also.

This last formula serves as a powerful labor-saving device in the analysis of complex engineering problems since it enables the engineer to subdivide the "method of analysis" itself into several steps and develop each step as if the other steps were absent, then to recombine the individual steps into one resultant step.

(*c*) It is interesting to note that the "group property" of the process of impressing voltages, shown in Section V*a* (namely, the possibility of impressing several voltages and calculating their effect

separately as if the others were absent), has been used in electrical engineering under the name " superposition theorem." Hence the process of dividing a transformation into a whole series of successive transformations is equivalent to another type of superposition theorem.

IX. THE SUBDIVISIONS OF COMPLEX PROBLEMS

The labor-saving devices previously introduced, that enable the engineer to *subdivide his complex problems into several independent problems, each having fewer complexities,* may be separated roughly into three classes. Assuming a large number of analogous systems to be analyzed simultaneously:

1. The concept of *geometric objects* enables the engineer to subdivide his whole *physical* system arbitrarily into several independent *component physical systems,* each having either different types of geometric objects or just different types of performance. For instance, *any portion of a network may be detached and analyzed separately.* Or a transmission system may be arbitrarily divided into rotating machines, multiwinding transformers, transmission lines, and each subdivision may be analyzed separately as if the others were absent, then finally recombined into the original system.

2. The concept of *transformation tensors* enables the engineer to subdivide each of his great number of analogous physical systems into *two analytical parts*:

A. Those parts that are *identical* for all systems, like $z_{\alpha\beta}$.

B. Those parts that are *different* for each particular system, namely $C^{\alpha}_{\alpha'}$.

The establishment of the matrix $C^{\alpha}_{\alpha'}$ for each particular system is practically the only step that requires analysis on the part of the engineer in setting up the equations of performance.

3. The concept of *group property* enables the engineer to subdivide *the analytical part* of his work, namely the establishment of $C^{\alpha}_{\alpha'}$, arbitrarily into several independent steps, each step having its own method of analysis.

The physical and analytical *subdivision* of a group of problems into a series of simpler problems, and their *recombination* into the original group of problems, is made possible only by the introduction of the concept of a " group " of transformation matrices $C^{\alpha}_{\alpha'}$. Without its use:

1. Each physical system must be analyzed *separately* from the ground up as if the analogous systems had not been analyzed.

2. The physical system must be analyzed *as a whole* except in very

simple cases where the transformation matrix $\mathbf{C}$ would usually become the unit matrix.

X. TYPES OF SUBDIVISIONS OF C

The subdivision of the transformation matrix $\mathbf{C}$ into the product of several transformation matrices $\mathbf{C}_1$, $\mathbf{C}_2 \cdots$ may be prompted by various types of considerations:

1. The problem itself *automatically* subdivides the transformation matrix into two or more steps.

For instance, in a transformer system the division of the transformation into two steps:

A. Interconnection of coils by $\mathbf{C}_1$

B. Neglect of magnetizing currents by $\mathbf{C}_2$

is made self-evident by the fundamental difference in the nature of the types of transformation.

Another example of an *automatic* subdivision of $\mathbf{C}$ is that of an armature winding of a direct-current machine, as shown in Section VII.

2. The transformation matrix may cover only one type of transformation, but it may be so *complicated* that, to avoid confusion, it may be set up in several steps.

For instance, in a complex network it is possible to interconnect first, say, those coils that form branches by series connections and set up $\mathbf{C}_1$ for this step. Then the branches are treated as single coils and are interconnected by $\mathbf{C}_2$ into the final network. Their product $\mathbf{C}_1 \cdot \mathbf{C}_2$ is equivalent to setting up $\mathbf{C}$ in one step. The final interconnections may be set up in three or more steps if so desired. In rotating machinery this division is often used to avoid confusion.

3. The transformation may be of one type, and it may be simple to establish $\mathbf{C}$ in one step, $\mathbf{C}$ *is divided nevertheless into* $\mathbf{C}_1 \cdot \mathbf{C}_2 \cdots$ *for analytical reasons.*

For instance, it may be found that the product $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ is calculated much more quickly if it is performed in several steps as

$$\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} = (\mathbf{C}_{2t} \cdot \mathbf{C}_{1t}) \cdot \mathbf{z} \cdot (\mathbf{C}_1 \cdot \mathbf{C}_2) = \mathbf{C}_{2t} \cdot (\mathbf{C}_{1t} \cdot \mathbf{z} \cdot \mathbf{C}_1) \cdot \mathbf{C}_2 \quad 11.2$$

That is, *first* $\mathbf{z}$ *is multiplied twice by* $\mathbf{C}_1$ *as* $\mathbf{C}_{1t} \cdot \mathbf{z} \cdot \mathbf{C}_1 = \mathbf{z}'$ *and then twice by* $\mathbf{C}_2$. Such cases occur, for instance, in armature-winding reactance calculations, where the establishment of the resultant $\mathbf{C}$ is quite simple, *but the matrix of* $\mathbf{z}$ *itself may have several hundred rows and columns* and is imperative to reduce it *quickly* to a matrix with few rows and columns.

It is emphasized that a transformation matrix $\mathbf{C}$ may be subdivided for the purpose of analysis into a product $\mathbf{C}_1 \cdot \mathbf{C}_2 \cdot \mathbf{C}_3 \cdots$ *arbitrarily*

even if $\mathbf{C}_1$ or $\mathbf{C}_2$, etc., do not correspond to any physical set-up or even hypothetical division. The subdivision may be purely analytical.

XI. MULTIWINDING TRANSFORMER SYSTEMS

(*a*) Examples of networks whose analysis requires the application of at least *two* transformation tensors are the numerous types of multiwinding transformer systems. Their *resultant* transformation tensor is the product of the following two *component* transformation tensors:

1. $\mathbf{C}_1$ *showing the manner of interconnection of the windings into a mesh network.* It is set up in exactly the same manner as any other mesh network.
2. $\mathbf{C}_2$ *showing that the magnetizing current of each closed magnetic circuit is neglected.* It is set up in the manner shown in Section X, Chapter VI.

The resultant transformation tensor $\mathbf{C}$ *is*

$$\boxed{\mathbf{C} = \mathbf{C}_1 \cdot \mathbf{C}_2} \qquad 11.3$$

(*b*) To neglect the magnetizing currents, first the equations of constraints are set up in terms of the *new* currents $\mathbf{i}'$ existing *after* the interconnection, making the sum of the m.m.f.'s around each closed magnetic circuit zero, then they are expressed in the form $\mathbf{i}' = \mathbf{C}_2 \cdot \mathbf{i}''$, where $\mathbf{C}_2$ is a singular matrix containing as many less columns than rows as there are equations of constraints.

Since it is not so easy to set up the equations of constraint in terms of the currents $\mathbf{i}'$ existing *after* the interconnections, *it is easier to set up the equations of constraints first in terms of the old currents* $\mathbf{i}$ *existing before interconnection*, then to replace the old currents by the new currents from the relation $\mathbf{i} = \mathbf{C}_1 \cdot \mathbf{i}'$.

Hence *the transformation tensor* $\mathbf{C}_2$ *neglecting the magnetizing currents is set up in three steps:*

1. *The equations of constraints are set up in terms of the old currents* $\mathbf{i}$ *with all interconnections removed.*
2. *The old currents are replaced by the new currents from* $\mathbf{i} = \mathbf{C}_1 \cdot \mathbf{i}'$, *thereby expressing the equations of constraints in terms of the new currents.*
3. *The resultant equations of constraints are replaced by the relation* $\mathbf{i}' = \mathbf{C}_2 \cdot \mathbf{i}''$.

(*c*) There is one other difference between the analysis of transformer systems and other general network. In general static networks the design constants of the coils of the primitive network consist usually of the *self-impedances* Z_{aa}, Z_{bb}, etc., of each coil and of the *mutual*

impedances Z_{ab}, Z_{cd}, etc., between any two coils. However, in multiwinding transformer networks, where the magnetizing currents are neglected, *design constants of a new type are used for the primitive transformer, the so-called "leakage-impedances" between two coils,* Z_{a-b}. They are also called "through-impedances" or "bucking impedances," etc. In terms of these design constants the final equations have a simpler form.

Of course it does not make any difference in the use of the formulas what type of components the impedance tensor $\mathbf{z}$ of the primitive network has or in what manner the resultant transformation tensor $\mathbf{C}$ is set up. The impedance tensor $\mathbf{z}'$ of the resultant network is still found by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$, the voltages induced in the individual windings by $\mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}''$, and so on.

XII. IMPEDANCE TENSOR OF THE PRIMITIVE TRANSFORMER

(*a*) Let two single coils, *each with one turn*, be connected in opposing series. If the self- and mutual impedances of the individual coils are z_{11}, z_{22}, and z_{12}, then the resultant impedance of the two coils in opposing series is

$$z_{1-2} = z_{11} + z_{22} - 2z_{12} \qquad 11.4$$

This impedance is called the "*leakage impedance*" between the two coils.

It is possible to say that the "*self* leakage impedance" of each coil is zero, while the "*mutual* leakage impedance" between the two coils is $-z_{1-2}/2$. The negative sign occurs because the flux due to one of the coils opposes the flux of the other coil. The $\frac{1}{2}$ occurs because the voltage induced in *one* of the coils is one-half that induced across both coils.

(*b*) Hence, when n coils exist on one transformer core, *each coil with one turn,* then in terms of self and mutual leakage impedances *the impedance tensor of the multiwinding transformer is*

$$\mathbf{z}_l = -\tfrac{1}{2} \times \begin{array}{c|ccccc|} & 1 & 2 & 3 & & n \\ \hline 1 & 0 & z_{1-2} & z_{1-3} & \cdots & z_{1-n} \\ 2 & z_{1-2} & 0 & z_{2-3} & \cdots & z_{2-n} \\ 3 & z_{1-3} & z_{2-3} & 0 & \cdots & z_{3-n} \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ n & z_{1-n} & z_{2-n} & z_{3-n} & \cdots & 0 \\ \hline \end{array} \qquad 11.5$$

It should be noted that *all diagonal components are zero.* Also the presence of $-\frac{1}{2}$ should be noted, being a factor of each component.

(*c*) When each coil has a different number of turns, the mutual leakage impedance of two coils will be defined as

$$\boxed{Z_{2-3} = -n_2n_3(z_{22} + z_{33} - 2z_{23})/2} \qquad 11.6$$

The presence of $-\frac{1}{2}$ should be noted.

In terms of these *actual turn* leakage impedances *the impedance tensor of a multiwinding transformer with isolated windings (the " primitive transformer ") is*

$$\mathbf{z} = \begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & \cdots & n \\ \hline 1 & 0 & Z_{1-2} & Z_{1-3} & \cdots & Z_{1-n} \\ \hline 2 & Z_{1-2} & 0 & Z_{2-3} & \cdots & Z_{2-n} \\ \hline 3 & Z_{1-3} & Z_{2-3} & 0 & \cdots & Z_{3-n} \\ \hline & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline n & Z_{1-n} & Z_{2-n} & Z_{3-n} & \cdots & 0 \\ \hline \end{array} \qquad 11.7$$

having all zero components along the main diagonal line, and containing fewer design constants.

This impedance tensor **z** *will be assumed in the following pages as the impedance tensor of the " primitive multiwinding transformer " when its magnetizing current is to be neglected.* Each component is defined in equation 11.6.

(*d*) The *actual* mutual impedance Z_{23} between two coils is defined in terms of *single-turn* impedances as

$$Z_{23} = n_2n_3z_{23}$$

If these *actual* impedances are used in place of *leakage* impedances, then **z** of equation 11.7 contains no zero components, and it also contains n more design constants.

XIII. ANOTHER DEFINITION OF LEAKAGE IMPEDANCE *

(*a*) *In general practice it is customary to assume one of the windings as a reference winding*, say coil one, whose number of turns is n_1, and define the mutual leakage impedance of two coils as

$$Z'_{2-3} = n_1n_1(z_{22} + z_{33} - 2z_{23}) \qquad (11.8)$$

It should be noted that this definition assumes that *all coils have n_1 turns.*

* This section may be left out at the first reading.

In terms of these *equal turn* leakage impedances the impedance tensor is

$$\mathbf{z}_2 = -\tfrac{1}{2} \times \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \\ \mathrm{n} \end{array} \begin{array}{ccccc} 1 & 2 & 3 & & \mathrm{n} \\ \hline 0 & Z'_{1-2} & Z'_{1-3} & \dots & Z'_{1-n} \\ Z'_{1-2} & & Z'_{2-3} & \dots & Z'_{2-n} \\ Z'_{1-3} & Z'_{2-3} & 0 & \dots & Z'_{3-n} \\ \dots & \dots & \dots & \dots & \dots \\ Z'_{1-n} & Z'_{2-n} & Z' & \dots & 0 \end{array} \qquad 11.9$$

(*b*) It is possible to pass from the tensor (11.9) to the tensor (11.7) by the transformation tensor

$$\mathbf{C} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \\ \mathrm{n} \end{array} \begin{array}{ccccc} 1 & 2 & 3 & \dots & \mathrm{n} \\ \hline 1 & 0 & 0 & \dots & 0 \\ 0 & n_2/n_1 & 0 & \dots & 0 \\ 0 & 0 & n_3/n_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & n_n/n_1 \end{array} \qquad 11.10$$

so that

$$\mathbf{z} = \mathbf{C}_t \cdot \mathbf{z}_2 \cdot \mathbf{C}$$

$$\mathbf{z} = -\tfrac{1}{2} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \\ \mathrm{n} \end{array} \begin{array}{ccccc} 1 & 2 & 3 & \dots & \mathrm{n} \\ \hline 0 & \frac{n_2}{n_1} Z'_{1-2} & \frac{n_3}{n_1} Z'_{1-3} & \dots & \frac{n_n}{n_1} Z'_{1-n} \\ \frac{n_2}{n_1} Z'_{1-2} & 0 & \frac{n_2 n_3}{n_1 n_1} Z'_{2-3} & \dots & \frac{n_2 n_n}{n_1 n_1} Z'_{2-n} \\ \frac{n_3}{n_1} Z'_{1-3} & \frac{n_2 n_3}{n_1 n_1} Z'_{2-3} & 0 & \dots & \frac{n_3 n_n}{n_1 n_1} Z'_{3-n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{n_n}{n_1} Z'_{1-n} & \frac{n_2 n_n}{n_1 n_1} Z'_{2-n} & \frac{n_3 n_n}{n_1 n_1} Z'_{3-n} & \dots & 0 \end{array} \qquad 11.11$$

since by equations 11.6 and 11.8

$$Z_{2-3} = -Z'_{2-3}\left(\frac{1}{2}\right)\frac{n_2 n_3}{n_1 n_1} \quad \text{or} \quad Z'_{2-3} = -Z_{2-3}\frac{2 n_1 n_1}{n_2 n_3} \qquad 11.12$$

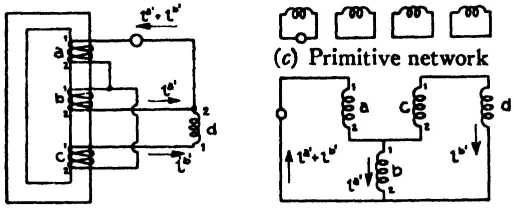
Hence *it may be assumed that the impedance tensor* **z** *of the primitive multiwinding transformer is either equation* 11.7 *or equation* 11.11.

The first tensor contains the *actual turn* leakage impedances Z_{2-3}, the second tensor contains the usual *equal turn* leakage impedances Z'_{2-3}.

(*c*) It is possible to use other types of leakage-impedances in place of Z'_{2-3} and Z_{2-3}. However, they are not considered here. Of course **z** may contain also the usual self- and mutual impedances Z_{22}, Z_{23}, etc., in place of the leakage impedances Z_{2-3}, etc. But whatever components are used for **z**, *the following analysis is equally valid for all types of components.*

XIV. FORKED AUTOTRANSFORMER

(*a*) As a simple example, consider the three-winding transformer of Fig. (11.1*a*) whose three coils *a*, *b*, *c*, and the load *d* are connected



(*c*) Primitive network

(*a*) Forked autotransformer (*b*) Connection diagram

FIG. 11.1

into a two-mesh network as shown in the conventional diagram of Fig. 11.1*b*.

The impedance tensor of *the primitive network* Fig. (11.1*c*) is

$$\mathbf{z} = \begin{array}{c|cccc} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \hline \mathbf{a} & 0 & Z_{a-b} & Z_{a-c} & 0 \\ \mathbf{b} & Z_{a-b} & 0 & Z_{b-c} & 0 \\ \mathbf{c} & Z_{a-c} & Z_{b-c} & 0 & 0 \\ \mathbf{d} & 0 & 0 & 0 & Z \end{array} \qquad 11.13$$

containing leakage impedances (any other actual or leakage impedances may be used).

Its impressed voltage vector is

$$\mathbf{e} = \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \hline e_a & 0 & 0 & 0 \end{array} \qquad 11.14$$

(*b*) *The transformation tensor* $\mathbf{C}_1$, *representing the interconnection,* is set up by equating the old and the new currents *in each coil* of Fig. (11.1*b*) as

$$
\begin{aligned}
i^a &= i^{a'} + i^{b'} \\
i^b &= i^{a'} \\
i^c &= \quad\;\; - i^{b'} \\
i^d &= i^{b'}
\end{aligned}
\qquad
C_1 = \begin{array}{c|c|c|}
 & a' & b' \\
\hline
a & 1 & 1 \\
\hline
b & 1 & \\
\hline
c & & -1 \\
\hline
d & & 1 \\
\hline
\end{array}
\tag{11.15}
$$

(*c*) *The transformation tensor* C_2, *neglecting the magnetizing current,* is set up as follows:

1. The m.m.f. around the transformer is made equal to zero. *The currents of the primitive network, that is the currents flowing before the interconnection, are used* instead of the new currents, so that

$$n_a i^a + n_b i^b + n_c i^c = 0 \tag{11.16}$$

This is the equation of constraint in terms of the *old* currents

2. *In the equation of constraint the old currents are replaced by the new currents by equation* 11.15 as

$$n_a(i^{a'} + i^{b'}) + n_b(i^{a'}) + n_c(-i^{b'}) = 0 \tag{11.17}$$

or

$$(n_a + n_b)i^{a'} + (n_a - n_c)i^{b'} = 0 \tag{11.18}$$

This is the equation of constraint in terms of the *new* currents.

3. Eliminating one (any one) of the new currents

$$i^{a'} = \frac{n_c - n_a}{n_a + n_b} i^{b'} \tag{11.19}$$

4. Leaving the other currents unchanged, *the equation of constraint of equation* 11.16 *is replaced by the set of equation* $i' = C_2 \cdot i''$

$$
\begin{aligned}
i^{a'} &= \frac{n_c - n_a}{n_a - n_b} i^{b''} \\
i^{b'} &= \qquad\quad\;\; i^{b''}
\end{aligned}
\qquad
C_2 = \begin{array}{c|c|}
 & b'' \\
\hline
a' & \dfrac{n_c - n_a}{n_a + n_b} \\
\hline
b' & 1 \\
\hline
\end{array}
=
\begin{array}{c|c|}
 & b'' \\
\hline
a' & n \\
\hline
b' & 1 \\
\hline
\end{array}
\tag{11.20}
$$

(*d*) *The resultant transformation tensor* C is

$$
C_1 \cdot C_2 = \begin{array}{c|c|c|}
 & a' & b' \\
\hline
a & 1 & 1 \\
\hline
b & 1 & \\
\hline
c & & -1 \\
\hline
d & & 1 \\
\hline
\end{array}
\cdot
\begin{array}{c|c|}
 & b'' \\
\hline
a' & n \\
\hline
b' & 1 \\
\hline
\end{array}
=
\begin{array}{c|c|}
 & b'' \\
\hline
a & n + 1 \\
\hline
b & n \\
\hline
c & -1 \\
\hline
d & 1 \\
\hline
\end{array}
= C
\tag{11.21}
$$

The impedance tensor of the forked autotransformer is, by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$,

$$\mathbf{z}' = \begin{array}{c|c|} & \mathbf{b}'' \\ \hline \mathbf{b}'' & 2(n+1)nZ_{a-b} - 2(n+1)Z_{a-c} - 2nZ_{b-c} + Z \\ \hline \end{array} \qquad 11.22$$

acting as a single impedance.

The impressed voltage vector is by $\mathbf{C}_t \cdot \mathbf{e}$

$$\mathbf{e}' = \begin{array}{|c|} \mathbf{b}'' \\ \hline (n+1)e_a \\ \hline \end{array} \qquad 11.23$$

The equation of voltage is $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}''$, from which the current is by $\mathbf{z}'^{-1} \cdot \mathbf{e}'$

$$\mathbf{i}'' = \begin{array}{|c|} \mathbf{b}'' \\ \hline \dfrac{(n+1)e_a}{2(n+1)nZ_{a-b} - 2(n+1)Z_{a-c} - 2nZ_{b-c} + Z} \\ \hline \end{array} \qquad 11.24$$

(*f*) *The currents flowing in the individual coils* are by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}''$

$$\mathbf{i} = \begin{array}{|c|c|c|c|} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \hline (n+1)i^{b''} & ni^{b''} & -i^{b''} & i^{b''} \\ \hline \end{array} \qquad 11.25$$

The voltages induced in the individual coils are, by $\mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}''$, where $\mathbf{z} \cdot \mathbf{C}$ has been calculated in finding $\mathbf{z}'$

$$\mathbf{e}_c = \begin{array}{c|c|} \hline \mathbf{a} & i^{b''}(nZ_{a-b} - Z_{a-c}) \\ \hline \mathbf{b} & i^{b''}[(n+1)Z_{a-b} - Z_{b-c}] \\ \hline \mathbf{c} & i^{b''}[(n+1)Z_{a-c} + nZ_{b-c}] \\ \hline \mathbf{d} & i^{b''}Z \\ \hline \end{array} \qquad 11.26$$

XV. LOAD-RATIO CONTROL AND REGULATING UNITS

(*a*) As an example of *several* interconnected multiwinding transformers consider Fig. 11.2 where one three-winding and two two-

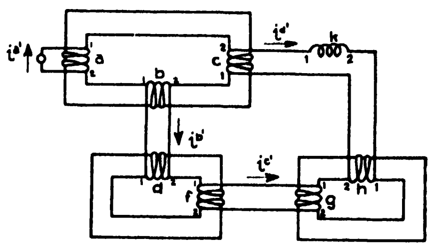


FIG. 11.2.—Load-ratio Control and Regulating Units

winding transformers are interconnected with a load into a four-mesh network.

(*b*) The impedance tensor of the primitive network is

$$Z = \begin{array}{c|cccccccc|} & \text{a} & \text{b} & \text{c} & \text{d} & \text{f} & \text{g} & \text{h} & \text{k} \\ \hline \text{a} & & Z_{a-b} & Z_{a-c} & & & & & \\ \text{b} & Z_{a-b} & & Z_{b-c} & & & & & \\ \text{c} & Z_{a-c} & Z_{b-c} & & & & & & \\ \text{d} & & & & & Z_{d-f} & & & \\ \text{f} & & & & Z_{d-f} & & & & \\ \text{g} & & & & & & & Z_{g-h} & \\ \text{h} & & & & & & Z_{g-h} & & \\ \text{k} & & & & & & & & Z \\ \hline \end{array} \qquad 11.27$$

and its impressed voltage vector is

$$\mathbf{e} = \begin{array}{|cccccccc|} \text{a} & \text{b} & \text{c} & \text{d} & \text{f} & \text{g} & \text{h} & \text{k} \\ \hline e_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad 11.28$$

(*c*) The transformation tensor of the four-mesh network, by equating old and new current in each coil, is

$$\begin{aligned} i^a &= i^{a'} \\ i^b &= i^{b'} \\ i^c &= i^{d'} \\ i^d &= -i^{b'} \\ i^f &= -i^{c'} \\ i^g &= -i^{c'} \\ i^h &= i^{d'} \\ i^k &= i^{d'} \end{aligned} \qquad C_1 = \begin{array}{c|cccc|} & \text{a}' & \text{b}' & \text{c}' & \text{d}' \\ \hline \text{a} & 1 & & & \\ \text{b} & & 1 & & \\ \text{c} & & & & 1 \\ \text{d} & & -1 & & \\ \text{f} & & & -1 & \\ \text{g} & & & 1 & \\ \text{h} & & & & 1 \\ \text{k} & & & & 1 \\ \hline \end{array} \qquad 11.29$$

(*d*) *There are three equations of constraint, one for each transformer.* In terms of the *old* currents flowing before interconnection (making the m.m.f. around each transformer of Fig. (11.2) equal to zero), they are

$$\begin{aligned} n_a i^a + n_b i^b + n_c i^c &= 0 \\ n_d i^d + n_f i^f &= 0 \\ n_g i^g + n_h i^h &= 0 \end{aligned} \qquad 11.30$$

Expressed in terms of the four *new* currents flowing after interconnection they become (by substituting equation 11.29)

$$\begin{aligned} n_a i^{a'} + n_b i^{b'} + n_c i^{d'} &= 0 \\ -n_d i^{b'} - n_f i^{c'} &= 0 \\ n_g i^{c'} + n_h i^{d'} &= 0 \end{aligned} \qquad 11.31$$

Three of the currents, say $i^{a'}$, $i^{b'}$ and $i^{c'}$, may be expressed in terms of the fourth current $i^{d'}$ as

$$\begin{aligned} i^{c'} &= -(n_h/n_g)i^{d'} \\ i^{b'} &= -(n_f/n_d)i^{c'} = (n_f/n_d)(n_h/n_g)i^{d'} \\ i^{a'} &= -(n_b/n_a)i^{b'} - (n_c/n_a)i^{d'} = -\left(\frac{n_b}{n_a}\frac{n_f}{n_d}\frac{n_h}{n_g} + \frac{n_c}{n_a}\right)i^{d'} \end{aligned} \qquad 11.32$$

Hence the three equations of constraint may be expressed by the transformation $\mathbf{i}' = \mathbf{C}_2 \cdot \mathbf{i}''$

$$\begin{aligned} i^{a'} &= -\left(\frac{n_b}{n_a}\frac{n_f}{n_d}\frac{n_h}{n_g} + \frac{n_c}{n_a}\right)i^{d''} \\ i^{b'} &= \frac{n_f}{n_d}\frac{n_h}{n_g}\, i^{d''} \\ i^{c'} &= -\frac{n_h}{n_g}\, i^{d''} \\ i^{d'} &= i^{d''} \end{aligned} \qquad \mathbf{C}_2 = \begin{array}{c|c|} & \mathbf{d}'' \\ \mathbf{a}' & -\left(\frac{n_b}{n_a}\frac{n_f}{n_d}\frac{n_h}{n_g} + \frac{n_c}{n_a}\right) \\ \mathbf{b}' & \frac{n_f}{n_d}\frac{n_h}{n_g} \\ \mathbf{c}' & -\frac{n_h}{n_g} \\ \mathbf{d}' & 1 \end{array} = \begin{array}{c|c|} & \mathbf{d}'' \\ \mathbf{a}' & -n_1 \\ \mathbf{b}' & n_2 \\ \mathbf{c}' & -n_3 \\ \mathbf{d}' & 1 \end{array} \qquad 11.33$$

(*e*) The resultant transformation tensor is

$$\mathbf{C}_1 \cdot \mathbf{C}_2 = \begin{array}{c|cccc|} & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \mathbf{d}' \\ \mathbf{a} & 1 & & & \\ \mathbf{b} & & 1 & & \\ \mathbf{c} & & & & 1 \\ \mathbf{d} & & -1 & & \\ \mathbf{f} & & & -1 & \\ \mathbf{g} & & & 1 & \\ \mathbf{h} & & & & 1 \\ \mathbf{k} & & & & 1 \end{array} \cdot \begin{array}{c|c|} & \mathbf{d}'' \\ \mathbf{a}' & -n_1 \\ \mathbf{b}' & n_2 \\ \mathbf{c}' & -n_3 \\ \mathbf{d}' & 1 \end{array} = \begin{array}{c|c|} & \mathbf{d}'' \\ \mathbf{a} & -n_1 \\ \mathbf{b} & n_2 \\ \mathbf{c} & 1 \\ \mathbf{d} & -n_2 \\ \mathbf{f} & n_3 \\ \mathbf{g} & -n_3 \\ \mathbf{h} & 1 \\ \mathbf{k} & 1 \end{array} = \mathbf{C} \qquad 11.34$$

(*f*) The new components of the impedance tensor are found by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ and those of the impressed-voltage vector by $\mathbf{C}_t \cdot \mathbf{e}$.

$$\mathbf{Z}' = \begin{array}{c|c|} & \mathbf{d}'' \\ \mathbf{d}'' & \begin{array}{c} -2n_1n_2Z_{a-b} - 2n_1Z_{a-c} + 2n_2Z_{b-c} - \\ -2n_2n_3Z_{d-f} - 2n_3Z_{g-h} + Z \end{array} \end{array} \qquad 11.35 \qquad \mathbf{e}' = \begin{array}{|c|} \mathbf{d}'' \\ -n_1e_a \end{array} \qquad 11.39$$

The current $i^{d''}$ is found by $\mathbf{i}'' = \mathbf{z}'^{-1} \cdot \mathbf{e}'$.

(*g*) *The currents in the individual coils are* found by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}''$ (equation 11.34)

$$\mathbf{i} = \begin{array}{cccccccc} a & b & c & d & f & g & h & k \\ \hline -n_1 i^{d''} & n_2 i^{d''} & i^{d''} & -n_2 i^{d''} & n_3 i^{d''} & -n_3 i^{d''} & i^{d''} & i^{d''} \end{array} \qquad 11.37$$

The voltage drops in the individual coils are found by $\mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}''$.

(*h*) Table 11.1 shows the component and resultant transformation matrices of some commonly used unbalanced transformer connections.

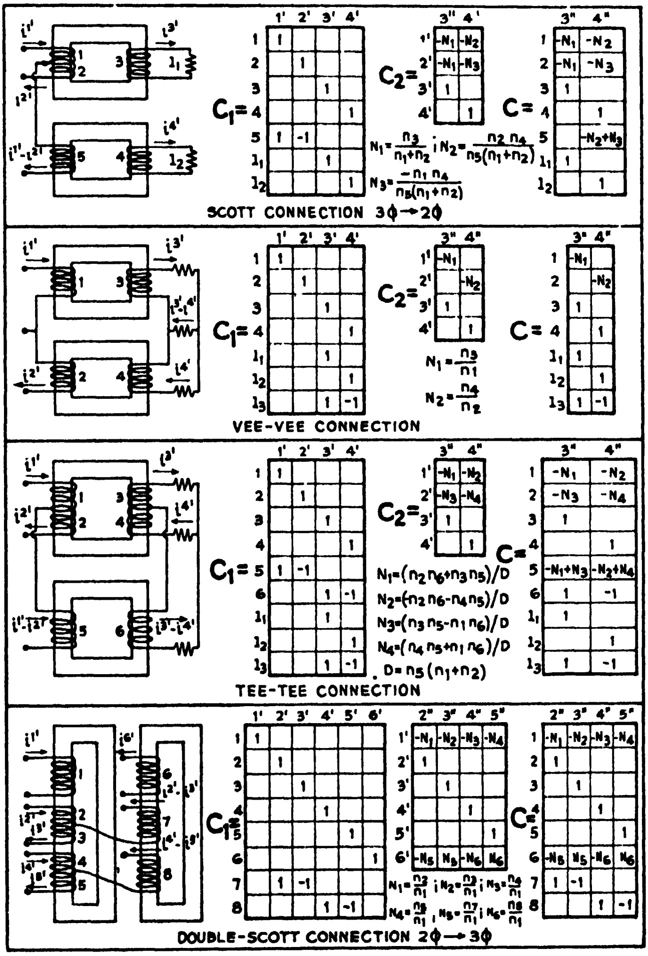
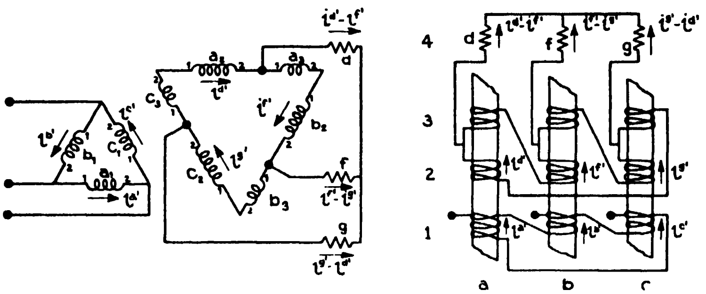


Table 11.1.—Unbalanced Transformer Connections and their Transformation Matrices

First column—$\mathbf{C}_1$ showing interconnection of coils
Second column—$\mathbf{C}_2$ neglecting magnetizing currents
Third column—$\mathbf{C}$ representing their resultant

XVI. UNBALANCED INSCRIBED DELTA

(*a*) Let the three three-winding transformers of Fig. 11.3 supply an unbalanced load. For greater generalization each winding may be assumed to be different (unbalanced *three-phase* multiwinding transformer circuits will be worked out by a quicker procedure in Chapter



(*a*) Connection diagram (*b*) Actual connections

FIG. 11.3.—Unbalanced Inscribed Delta

XIX. This example of a slower process serves as a check on the quicker procedure).

(*b*) The impedance tensor of the primitive network is

$$
\mathbf{z} = \begin{array}{c|ccc|ccc|ccc|ccc|}
 & a_1 & b_1 & c_1 & a_2 & b_2 & c_2 & a_3 & b_3 & c_3 & d & f & g \\
\hline
a_1 & & & & Z_{a_1-a_2} & & & Z_{a_1-a_3} & & & & & \\
b_1 & & & & & Z_{b_1-b_2} & & & Z_{b_1-b_3} & & & & \\
c_1 & & & & & & Z_{c_1-c_2} & & & Z_{c_1-c_3} & & & \\
\hline
a_2 & Z_{a_1-a_2} & & & & & & Z_{a_2-a_3} & & & & & \\
b_2 & & Z_{b_1-b_2} & & & & & & Z_{b_2-b_3} & & & & \\
c_2 & & & Z_{c_1-c_2} & & & & & & Z_{c_2-c_3} & & & \\
\hline
a_3 & Z_{a_1-a_3} & & & Z_{a_2-a_3} & & & & & & & & \\
b_3 & & Z_{b_1-b_3} & & & Z_{b_2-b_3} & & & & & & & \\
c_3 & & & Z_{c_1-c_3} & & & Z_{c_2-c_3} & & & & & & \\
\hline
d & & & & & & & & & & Z_{dd} & Z_{df} & Z_{dg} \\
f & & & & & & & & & & Z_{fd} & Z_{ff} & Z_{fg} \\
g & & & & & & & & & & Z_{gd} & Z_{gf} & Z_{gg} \\
\hline
\end{array} \tag{11.38}
$$

or expressed as a compound tensor

$$
\mathbf{Z} = \begin{array}{c|c|c|c|c|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & & \mathbf{Z}_{1-2} & \mathbf{Z}_{1-3} & \\
\hline
2 & \mathbf{Z}_{1-2} & & \mathbf{Z}_{2-3} & \\
\hline
3 & \mathbf{Z}_{1-3} & \mathbf{Z}_{2-3} & & \\
\hline
4 & & & & \mathbf{Z}_4 \\
\hline
\end{array} \qquad 11.39
$$

where $\mathbf{Z}_{1-2}$, etc., have diagonal matrices.

(*c*) The transformation tensor of the interconnection is:

$$
\begin{array}{l}
i^{a_1} = i^{a'} \\
i^{b_1} = i^{b'} \\
i^{c_1} = i^{c'} \\
i^{a_2} = i^{d'} \\
i^{b_2} = i^{f'} \\
i^{c_2} = i^{g'} \\
i^{a_3} = i^{f'} \\
i^{b_3} = i^{g'} \\
i^{c_3} = i^{d'} \\
i^{d} = i^{d'} - i^{f'} \\
i^{f} = i^{f'} - i^{g'} \\
i^{g} = -i^{d'} + i^{g'}
\end{array}
\qquad
\mathbf{C}_1 = \begin{array}{c|c|c|c|c|c|c|}
 & a' & b' & c' & d' & f' & g' \\
\hline
a_1 & 1 & & & & & \\
\hline
b_1 & & 1 & & & & \\
\hline
c_1 & & & 1 & & & \\
\hline
a_2 & & & & 1 & & \\
\hline
b_2 & & & & & 1 & \\
\hline
c_2 & & & & & & 1 \\
\hline
a_3 & & & & & 1 & \\
\hline
b_3 & & & & & & 1 \\
\hline
c_3 & & & & 1 & & \\
\hline
d & & & & 1 & -1 & \\
\hline
f & & & & & 1 & -1 \\
\hline
g & & & & -1 & & 1 \\
\hline
\end{array}
=
\begin{array}{c|c|c|}
 & 1' & 2' \\
\hline
1 & \mathbf{I} & \\
\hline
2 & & \mathbf{I} \\
\hline
3 & & \mathbf{C}_1 \\
\hline
4 & & \mathbf{C}_2 \\
\hline
\end{array} \qquad 11.40
$$

(*d*) The equations of constraints in terms of the twelve *old* currents are

$$
\begin{aligned}
n_{a1} i^{a1} + n_{a2} i^{a2} + n_{a3} i^{a3} &= 0 \\
n_{b1} i^{b1} + n_{b2} i^{b2} + n_{b3} i^{b3} &= 0 \qquad 11.41 \\
n_{c1} i^{c1} + n_{c2} i^{c2} + n_{c3} i^{c3} &= 0
\end{aligned}
$$

In terms of the six *new* currents they are

$$
\begin{aligned}
n_{a1} i^{a'} + n_{a2} i^{d'} + n_{a3} i^{f'} &= 0 \\
n_{b1} i^{b'} + n_{b2} i^{f'} + n_{a3} i^{g'} &= 0 \qquad 11.42 \\
n_{c1} i^{c'} + n_{c2} i^{g'} + n_{c3} i^{d'} &= 0
\end{aligned}
$$

Eliminating, say, $i^{a'}$, $i^{b'}$, and $i^{c'}$, the equations of transformation $\mathbf{i}' = \mathbf{C}_2 \cdot \mathbf{i}''$ are

$$
\begin{aligned}
i^{a'} &= -(n_{a2}/n_{a1})\, i^{d''} - (n_{a3}/n_{a1})\, i^{f''} \\
i^{b'} &= -(n_{b2}/n_{b1})\, i^{f''} - (n_{a3}/n_{b1})\, i^{g''} \\
i^{c'} &= -(n_{c2}/n_{c1})\, i^{g''} - (n_{c3}/n_{c1})\, i^{d''} \\
i^{d'} &= i^{d''} \\
i^{f'} &= i^{f''} \\
i^{g'} &= i^{g''}
\end{aligned}
$$

$\mathbf{C}_2 =$

	d''	f''	g''
a'	n_{a2}/n_{a1}	$-n_{a3}/n_{a1}$	
b'		$-n_{b2}/n_{b1}$	$-n_{b3}/n_{b1}$
c'	$-n_{c3}/n_{c1}$		$-n_{c2}/n_{c1}$
d'	1		
f'		1	
g'			1

$=$

	4''
1'	$-\mathbf{n}$
2'	$\mathbf{I}$

11.43

(*e*) The resultant transformation tensor is

$\mathbf{C}_1 \cdot \mathbf{C}_2 = \mathbf{C} =$

	d''	f''	g''
a_1	$-n_{a2}/n_{a1}$	$-n_{a3}/n_{a1}$	
b_1		$-n_{b2}/n_{b1}$	$-n_{b3}/n_{b1}$
c_1	$-n_{c3}/n_{c1}$		$-n_{c2}/n_{c1}$
a_2	1		
b_2		1	
c_2			1
a_3		1	
b_3			1
c_3	1		
d	1	−1	
f		1	−1
g	−1		1

$=$

	4''
1	$-\mathbf{n}$
2	$\mathbf{I}$
3	$\mathbf{C}_1$
4	$\mathbf{C}_2$

11.44

(*f*) The impedance tensor $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ is found as

$$
\mathbf{z}' = \begin{array}{c|c|} & 4'' \\ \hline 4'' & -\mathbf{n}_t \cdot \mathbf{Z}_{1-2} - \mathbf{Z}_{1-2} \cdot \mathbf{n} - \mathbf{n}_t \cdot \mathbf{Z}_{1-3} \cdot \mathbf{C}_1 - \mathbf{C}_{1t} \cdot \mathbf{Z}_{1-3} \cdot \mathbf{n} + \mathbf{Z}_{2-3} \cdot \mathbf{C}_1 + \mathbf{C}_{1t} \cdot \mathbf{Z}_{2-3} + \mathbf{C}_{2t} \cdot \mathbf{Z}_4 \cdot \mathbf{C}_2 \\ \hline \end{array}
\tag{11.45}
$$

representing a 2-tensor with three rows and columns. The multiplication of their matrices has yet to be performed.

Assuming identical transformers, $Z_{a_1-a_2} = Z_{b_1-b_2} = Z_{c_1-c_2} = Z_{1-2}$

	d″	f″	g″
d″	$-2Z_{1-2}\frac{n_2}{n_1} + 2Z_{dd} - Z_{fd} - Z_{gd}$	$-2Z_{1-2}\frac{n_3}{n_1} + 2Z_{2-3} - 2Z_{1-3}\frac{n_2}{n_1} + 2Z_{df} - Z_{ff} - Z_{gf}$	$-2Z_{1-3}\frac{n_3}{n_1} + 2Z_{dg} - Z_{fg} - Z_{gg}$
z′ = f″	$-2Z_{1-3}\frac{n_3}{n_1} + 2Z_{fd} - Z_{gd} - Z_{dd}$	$-2Z_{1-2}\frac{n_2}{n_1} + 2Z_{ff} - Z_{gf} - Z_{df}$	$-2Z_{1-2}\frac{n_3}{n_1} + 2Z_{2-3} - 2Z_{1-3}\frac{n_2}{n_1} + 2Z_{fg} - Z_{gg} - Z_{dg}$
g″	$-2Z_{1-2}\frac{n_3}{n_1} + 2Z_{2-3} - 2Z_{1-3}\frac{n_2}{n_1} + 2Z_{gd} - Z_{dd} - Z_{fd}$	$-2Z_{1-2}\frac{n_3}{n_1} + 2Z_{gf} - Z_{df} - Z_{ff}$	$-2Z_{1-2}\frac{n_2}{n_1} + 2Z_{gg} - Z_{dg} - Z_{fg}$

11.46

(*g*) If i″ is calculated, the induced voltages are found by $\mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}''$, etc.

XVII. UNBALANCED VOLTAGES IN D-C WINDINGS

(*a*) As another example where at least two transformation tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ have to be set up in succession to find the resultant **C**, *let the generated voltages appearing between the six brushes of the wave winding of Fig.* 11.4 *be calculated when the excitation from one of the poles is removed.*

(*b*) The transformation tensor **C** for the six brush circuits I–VI (assuming brushes of infinitesimal thickness) is set up in three steps.

1. The effect of connecting the 126 coils into one continuous winding is shown by $\mathbf{C}_1$ in Table 11.2. The winding has a coil pitch 1–21 and a commutator pitch 1–42.

The instantaneous position of the six brushes with respect to the commutator is shown along the left-hand column of $\mathbf{C}_1$ representing their magnetic order. If a horizontal line is drawn from each of them until they reach + 1, then a vertical line is drawn from + 1 to the top of $\mathbf{C}_1$, *the position of the six " reflected " brushes along the upper horizontal line shows their electrical order.* That is, *all coils lying between two reflected brushes are short-circuited by them.*

2. The effect of short-circuiting all coils lying between any two reflected brushes is shown by another transformation tensor $\mathbf{C}_2$ in Table 11.2 which indicates that the original 126 coils are connected into six coils.

3. The product $C_1 \cdot C_2$ gives the actual transformation tensor C. Of course, the product may be found without the actual use of the arrow rule by simply writing in *one column* all $+1$ and -1 that lie between two reflected brushes.

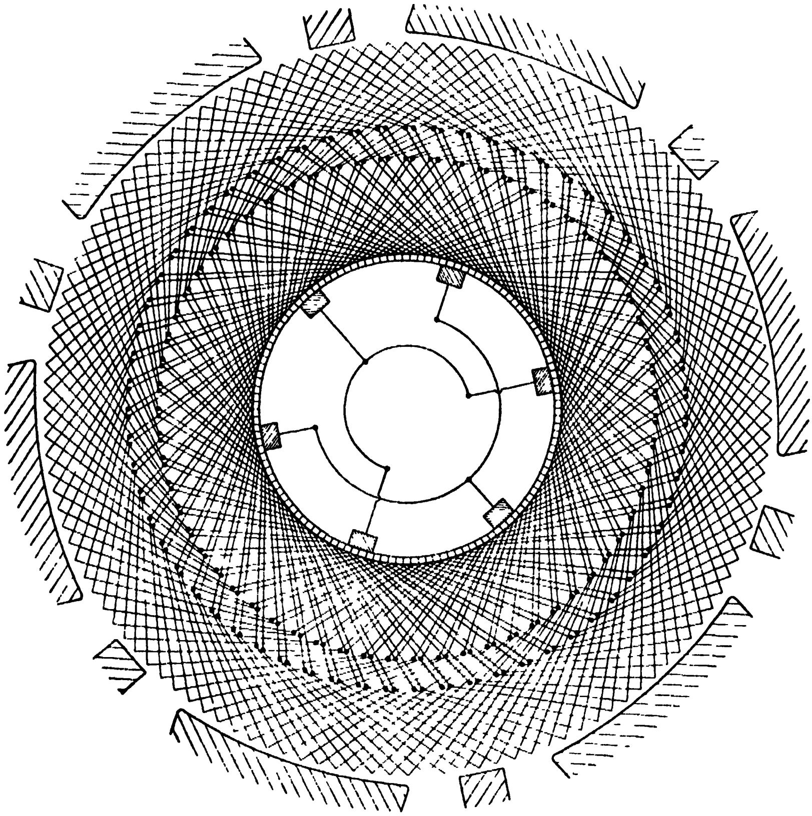


FIG. 11.4.—A Six-pole Direct-current Wave Winding with 126 Coils

(*c*) The generated voltage **e** represents the flux-density distribution along the six poles, shown along the left-hand side of C_1, also the instantaneous voltage generated in each coil.

The instantaneous generated voltage appearing in the six brush circuits is found by $C_t \cdot \mathbf{e}$. When all poles are equally excited in each circuit the same generated voltage exists, but if one of the poles is unexcited, *in each circuit a different set of coils remains unexcited* as shown in the last column of Table 11.2. In the matrix the coefficients of the various columns have been factored out, so that the columns contain only 0, $+1$, or -1 (in some components where $+1$ and -1

cancel each other, both coefficients are shown, instead of being replaced by zero).

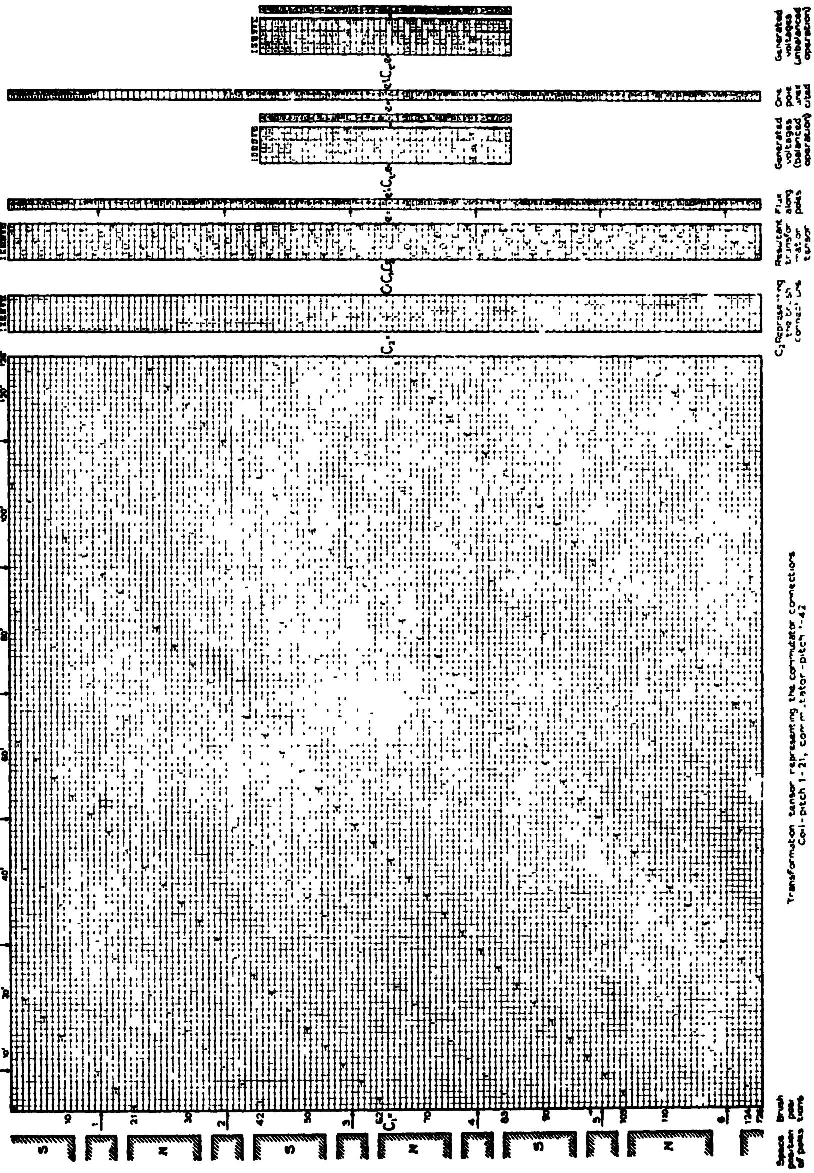


TABLE 11.2.—Calculation of the Generated Voltages in the Wave Winding of Fig. 11.4 under Balanced and Unbalanced Conditions

CHAPTER XII

REACTANCE CALCULATION OF WINDINGS

I. GENERALITY OF THE METHOD

The reactance calculation of armature windings offers another example where the transformation tensor C is set up in several steps. It also illustrates how *the concept of " primitive network " opens up a new and simple method of attack that has not hitherto been used in winding calculations.*

In all reactance calculations shown in the literature the method of attack consists of analyzing the *resultant* flux wave produced by the *actual interconnected winding.* For every different interconnection of coils the analysis has to be started all over again. Also, in general, various types of attacks are used for the calculation of the various types of reactances. To calculate *harmonic* reactances a Fourier's analysis is employed; to calculate *air-gap* reactances the area measurements are undertaken; still another method is used for the calculation of slot-leakage reactances, and so on. All these methods require a constant *physical* analysis to the end, and no part of the engineer's work can be delegated to the computers.

The method to be shown represents a radical departure from previous procedures in its method of reasoning, in its selection of the design constants, and in the outward form of the final formulas arrived at. Of course it gives the same numerical answer as other methods do. The final formulas come out in a *tabulated form,* so that the *numerical* substitutions can be performed with the aid of an adding machine. A large part of the calculations themselves may be done by a computer who knows how to multiply matrices but knows nothing about reactances.

The method of attack is quite general. *The slots may be unevenly spaced* and may contain different numbers of coil sides; the coils may contain different numbers of turns, may have different pitches, and may be connected in any arbitrary manner into groups, the groups into phases, etc. The phases in turn may be connected in any desired manner to the line. The air-gap is assumed to be smooth, however, and

its non-uniformity has to be taken care of by the usual assumptions.

The reactances to be calculated are:

1. The fundamental air-gap reactance.
2. *Any* harmonic air-gap reactance.
3. That total air-gap reactance.
4. The slot-leakage reactance.
5. The end-leakage reactance.

The same final formulas are valid for the calculation of all the above five reactances. The formulas also do not change if the pitch, the number of turns of the individual coils, or the slot shape or air gap varies; hence the effect of any variation in them upon the winding reactances can quickly be evaluated.

For the different types of windings, especially when several analogous types are analyzed simultaneously, various labor-saving devices can be introduced. Here only a *general* method of attack is given, but many of the steps may be left out in particular problems.

II. THE METHOD OF ATTACK

(*a*) When any armature winding is given, *the first step is to assume all coil connections removed,* so that only individual coils are left, representing the " primitive " winding. The self- and mutual inductances (or reactances) of the various coils will be denoted as A, B, C, $D \cdots$. *It will be assumed that these coil reactances represent any one of the five types of reactance* mentioned in the previous section.

Once the coil reactances A, B, $C \cdots$ are known, the following three steps are taken:

1. Set up the components of the impedance tensor $\mathbf{z}$ along the individual coils.

2. Set up the transformation tensor $\mathbf{C}$ of the coil connections.

3. Find the components of the impedance tensor $\mathbf{z}'$ along the actual windings by the formula $\mathbf{z}' = \mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$, using the labor-saving method of " compound tensors " if necessary, or any other suitable labor-saving device.

The final formulas of the winding reactances in terms of A, B, $C \cdots$ are the same for any one of the five types of reactance. The final winding formulas are also unchanged if the reactance of the individual coils is changed by using different pitch, or different number of turns, or

different slot proportions, or air gap, or coil-end shapes, etc. *The final formulas change only if the manner of interconnection of the coils,* **C**, *changes.*

(*b*) If the windings with $\mathbf{z}'$ are *again* interconnected by a new transformation tensor $\mathbf{C}'$ then the resultant components of the impedance tensor $\mathbf{z}''$ are found again by the same formula $\mathbf{z}'' = \mathbf{C}'_t \cdot \mathbf{z}' \cdot \mathbf{C}'$.

The individual coils themselves may be assumed to consist of *conductors* (coil sides) interconnected by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ (in the calculation of slot-leakage or end-leakage reactances).

When the windings are *complex* or when they show certain *repetitions of patterns, etc., the coils may be interconnected into the actual windings in two or more steps instead of in one step.* In other words, first it is assumed that the individual coils are interconnected into a larger number of groups with the aid of $\mathbf{C}_1$ and the corresponding $\mathbf{z}_1$ is calculated. Then these groups again are interconnected into smaller number of groups by $\mathbf{C}_2$ and the corresponding $\mathbf{z}_2$ is calculated. These steps are repeated until the actual winding is reached, each step reducing the number of rows and columns of **C** and **z**.

III. INTERCONNECTION OF NEIGHBORING COILS

Of the large number of possible short cuts one is shown here, because of its frequent occurrence in the calculation of standard windings. In standard windings usually two or more *neighboring* coils are first interconnected into groups (say into pole-phase groups), then these groups are interconnected in various manners into phases, etc. In such cases it is advantageous to *set up* **C** *in two steps* as suggested by the coil arrangement.

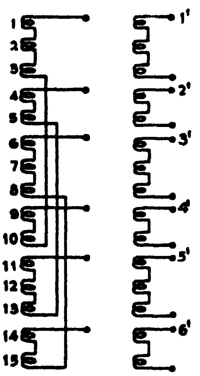


FIG. 12.1.—Interconnection of Neighboring Coils

The interconnection of, say, groups of three *neighboring* coils reduces the number of rows and columns of **z** to one-third of its former value. This reduction can be performed very quickly because of the simplicity of **C** without the formal steps of setting up first **C**, then finding $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$.

For instance, let fifteen coils be interconnected as shown in Fig. 12.1 into three windings. The first step is to interconnect them into six groups as

$$
C = \begin{array}{c|cccccc|}
 & 1' & 2' & 3' & 4' & 5' & 6' \\
\hline
1 & 1 & & & & & \\
2 & 1 & & & & & \\
3 & 1 & & & & & \\
4 & & 1 & & & & \\
5 & & 1 & & & & \\
6 & & & 1 & & & \\
7 & & & 1 & & & \\
8 & & & 1 & & & \\
9 & & & & 1 & & \\
10 & & & & 1 & & \\
11 & & & & & 1 & \\
12 & & & & & 1 & \\
13 & & & & & 1 & \\
14 & & & & & & 1 \\
15 & & & & & & 1 \\
\hline
\end{array}
\tag{12.1}
$$

The z of the individual coils is

$$
z = \begin{array}{c|ccccccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\hline
1 & A & B & C & D & E & F & G & H & H & G & F & E & D & C & B \\
2 & B & A & B & C & D & E & F & G & H & H & G & F & E & D & C \\
3 & C & B & A & B & C & D & E & F & G & H & H & G & F & E & D \\
4 & & & & A & B & C & D & E & F & G & H & H & G & F & E \\
5 & & & & B & A & B & C & D & E & F & G & H & H & G & F \\
6 & & & & & & A & B & C & D & E & F & G & H & H & G \\
7 & & & & & & B & A & B & C & D & E & F & G & H & H \\
8 & & & & & & C & B & A & B & C & D & E & F & G & H \\
9 & & & & & & & & & A & B & C & D & E & F & G \\
10 & & & & & & & & & B & A & B & C & D & E & F \\
11 & & & & & & & & & & & A & B & C & D & E \\
12 & & & & & & & & & & & B & A & B & C & D \\
13 & & & & & & & & & & & C & B & A & B & C \\
14 & & & & & & & & & & & & & & A & B \\
15 & & & & & & & & & & & & & & B & A \\
\hline
\end{array}
\tag{12.2}
$$

Since the matrix of this tensor is symmetrical, there is no need of filling in the other half. If should be divided into compound matrices by heavy lines according to the number of neighboring coils in each group.

If the multiplication $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ is performed it is found that each new components of the tensor $\mathbf{I}'$ consists of all components of each small matrix added up, namely,

$$\mathbf{z}' = \begin{array}{c|cccccc|} & 1' & 2' & 3' & 4' & 5' & 6' \\ \hline 1' & a & b & c & d & c & b \\ 2' & b & e & b & f & d & f \\ 3' & c & b & a & b & c & d \\ 4' & d & f & b & e & b & f \\ 5' & c & d & c & b & a & b \\ 6' & b & f & d & f & b & e \\ \hline \end{array} \qquad \begin{aligned} a &= 3A + 4B + 2C \\ b &= B + 2C + 2D + E \\ c &= D + 2E + 3F + 2G + H \\ d &= 2G + 4H \\ e &= 2A + 2B \\ f &= E + 2F + G \end{aligned} \tag{12.3}$$

Since the first multiplication $\mathbf{C}_t \cdot \mathbf{z}$ adds up the components of each *column* in each matrix, and the second multiplication $(\mathbf{C}_t \cdot \mathbf{z}) \cdot \mathbf{C}$ adds up the components of each *row* in each matrix, *hence in connecting neighboring coils into groups it is sufficient to add up the letters in each component matrix of* $\mathbf{z}$, *without the formal calculation of* $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$.

The winding of Fig. 12.2 representing the winding of Fig. 12.1 after the neighboring coils have been interconnected is connected again into three phases by $\mathbf{C}'$

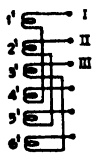


FIG. 12.2.—Connecting Groups Into Phases

$$\mathbf{C}' = \begin{array}{c|ccc|} & \mathrm{I} & \mathrm{II} & \mathrm{III} \\ \hline 1' & 1 & & \\ 2' & & 1 & \\ 3' & & & 1 \\ 4' & -1 & & \\ 5' & & -1 & \\ 6' & & & -1 \\ \hline \end{array} \tag{12.4}$$

This second $\mathbf{C}'$ *is identical for all machines having the same number of pole-phase groups, whether the groups are identical or different, that is, whether the windings have integer or fractional number of slots per pole-phase group.*

The final components of the impedance tensor $\mathbf{z}''$ are by $\mathbf{C}_t' \cdot \mathbf{z}' \cdot \mathbf{C}'$

$$\mathbf{z}'' = \begin{array}{c|c|c|c|} & \text{I} & \text{II} & \text{III} \\ \hline \text{I} & a + e - 2d & 2b - c - f & c + f - 2b \\ \hline \text{II} & 2b - c - f & a + e - 2d & 2b - c - f \\ \hline \text{III} & c + f - 2b & 2b - c - f & a + e - 2d \\ \hline \end{array} \qquad 12.5$$

Similar labor-saving devices can be introduced in practically all types of windings.

IV. REACTANCE CALCULATION OF INDIVIDUAL COILS

Types of Reactances. (*a*) For purposes of calculation the *total reactance* of an individual armature coil (or of a whole armature winding) is divided into three parts:

1. *Total Air-gap Reactance,* due to all the fluxes passing across the air gap from the stator to the rotor, or vice versa (Fig. 12.3).

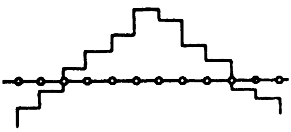


FIG. 12.3.—Flux-density Wave Producing Total Air-gap Reactance

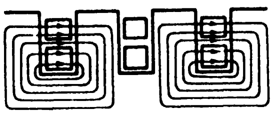


FIG. 12.4.—Flux Producing Slot-leakage Reactance

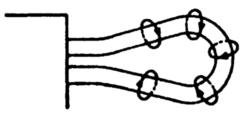


FIG. 12.5.—Flux Producing End-leakage Reactance

2. *Slot-leakage Reactance,* due to all fluxes that pass across the slots, without getting across the air gap (Fig. 12.4).

3. *End-leakage Reactance,* due to all fluxes linking the end connections (Fig. 12.5).

The sum of these three types of reactances is the *total reactance* of the individual coil or of the whole winding.

(*b*) The *total air-gap reactance* (Fig. 12.3) itself is divided into two parts:

1. *Fundamental Reactance,* due to the *sinusoidal* part of the flux wave shown in Fig. 12.3 (This is the *useful* part of the flux.)

2. *Differential-leakage Reactance,* due to the *remaining* part of the total air-gap flux. (*This flux causes the high-frequency tooth losses.*) This reactance is found by: Differential = Total − Fundamental.

The *differential-leakage reactance* itself is divided into the sum of *n*th *harmonic reactances,* namely, the sum of the second, third, etc., reactances.

Air-gap Reactances of Coils. Table 12.1 gives formulas for the calculation of the following self- and mutual reactances of individual coils:

1. *Total air-gap reactance,* assuming the two coils (*a*) outside of each other, (*b*) coupled, (*c*) inside of each other.

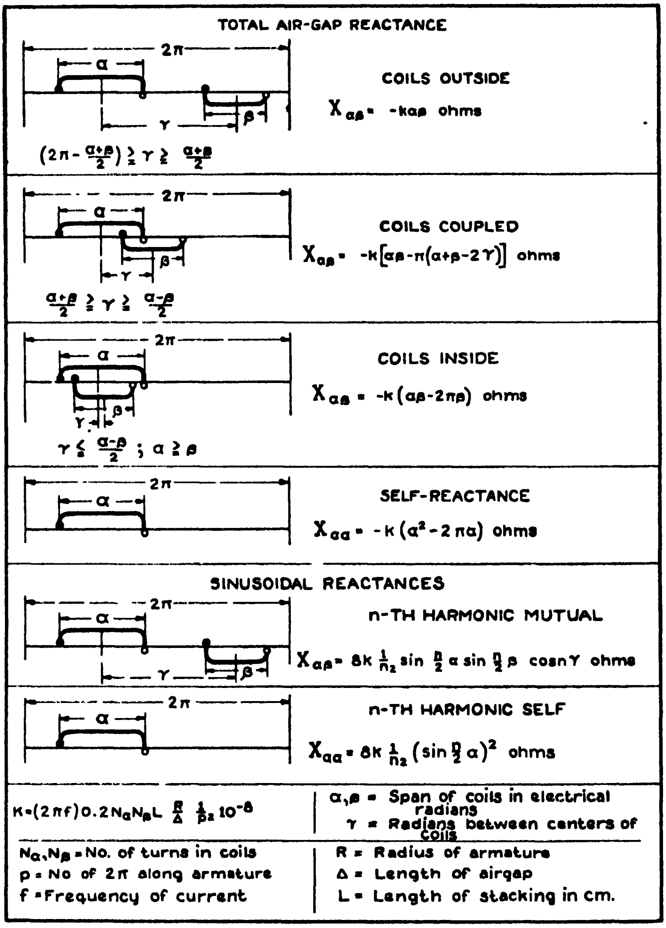


Table 12.1.—Mutual-reactance Formulas of Two Arbitrary Coils

2. *Nth Harmonic Reactance.* The special case $n = 1$ gives the *fundamental* reactance.

The self-inductance of a coil is found by assuming the two coils equal and at zero distance.

If the winding repeats itself after every pair of poles, then the dis-

tance covering *one* pair of poles is considered as 2π radians. *If*, however, *there are also fluxes having fewer pairs of poles than the fundamental flux, then the distance covered by the longest wave length is assumed as 2π radians.* When in doubt, assume the whole machine circumference as 2π radians.

The number of turns in the coils, N_α and N_β, may be explicitly given in $X_{\alpha\beta}$ instead of including them in the constant k.

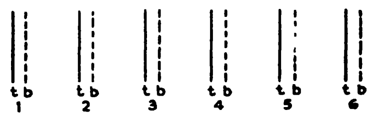
Since in an armature winding many of the coils are identical and similarly arranged, only a very few of these self- and mutual coil reactances have to be calculated. These various reactances will be denoted by $A, B, C, D \cdots$.

Slot-leakage Reactance of Coils. The calculation of the self- and mutual slot-leakage reactances of *conductors* lying in the same slot is given in design books. The steps in going from conductor reactances to coil reactances can be made with the aid of a "*transformation tensor*" **C** *showing how the conductors are interconnected into coils.*

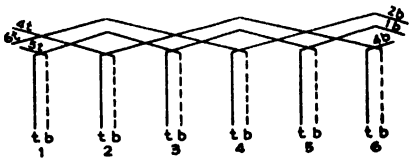
That is, if **z** represents the impedance tensor of all *conductors* and **C** is the transformation tensor showing the connection of the conductors into coils, as in equation 12.6 for Fig. 12.6, the impedance tensor of the *coils* is found by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ showing their self- and mutual slot-leakage reactances. Since the conductors in *different* slots have no mutual inductances, the coil reactances usually can be read off the winding diagrams, without setting up a transformation tensor **C**.

		1	2	3	4	5	6
1	t	1					
	b					−1	
2	t		1				
	b				−1		
3	t			1			
	b	−1					
4	t				1		
	b						−1
5	t					1	
	b			−1			
6	t						1
	b		−1				

12.6



(*a*) Conductors lying in slots



(*b*) Conductors connected into coils

FIG. 12.6

End-leakage Reactance of Coils. Similarly the end-winding reactances may be found if the self and mutual reactances of the individual coils are known.

V. STANDARD THREE-PHASE WINDING

Types of Reactances. As a first example the reactances of a standard three-phase winding will be calculated, since their value can be easily checked by other methods or tables.

Let the winding of Fig. 12.7 be given. It has twelve coils, pitch 1–6, the coils being connected as shown. The following single-phase reactances will be calculated:

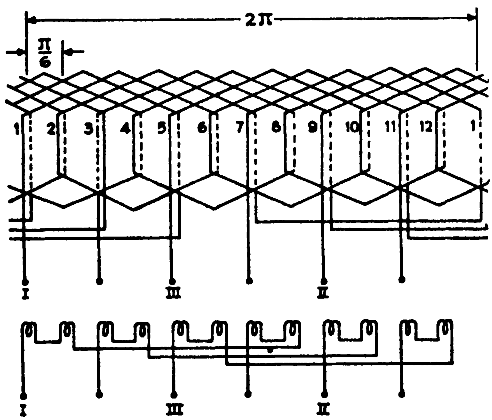


FIG. 12.7.—Standard Three-phase Winding

1. Total air-gap reactance.
2. Fundamental reactance.
3. Fifth-harmonic reactance.
4. Seventh-harmonic reactance.
5. Slot-leakage reactance.
6. End-leakage reactance.

Total Air-gap Reactances. It is sufficient to calculate the self- and mutual reactances of coil 1 with coils 2 to 6, that, is the *six different values* A, B, C, D, E, F, since those of the other coils are these same six values repeated in a different order.

The pitch of every coil is, from Fig. 12.8, $\alpha = \beta = 150° = (5/6)\pi$ radians. The distance γ between coils is 0, $\pi/6$, $2\pi/6$, $3\pi/6$, $4\pi/6$, $5\pi/6$, $7\pi/6$, etc., radians or 0, 30, 60, 90, 120, 150, etc., degrees.

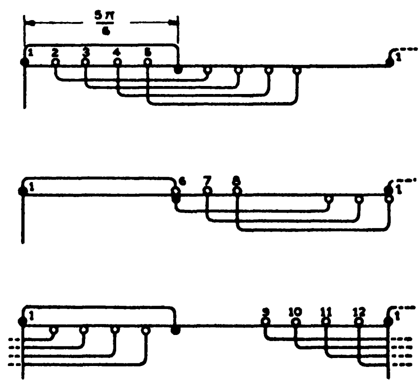


FIG. 12.8.—Calculation of Mutual Reactances

1. The *self-inductance* of coil 1 is

$$A = -k(\alpha^2 - 2\pi\alpha)$$
$$= -k[(25/36)\pi^2 - (10/6)\pi^2] = 9.583k$$

2. The *mutual inductance* of coil 1 with coils 2, 3, 4, and 5, that is with those coils that couple it, is:

$$M_{\alpha\beta} = k(9.583 - 6.28\gamma)$$

where γ is 0.5236, 1.0472, 1.571, and 2.0944. Substituting the various values of γ, the mutual inductances are: $B = 6.29k$, $C = 3.02k$, $D = -0.28k$, $E = -3.57k$.

3. The *mutual inductance* of coil 1 with coils 6, 7, and 8 is $F = -k\alpha\beta = -k6.85 = G = H$.

4. The *mutual inductance* of coil 9 with coil 1 is the same as that of coil 5 with 1. Similarly that of coil 10 with 1 is the same as that of coil 4 with 1, and so on.

Hence the self- and mutual inductances of coil 1 with all the other coils can be arranged in a row as

	1	2	3	4	5	6	7	8	9	10	11	12
1	A	B	C	D	E	F	G	F	E	D	C	B

where the values of A, B, C, etc., are given in the first column of Table 12.2 and where the constant k is defined in Table 12.1, being a function of the number of turns, air-gap length, etc.

TABLE 12.2

COIL REACTANCES

	Air gap	Fundamental	Fifth-harmonic	Seventh-harmonic	Slot-leakage	End
A	$9.58k$	$7.47k$	$0.0216k$	$0.0294k$	$L + N$	A'
B	$6.29k$	$6\ 47k$	$-0.0187k$	$-0.0254k$	0	B'
C	$3.02k$	$3.73k$	$0.0108k$	$0.0147k$	0	C'
D	$-0.28k$	$0.\ \ k$	$0.\ \ k$	$0.\ \ k$	0	D'
E	$-3.57k$	$-3.73k$	$-0.0108k$	$-0.0147k$	0	E'
F	$-6.85k$	$-6.47k$	$0\ 0187k$	$0\ 0254k$	$-M$	F'
G	$-6.85k$	$-7.47k$	$-0\ 0216k$	$-0\ 0294k$	0	G'

The self- and mutual inductances of coil 2 form a similar line except the components are shifted and hence all the self- and mutual air-gap inductances of the individual coils can be arranged in the impedance tensor

$$\mathbf{z} = \begin{array}{c|cccccccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline 1 & A & B & C & D & E & F & G & F & E & D & C & B \\ 2 & B & A & B & C & D & E & F & G & F & E & D & C \\ & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 12 & B & C & D & E & F & G & F & E & D & C & B & A \\ \hline \end{array} \qquad 12.7$$

This tensor represents also all the other types of coil reactances to be calculated presently.

Sinusoidal Reactances. (1). Considering the *fundamental* sine-wave reactances let in the formulas in Table 12.1, $n = 1$. Then

$$8k \sin \frac{\alpha}{2} \sin \frac{\beta}{2} = k8(\sin \frac{5}{12} \pi)^2 = 8k(\sin 75°)^2 = 7.464k$$

and $m_{\alpha\beta} = 7.47k \cos \gamma$, where γ varies from 0° to 330° by steps of 30°, giving for the various fundamental self- and mutual inductances of the individual coils the second column of Table 12.2.

2. Considering the *fifth-harmonic* self- and mutual inductances when $n = 5$, then $8k \frac{1}{25}(\sin \frac{5}{2} \frac{5}{6}\pi)^2 = 0.32k(\sin 15°)^2 = 0.0216k$, and $M_{\alpha\beta} = 0.0216k \cos 5\gamma$, where γ varies from 0° to 330° by steps of 30°. The various fifth-harmonic self- and mutual inductances are giveni n Table 12.2.

3. Considering the *seventh-harmonic* reactances when $n = 7$, then $8k \frac{1}{49}(\sin \frac{7}{2} \frac{5}{6}\pi)^2 = 0.163k\ (\sin 165°)^2 = 0.0294k$, and $M_{\alpha\beta} = 0.0294k \cos 7\gamma$, where γ varies from 0° to 330° by steps of 30°. The various reactances are given in Table 12.2.

Slot-leakage Reactances. In one slot lie two coil sides as shown in Fig. 12.9. Let the self-inductances of the upper conductor due to slot-leakage fluxes be L, of the bottom conductor N, and their mutual inductance M, calculated by methods given in textbooks. The problem is to find the slot-leakage reactances of the various *coils*, if those of the *coil sides* are L, N, and M.

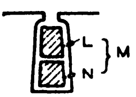


FIG 12.9.—Two Coil Sides in a Slot

It is unnecessary to set up a **C**. By inspection it can be seen that every coil has a mutual inductance with those coils only that lie in the same slot, consequently $\mathbf{z}'$ can be set up immediately.

The slot-leakage reactances of coils is shown in the fifth column of Table 12.2.

Let it be assumed likewise that the *end-leakage reactances* of the individual coils have also been calculated by some method. They are shown in the last column of Table 12.2.

Reactances of the Interconnected Windings. From Fig. 12.7 it can be seen that the first step is to interconnect *neighboring* coils into six pole-phase groups by dividing $\mathbf{z}$ into small matrices with two rows and columns. The product $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ is found by simply adding up the components of each small matrix of $\mathbf{z}$ giving

$$\mathbf{z}' = \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{array}{|c|c|c|c|c|c|} \hline a & b & c & d & c & b \\ \hline b & a & b & c & d & c \\ \hline c & b & a & b & c & d \\ \hline d & c & b & a & b & c \\ \hline c & d & c & b & a & b \\ \hline b & c & d & c & b & a \\ \hline \end{array} \end{array} \qquad \begin{aligned} a &= 2A + 2B \\ b &= B + 2C + D \\ c &= D + 2E + F \\ d &= 2F + 2G \end{aligned} \qquad 12.8$$

The next step is to interconnect the six pole-phase groups into three phase windings as shown in Fig. 12.7. *In three-phase windings it is advantageous to reverse the direction of the second winding* so that the three windings are symmetrically placed at 120° apart. In that case

$$\mathbf{C}' = \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{I} & \mathrm{II} & \mathrm{III} \end{array} \\ \begin{array}{|c|c|c|} \hline 1 & & \\ \hline & -1 & \\ \hline & & 1 \\ \hline -1 & & \\ \hline & 1 & \\ \hline & & -1 \\ \hline \end{array} \end{array}$$

The components of $\mathbf{z}'' = \mathbf{C}'_t \cdot \mathbf{z}' \cdot \mathbf{C}'$ are

$$\mathbf{z}'' = \begin{array}{c} \\ \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{I} & \mathrm{II} & \mathrm{III} \end{array} \\ \begin{array}{|c|c|c|} \hline a' & b' & b' \\ \hline b' & a' & b' \\ \hline b' & b' & a' \\ \hline \end{array} \end{array} \qquad \begin{aligned} a' &= 2a - 2d = 4(A + B - F - G) \\ b' &= 2c - 2b = 2(F - B) + 4(E - C) \end{aligned} \qquad 12.9$$

where " a' " represents the self-inductance of each phase winding and " b' " the mutual inductance between any two phase windings. Out of the possible nine reactances only two are different.

Now substituting the values of A, B, C, D, E, F, given in Table 12.2, Table 12.3 *gives the various single-phase reactances of the phase windings.*

TABLE 12.3

SINGLE-PHASE REACTANCES

	Total	Funda-mental	Fifth-harmonic	Seventh-harmonic	Slot-leakage	End-leakage
a'	$118.28k$	$111.52k$	$0.0232k$	$0.032k$	$4(L+N+M)$	$4(A'+B'-F'-G')$
b'	$-52.64k$	$-55.76k$	$-0.0116k$	$-0.016k$	$-2M$	$2(F'-B')+4(E'-C')$

The various types of *three-phase reactances* can be calculated again by the formula $\mathbf{z}'' = \mathbf{C}'_t \cdot \mathbf{z}' \cdot \mathbf{C}'$, where $\mathbf{C}'$ represents the manner of interconnection of the individual phase windings, as will be shown in Fig. 12.13.

VI. CAPACITOR-MOTOR WINDING

(*a*) As an example of a winding in which the coils have different pitches, consider the four-pole 36-slot unbalanced two-phase winding

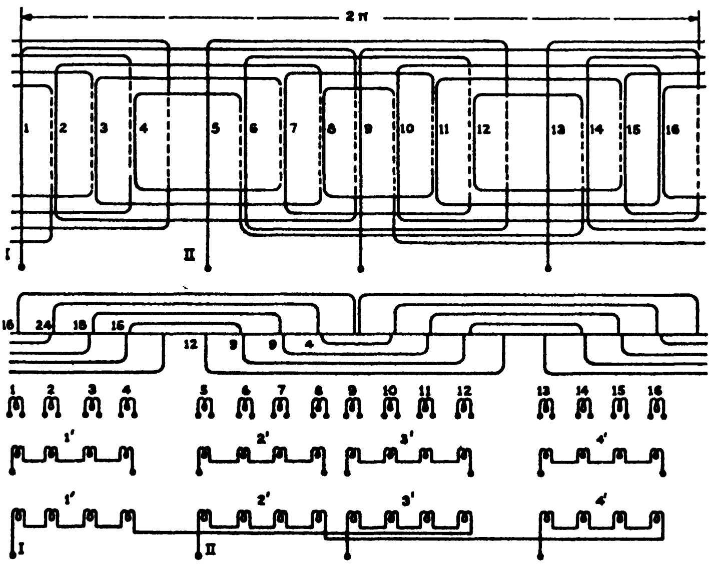


FIG. 12.10.—Capacitor-motor Winding

shown in Fig. 12.10. The main winding coils have 16, 24, 18, 16 turns, and the capacitor winding coils have 12, 9, 9, 4 turns. Because of the winding symmetry, 18 slots are considered as 2π radians.

Since half the coils along 2π radians have different pitches and number of turns, in the **z** tensor of the generalized winding the rows do not repeat themselves but all are different. As an example, only the fundamental and third-harmonic reactances for which **z** reduces to a simpler form will be calculated.

The *fundamental* reactances are calculated by the formula

$$X_{\alpha\beta} = \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\gamma N_\alpha N_\beta k$$

The *third harmonic* reactances are calculated by

$$X_{\alpha\beta} = \frac{1}{9}\sin\frac{3}{2}\sin\frac{3\beta}{2}\cos 3\gamma N_\alpha N_\beta k.$$

It is found that there is no mutual reactance between the coils of the main and the starting windings ($\cos\gamma = \cos 90° = 0$), so that for both types of reactances the impedance tensor of the individual coils is

$$\mathbf{z} = \begin{array}{c|cccc|} & 1' & 2' & 3' & 4' \\ \hline 1' & \mathbf{A} & & -\mathbf{A} & \\ 2' & & \mathbf{B} & & -\mathbf{B} \\ 3' & -\mathbf{A} & & \mathbf{A} & \\ 4' & & -\mathbf{B} & & \mathbf{B} \\ \hline \end{array} \qquad 12.10$$

where for the *fundamental* reactance

A =

	1	2	3	4
1	256	361	220.5	128
2	361	507	311	180.5
3	220.5	311	190	107.5
4	128	180.5	107.5	64

B =

	5	6	7	8
5	248	327	182	86.1
6	327	432	240	113.8
7	182	240	134	63.4
8	86.1	113.8	63.4	30

12.11

and for the *third-harmonic* reactance

A =

	1	2	3	4
1	28.4	21.3	−16	−28.4
2	21.3	16	−20	−21.2
3	−16	−20	9	16
4	−28.4	−21.2	16	28.4

B =

	5	6	7	8
5	21.3	0	−24	−21.3
6	0	0	0	0
7	−24	0	27	24
8	−21.3	0	24	21.3

12.12

each number being multiplied by the constant k.

For the total air-gap reactance **z** has a more complex form.

(*b*) The first step is to interconnect the neighboring coils into four groups as shown. The **z**′ of this arrangement is found by ***summing up the components of each matrix*** of **z**, namely, those of **A** and **B**, giving a similar matrix with **A** and **B** replaced by the numbers A and B. For the fundamental reactances $A = 3634$, $B = 3368.5$. For the third harmonic $A = -14.8$ and $B = 27$.

Interconnecting the four groups into phases, as shown in Fig. 12.11, $\mathbf{C}'_t \cdot \mathbf{z}' \cdot \mathbf{C}'$ gives $\mathbf{z}''$ (where $\mathbf{z}'$ is the same as **z** with **A** and **B** replaced by the numbers A and B.)

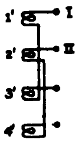


FIG. 12.11

$$\mathbf{C}' = \begin{array}{c|cc|} & \mathrm{I} & \mathrm{II} \\ \hline 1' & 1 & \\ 2' & & 1 \\ 3' & -1 & \\ 4' & & -1 \\ \hline \end{array} \qquad 12.13 \qquad \mathbf{z}'' = \begin{array}{c|cc|} & \mathrm{I} & \mathrm{II} \\ \hline \mathrm{I} & 4A & 0 \\ \mathrm{II} & 0 & 4B \\ \hline \end{array} \qquad 12.14$$

For the fundamental and third-harmonic reactances these are respectively

$$\mathbf{z}'' = \begin{array}{c|cc|} & \mathrm{I} & \mathrm{II} \\ \hline \mathrm{I} & 14{,}536 & 0 \\ \mathrm{II} & 0 & 11{,}474 \\ \hline \end{array} \qquad 12.15 \qquad \mathbf{z}'' = \begin{array}{c|cc|} & \mathrm{I} & \mathrm{II} \\ \hline \mathrm{I} & -59.2 & 0 \\ \mathrm{II} & 0 & 108 \\ \hline \end{array} \qquad 12.16$$

each number being multiplied by the constant k.

VII. EXAMPLE OF DOUBLE WINDING FOR TURBO-ALTERNATOR

The Impedance Tensor z of the Primitive Winding. (*a*) Let the winding of Fig. 12.12 be given with forty-two slots, covering 2π electrical radians. All coils are identical, having a pitch from first to nineteenth slot ($18/21 = 6/7$ pitch). There are altogether six identical groups, each containing seven coils.

Coil 1 will have a self-inductance A and a different mutual inductance with each of the other coils. The coils to the right of 1, $(2, 3 \cdots)$ have the same mutual inductance as the coils to the left $(42, 41 \cdots)$, hence *altogether there are twenty-one different mutual inductances of coil* 1, shown in the first row of the accompanying **z**.

Coil 2 will have the *same* self- and mutual inductances with all the other coils as coil 1, except that the above row is shifted to the right by one block. For coil 3 the above row is shifted to the right by two

blocks, and so on, so that for the forty-two coils the self- and mutual inductances are $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2$ where $\mathbf{z}_1$ and $\mathbf{z}_2$ are

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	A	B	C	D	E	F	G	H	I	J	K	L	M	N	P	Q	R	S	T	U	V
2	B	A	B	C	D	E	F	G	H	I	J	K	L	M	N	P	Q	R	S	T	U
3	C	B	A	B	C	D	E	F	G	H	I	J	K	L	M	N	P	Q	R	S	T
4	D	C	B	A	B	C	D	E	F	G	H	I	J	K	L	M	N	P	Q	R	S
5	E	D	C	B	A	B	C	D	E	F	G	H	I	J	K	L	M	N	P	Q	R
6	F	E	D	C	B	A	B	C	D	E	F	G	H	I	J	K	L	M	N	P	Q
7	G	F	E	D	C	B	A	B	C	D	E	F	G	H	I	J	K	L	M	N	P
8	H	G	F	E	D	C	B	A	B	C	D	E	F	G	H	I	J	K	L	M	N
9	I	H	G	F	E	D	C	B	A	B	C	D	E	F	G	H	I	J	K	L	M
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
42	B	C	D	E	F	G	H	I	J	K	L	M	N	P	Q	R	S	T	U	V	W

	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
1	W	V	U	T	S	R	Q	P	N	M	L	K	J	I	H	G	F	E	D	C	B
2	V	W	V	U	T	S	R	Q	P	N	M	L	K	J	I	H	G	F	E	D	C
3	U	V	W	V	U	T	S	R	Q	P	N	M	L	K	J	I	H	G	F	E	D
4	T	U	V	W	V	U	T	S	R	Q	P	N	M	L	K	J	I	H	G	F	E
5	S	T	U	V	W	V	U	T	S	R	Q	P	N	M	L	K	J	I	H	G	F
6	R	S	T	U	V	W	V	U	T	S	R	Q	P	N	M	L	K	J	I	H	G
7	Q	R	S	T	U	V	W	V	U	T	S	R	Q	P	N	M	L	K	J	I	H
8	P	Q	R	S	T	U	V	W	V	U	T	S	R	Q	P	N	M	L	K	J	I
9	N	P	Q	R	S	T	U	V	W	V	U	T	S	R	Q	P	N	M	L	K	J
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
42	V	U	T	S	R	Q	P	N	M	L	K	J	I	H	G	F	E	D	C	B	A

12.17

(Because of printing exigencies, $\mathbf{z}$ is expressed as the sum of two tensors.)

Since each row repeats itself, it is sufficient to write down only the first few lines of **z**.

(*b*) Since there are six identical groups, each group containing seven coils, the tensor **z** containing forty-two rows and columns will be divided into component matrices as shown, each containing seven rows and columns. *Of the thirty-six component matrices only four are different.* Hence **z** as a compound tensor can be written as

$$
\mathbf{z} = \begin{array}{c|c|c|c|c|c|c|}
 & \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} & \mathrm{V} & \mathrm{VI} \\
\hline
\mathrm{I} & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{C}_t & \mathbf{B}_t \\
\hline
\mathrm{II} & \mathbf{B}_t & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{C}_t \\
\hline
\mathrm{III} & \mathbf{C}_t & \mathbf{B}_t & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\
\hline
\mathrm{IV} & \mathbf{D} & \mathbf{C}_t & \mathbf{B}_t & \mathbf{A} & \mathbf{B} & \mathbf{C} \\
\hline
\mathrm{V} & \mathbf{C} & \mathbf{D} & \mathbf{C}_t & \mathbf{B}_t & \mathbf{A} & \mathbf{B} \\
\hline
\mathrm{VI} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{C}_t & \mathbf{B}_t & \mathbf{A} \\
\hline
\end{array} \qquad 12.18
$$

where **A**, **B**, **C**, and **D** are the first four small matrices.

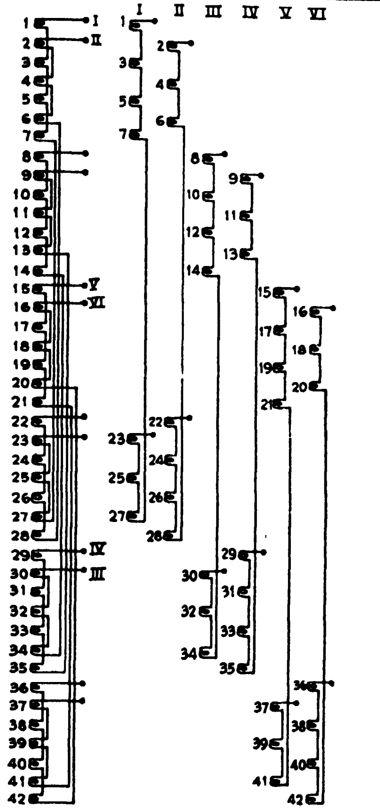
The Transformation Tensor C. There are forty-two coils connected into six groups, hence the transformation tensor **C** contains forty-two rows and six columns as shown in Fig. 12.12. (It is possible to interconnect the coils first into twelve groups and then only into six groups.)

The *first* column shows that coils 1, 3, 5, 7 are connected series opposing with coils 23, 25, 27. The *second* column shows that coils 2, 4, 6 are connected series opposing with coils 22, 24, 26, 28. The same connections repeat.

Looking at **C**, it can be seen that as a "*compound tensor*" it can be written as

$$
\mathbf{C} = \begin{array}{|c|c|c|}
\hline
\mathbf{M} & & \\
\hline
 & -\mathbf{M} & \\
\hline
 & & \mathbf{M} \\
\hline
\mathbf{N} & & \\
\hline
 & -\mathbf{N} & \\
\hline
 & & \mathbf{N} \\
\hline
\end{array}
\qquad
\mathbf{M} = \begin{array}{|c|c|}
\hline
1 & \\
\hline
 & 1 \\
\hline
1 & \\
\hline
 & 1 \\
\hline
1 & \\
\hline
 & 1 \\
\hline
1 & \\
\hline
\end{array}
\qquad
\mathbf{N} = \begin{array}{|c|c|}
\hline
 & -1 \\
\hline
-1 & \\
\hline
 & -1 \\
\hline
-1 & \\
\hline
 & -1 \\
\hline
-1 & \\
\hline
 & -1 \\
\hline
\end{array} \qquad 12.19
$$

It should be noted that **M** contains seven rows, just as many as **A**, **B**, **C**, **D** in the previous section.



$\mathbf{C} =$

	I	II	III	IV	V	VI
1	1					
2		1				
3	1					
4		1				
5	1					
6		1				
7	1					
8			-1			
9				-1		
10			-1			
11				-1		
12			-1			
13				-1		
14			-1			
15					1	
16						1
17					1	
18						1
19					1	
20						1
21					1	
22		-1				
23	-1					
24		-1				
25	-1					
26		-1				
27	-1					
28		-1				
29				1		
30			1			
31				1		
32			1			
33				1		
34			1			
35				1		
36						-1
37					-1	
38						-1
39					-1	
40						-1
41					-1	
42						-1

FIG. 12.12.—Connection Diagram and Transformation Matrix C of a Double Winding for Turbo-Alternators

The Winding Impedance Tensor z′. The impedance tensor $\mathbf{z}'$ of the windings is found by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C} =$

$$\mathbf{z}' = \begin{array}{|c|c|c|}\hline \mathbf{X} & \mathbf{Y} & \mathbf{W} \\ \hline \mathbf{Y}_t & \mathbf{X} & \mathbf{Y} \\ \hline \mathbf{W}_t & \mathbf{Y}_t & \mathbf{X} \\ \hline \end{array} \qquad \begin{aligned} \mathbf{X} &= (\mathbf{M}_t\cdot\mathbf{A}+\mathbf{N}_t\cdot\mathbf{D})\cdot\mathbf{M}+(\mathbf{M}_t\cdot\mathbf{D}+\mathbf{N}_t\cdot\mathbf{A})\cdot\mathbf{N} \\ \mathbf{Y} &= -(\mathbf{M}_t\cdot\mathbf{B}+\mathbf{N}_t\cdot\mathbf{C}_t)\cdot\mathbf{M}+(\mathbf{M}_t\cdot\mathbf{C}_t+\mathbf{N}_t\cdot\mathbf{B})\cdot\mathbf{N} \\ \mathbf{W} &= (\mathbf{M}_t\cdot\mathbf{C}+\mathbf{N}_t\cdot\mathbf{B}_t)\cdot\mathbf{M}+(\mathbf{M}_t\cdot\mathbf{B}_t+\mathbf{N}_t\cdot\mathbf{C})\cdot\mathbf{N} \end{aligned} \tag{12.20}$$

The remaining work consists of calculating the three matrices **X**, **Y**, *and* **W**. Table 12.4 gives the calculation of **X**.

Since every component in a matrix consists of the sum of letters as $A + B + C + D + E$, *the sign is left out* in Table 12.4, and a minus

sign is put in front of them where $-A$, $-B$, $-C$, $-D$, $-E$ occurs. It should be noted the matrices **X**, **Y**, and **W** are symmetrical, and that some of their components are equal. This equality of components serves as check on the correctness of the calculations. It should also be noted that in each component in Table 12.4 *the letters are identical along the diagonal lines*, serving as a further check and showing up *immediately* any mistake made during the calculations.

Hence **X**, **Y**, and **W** are the following matrices:

$$\mathbf{X} = \begin{array}{|c|c|}\hline a & b \\ \hline b & a \\ \hline\end{array} \qquad \mathbf{Y} = \begin{array}{|c|c|}\hline c & d \\ \hline d & c \\ \hline\end{array} \qquad \mathbf{W} = \begin{array}{|c|c|}\hline d & c \\ \hline c & d \\ \hline\end{array}$$

Substituting the above values of **X**, **Y**, and **W** into $\mathbf{z}'$, the final components of the impedance tensor $\mathbf{z}'$ are

$$\mathbf{z}' = \begin{array}{c|c|c|c|c|c|c|} & \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} & \mathrm{V} & \mathrm{VI} \\ \hline \mathrm{I} & a & b & c & d & d & c \\ \hline \mathrm{II} & b & a & d & c & c & d \\ \hline \mathrm{III} & c & d & a & b & c & d \\ \hline \mathrm{IV} & d & c & b & a & d & c \\ \hline \mathrm{V} & d & c & c & d & a & b \\ \hline \mathrm{VI} & c & d & d & c & b & a \\ \hline \end{array} \qquad 12.21$$

where $a = 7A + 10C + 6E + 2G - 4R - 8T - 12V$

$$b = 12B + 8D + 4F - 6S - 2Q - 10U - 7W$$

$$c = -(B+3D+5F+7H+3J-L-5N-6Q-4S-2U)$$

$$d = -(2C+4E+6G+5I+K-3M-7P-5R-3T-V)$$

representing the self- and mutual reactances of the six windings I–VI of Fig. 12.12 in terms of the reactances $A, B, C \cdots$ of the individual coils. *Out of the thirty-six possible reactances only four are different.*

If these six windings are again interconnected in any manner represented by $\mathbf{C}'$, the resultant new impedance tensor is formed by $\mathbf{C}'_t \cdot \mathbf{z}' \cdot \mathbf{C}' = \mathbf{z}''$.

The Evaluation of Reactances. Three types of reactances will be evaluated. They are: (1) total air gap, (2) fundamental, (3) slot-leakage reactances.

TABLE 12.4

SAMPLE CALCULATION OF A COMPOUND MATRIX X

$$\mathbf{M}_t\cdot\mathbf{A} = \begin{array}{|c|c|c|c|c|c|c|}\hline ACEG & BBDF & CACE & DBBD & ECAC & FDBB & GECA \\ \hline BDF & ACE & BBD & CAC & DBB & ECA & FDB \\ \hline\end{array}$$

$$\mathbf{M}_t\cdot\mathbf{A}\cdot\mathbf{M} = \begin{array}{|c|c|}\hline \begin{matrix} ACEG \\ CACE \\ ECAC \\ GECA \end{matrix} & \begin{matrix} BBDF \\ DBBD \\ FDBB \\ \end{matrix} \\ \hline \begin{matrix} BDF \\ BBD \\ DBB \\ FDB \end{matrix} & \begin{matrix} ACE \\ CAC \\ ECA \\ \end{matrix} \\ \hline\end{array} = \begin{array}{|c|c|}\hline 4A+6C+4E+2G & 6B+4D+2F \\ \hline 6B+4D+2F & 3A+4C+2E \\ \hline\end{array}$$

$$\mathbf{N}_t\cdot\mathbf{D} = \begin{array}{|c|c|c|c|c|c|c|}\hline -VTR & -WUX & -VVT & -UWU & -TVV & -SUW & -RTV \\ \hline -WUSQ & -VVTR & -UWUS & -TVVT & -SUWU & -RTVV & -QSUW \\ \hline\end{array}$$

$$\mathbf{N}_t\cdot\mathbf{D}\cdot\mathbf{M} = \begin{array}{|c|c|}\hline \begin{matrix} VTR \\ VVT \\ -TVV \\ RTV \end{matrix} & \begin{matrix} WUX \\ -UWU \\ SUW \\ \end{matrix} \\ \hline \begin{matrix} WUSQ \\ UWUS \\ -SUWU \\ QSUW \end{matrix} & \begin{matrix} VVTR \\ TVVT \\ RTVV \\ \end{matrix} \\ \hline\end{array} = \begin{array}{|c|c|}\hline -2R-4T-6V & -2S\ 4U-3W \\ \hline -2Q-4S-6U-4W & -2R-4T \\ \hline\end{array}$$

$$\mathbf{M}_t\cdot\mathbf{D} = \begin{array}{|c|c|c|c|c|c|c|}\hline WUSQ & VVTR & UWUS & TVVT & SUWU & RTVV & QSUW \\ \hline VTR & WUS & VVT & UWU & TVV & SUW & RTV \\ \hline\end{array}$$

$$\mathbf{M}_t\cdot\mathbf{D}\cdot\mathbf{N} = \begin{array}{|c|c|}\hline \begin{matrix} VVTR \\ -TVVT \\ RTVV \\ \end{matrix} & \begin{matrix} WUSQ \\ UWUS \\ -SUWU \\ QSUW \end{matrix} \\ \hline \begin{matrix} WUS \\ -UWU \\ SUW \\ \end{matrix} & \begin{matrix} VTR \\ VVT \\ TVV \\ RTV \end{matrix} \\ \hline\end{array} = \begin{array}{|c|c|}\hline -2R-4T-6V & -2Q-4S-6U-4W \\ \hline -2S-4U-3W & -2R-4T-6V \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline -BDF & -ACE & -BBD & -CAC & -DBB & -ECA & -FDB \\ \hline -ACEG & -BBDF & -CACE & -DBBD & -ECAC & -FDBB & -GECA \\ \hline\end{array}$$

$$\mathbf{N}_t\cdot\mathbf{A}\cdot\mathbf{N} = \begin{array}{|c|c|}\hline \begin{matrix} ACE \\ CAC \\ ECA \\ \end{matrix} & \begin{matrix} BDF \\ BBD \\ DBB \\ FDB \end{matrix} \\ \hline \begin{matrix} BBDF \\ DBBD \\ FDBB \\ \end{matrix} & \begin{matrix} ACEG \\ CACE \\ ECAC \\ GECA \end{matrix} \\ \hline\end{array} = \begin{array}{|c|c|}\hline 3A+4C+2E & 6B+4D+2F \\ \hline 6B+4D+2F & 4A+6C+4E+2G \\ \hline\end{array}$$

$$\mathbf{X} = \begin{array}{|c|c|}\hline \begin{matrix} 7A+10C+6E+2G-4R- \\ -8T-12V \end{matrix} & \begin{matrix} 12B+8D+4F-6S-2Q- \\ -10U-7W \end{matrix} \\ \hline \begin{matrix} 12B+8D+4F-6S-2Q- \\ -10U-7W \end{matrix} & \begin{matrix} 7A+10C+6E+2G-4R- \\ -8T-12V \end{matrix} \\ \hline\end{array}$$

(*a*) The self- and mutual reactances of the individual coils calculated by the formulas in Table 12.1 are given in Table 12.5. The constant k' was so selected that the fundamental self-inductance should be unity.

TABLE 12.5

	Total	Fundamental	Slot		Total	Fundamental	Slot
A	$1.273k'$	$1.000k'$	$1.875k'$	L	-0.088	-0.075	0
B	1.150	0.989	0	M	$-$ .212	$-$ .2235	0
C	1.025	.955	0	N	$-$ .336	$-$ .366	0
D	0.901	.902	0	P	$-$ .459	$-$ .500	0
E	.776	.825	0	Q	$-$ .583	$-$ .625	0
F	.653	.733	0	R	$-$ .707	$-$ .733	0
G	.529	.625	0	S	$-$ .829	$-$ .825	0
H	.405	.500	0	T	$-$ .953	$-$ .902	$-$ .625
I	.281	.366	0	U	$-$ .953	$-$ .955	0
J	.157	.2235	0	V	$-$ .953	$-$ .989	0
K	.035	.075	0	W	$-$ .953	-1.000	0

Since the pitch of all coils is the same, ***the slot-leakage reactance of the individual coils*** can be written down without calculation, analogously to that of a standard three-phase winding given previously. That is, if the self-inductance of the ***conductors in one slot*** is L_t and L_b and their mutual inductances is L_{tb}, then:

1. The self-inductance of each coil is $A = L_t + L_b$.

2. Only those coils have mutual inductance that lie in the same slot. The value of this mutual inductance is $-L_{tb}$.

Hence coil 1 has a mutual inductance only with coils 19 and 25, giving $T = -L_{tb}$, making all other mutual inductances $B, C, D, E \cdots$ equal to zero. The slot-leakage reactances of the conductors are assumed as $L_t = 0.5$, $L_b = 1.375$, and $L_{tb} = 0.625$.

(*b*) ***The four different self- and mutual reactances of the six interconnected windings***, contained in $\mathbf{z}'$, ***are given in Table*** 12.6.

Winding Interconnections. The reactances a, b, c, d calculated are the ***single-phase*** reactances of the windings. The windings themselves may be interconnected in various manner by a new $\mathbf{C}'$, three of the connections being shown in Fig. 12.13 in which case the reac-

tances $\mathbf{z}''$ of the interconnected windings are calculated again from the single-phase reactances by $\mathbf{C}'_t \cdot \mathbf{z} \cdot \mathbf{C}' = \mathbf{z}''$.

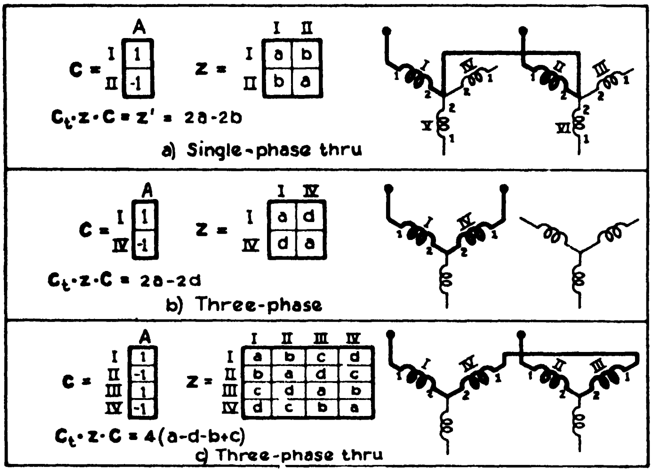


FIG. 12.13.—Various Interconnections of Double Windings

TABLE 12.6

WINDING REACTANCES

	Total	Fundamental	Slot
a	$46.76k'$	$44.77k'$	18.1
b	$45.96k'$	$44.77k'$	0
c	$-20.91k'$	$-22.39k'$	−0
d	$-20.96k'$	$-22.39k'$	−1.88

The self-impedances of the interconnected windings shown in Fig. 12.13 are given in Table 12.7.

TABLE 12.7

REACTANCES OF INTERCONNECTED WINDINGS

Reactance	Total	Fundamental	Slot leakage
$1-\phi \quad = a$	$46.76k'$	$44.77k'$	18.1
$1-\phi$ through $= 2(a-b)$	$1.6\ k'$	$0\quad k'$	36.2
$3-\phi \quad = 2(a-d)$	$135.44k'$	$134.34k'$	39.98
$3-\phi$ through $= 4(a-d-b+c)$	$3.4\ k'$	$0\quad k'$	79.96

VIII. PART WINDINGS FOR SYNCHRONOUS MOTOR STARTING

The Coil Reactances. (*a*) A particular twenty-four-pole, three-phase synchronous motor has 252 similar coils. For purposes of starting the coils are connected into twenty-four groups, as shown in Fig. 12.14. During starting some of the groups are interconnected and

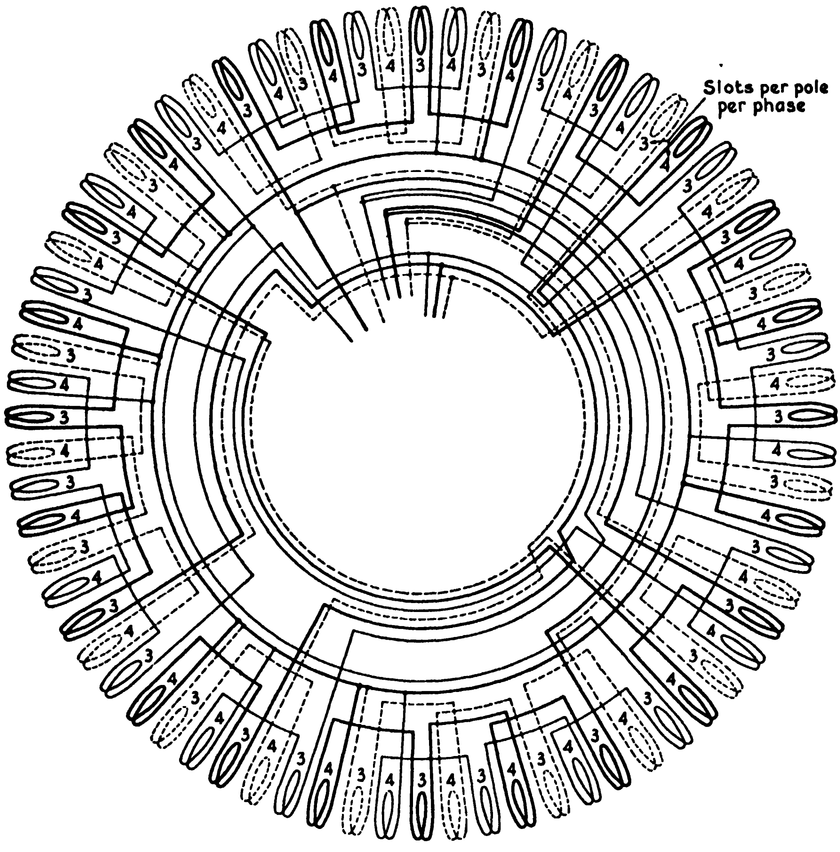


FIG. 12.14.—Connecting a Synchronous Motor Winding with 252 Coils into 24 Groups for Starting

energized in different order. Since two mechanically opposite windings are always energized, it is sufficient to analyze only half the machine, containing 126 coils in 2π radians. The connection diagram of the 126 coils into twelve windings is shown in Fig. 12.15.

Since there are sixty-three coils covering 2π radians, *the mutual inductances between the sixty-three coils are in general all different,*

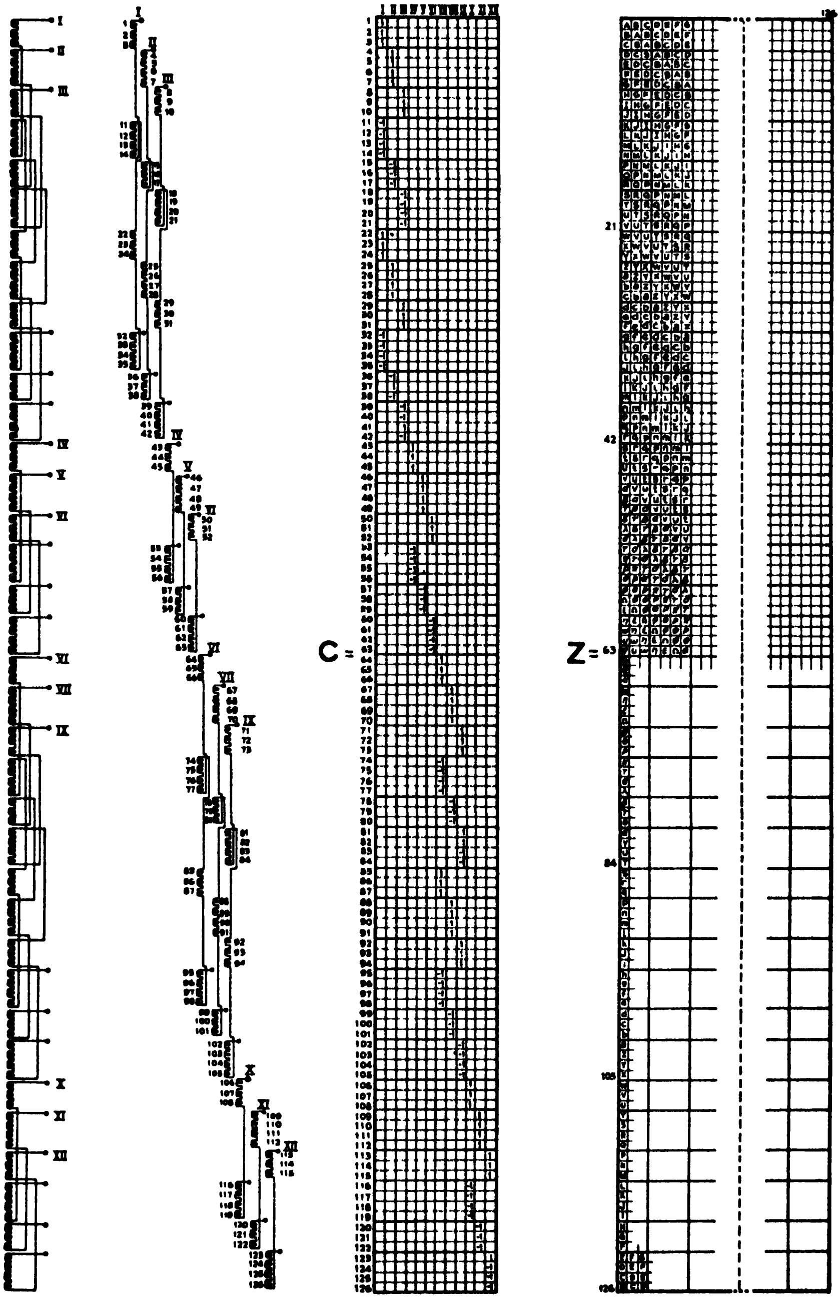


Fig. 12.15.—Connection Diagram and Transformation Matrix of the Winding of Fig. 12.14, ignoring Half the Winding

necessitating the use of sixty-four different letters in the impedance tensor $\mathbf{z}$, also shown in Fig. 12.15. *It is sufficient to fill in only half of the first seven columns*, as shown, since the second half repeats the first half in the reverse order, and also the other columns repeat the first seven columns.

(*b*) The first step is to interconnect only the *neighboring* coils in alternate groups of three and four. The resultant $\mathbf{z}'$ due to this interconnection is found without $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ by dividing $\mathbf{z}$ into 36×36 blocks and adding up the components of each block. The resulting $\mathbf{z}'$ is given in Fig. 12.16.

The connection diagram and transformation tensor of the thirty-six resultant groups are also shown in Fig. 12.16.

1. The transformation tensor $\mathbf{C}'$ may be represented as a compound tensor in which the only component is the matrix $\mathbf{M}$ having six rows.

$$\mathbf{C}' = \begin{array}{|c|c|c|c|}\hline \mathbf{M} & & & \\ \hline \mathbf{M} & & & \\ \hline & \mathbf{M} & & \\ \hline & & \mathbf{M} & \\ \hline & & \mathbf{M} & \\ \hline & & & \mathbf{M} \\ \hline \end{array} \qquad \mathbf{M} = \begin{array}{|c|c|c|}\hline 1 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline -1 & & \\ \hline & -1 & \\ \hline & & -1 \\ \hline \end{array} \tag{12.22}$$

2. The impedance tensor of the group reactances $\mathbf{z}'$ can be divided into component matrices, each having six rows and columns (that is, as many as there are rows in $\mathbf{M}$), so that $\mathbf{Z}'$ can be written as a compound tensor as

$$\mathbf{z}' = \begin{array}{|c|c|c|c|c|c|}\hline \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{C}_t & \mathbf{B}_t \\ \hline \mathbf{B}_t & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{C}_t \\ \hline \mathbf{C}_t & \mathbf{B}_t & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\ \hline \mathbf{D} & \mathbf{C}_t & \mathbf{B}_t & \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \hline \mathbf{C} & \mathbf{D} & \mathbf{C}_t & \mathbf{B}_t & \mathbf{A} & \mathbf{B} \\ \hline \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{C}_t & \mathbf{B}_t & \mathbf{A} \\ \hline \end{array} \tag{12.23}$$

Only four different matrices **A**, **B**, **C**, **D**, *each having six rows and columns and shown in Fig.* 12.16 *are needed to represent completely the tensor* $\mathbf{z}'$ *having thirty-six rows and columns.*

Winding Reactances. In finding $\mathbf{C}'_t \cdot \mathbf{z}' \cdot \mathbf{C}'$, the impedance tensor of the twelve windings is

$$\mathbf{z}'' = \begin{array}{|c|c|c|c|}\hline 2\mathbf{P}+\mathbf{Q}+\mathbf{Q}_t & \mathbf{Q}+\mathbf{R} & 2\mathbf{S}+\mathbf{R}+\mathbf{R}_t & \mathbf{Q}_t+\mathbf{R}_t \\ \hline \mathbf{Q}_t+\mathbf{R}_t & \mathbf{P} & \mathbf{Q}+\mathbf{R} & \mathbf{S} \\ \hline 2\mathbf{S}+\mathbf{R}\cdot\mathbf{R}_t & \mathbf{Q}_t+\mathbf{R}_t & 2\mathbf{P}+\mathbf{Q}+\mathbf{Q}_t & \mathbf{Q}+\mathbf{R} \\ \hline \mathbf{Q}+\mathbf{R} & \mathbf{S} & \mathbf{Q}_t+\mathbf{R}_t & \mathbf{P} \\ \hline \end{array} \qquad 12.24$$

where

$$\mathbf{P} = \mathbf{M}_t \cdot \mathbf{A} \cdot \mathbf{M} \qquad \mathbf{R} = \mathbf{M}_t \cdot \mathbf{C} \cdot \mathbf{M}$$

$$\mathbf{Q} = \mathbf{M}_t \cdot \mathbf{B} \cdot \mathbf{M} \qquad \mathbf{S} = \mathbf{M}_t \cdot \mathbf{D} \cdot \mathbf{M}$$

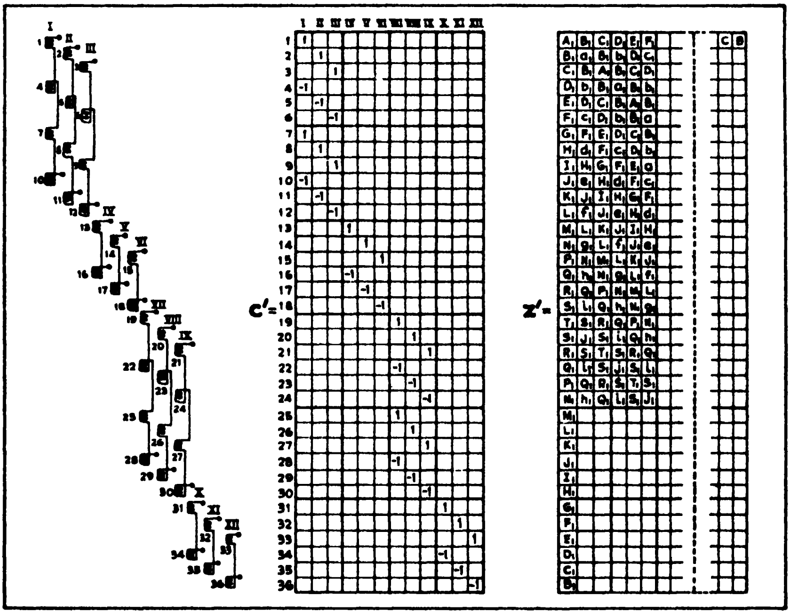


FIG. 12.16.—Connection-diagram and Transformation Matrix of the Winding of Fig. 12.15, after the Neighboring Coils are Interconnected

The use of compound matrices reduces the multiplication of a set of matrices with 36 rows and 12 columns (Fig. 12.16) to the multiplication of *four* set of matrices having only 6 rows and 3 columns. The saving of labor is considerable.

Substituting the values of **P**, **Q**, **R**, and **S** into **z**″, the final components of the impedance tensor are

$$
\mathbf{z}'' = \begin{array}{c|cccccccccccc|}
 & \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} & \mathrm{V} & \mathrm{VI} & \mathrm{VII} & \mathrm{VIII} & \mathrm{IX} & \mathrm{X} & \mathrm{XI} & \mathrm{XII} \\
\hline
\mathrm{I} & a' & b' & c' & i' & j' & k' & e' & f' & g' & i' & m' & n' \\
\mathrm{II} & b' & a' & d' & m' & i' & e' & f' & e' & h' & j' & i' & p' \\
\mathrm{III} & c' & d' & a' & n' & p' & i' & g' & h' & e' & k' & l' & i' \\
\mathrm{IV} & i' & m' & n' & q' & r' & s' & i' & j' & k' & u' & b' & w' \\
\mathrm{V} & j' & i' & p' & r' & q' & t' & m' & i' & l' & v' & u' & z' \\
\mathrm{VI} & k' & l' & i' & s' & t' & q' & n' & p' & i' & w' & z' & u' \\
\mathrm{VII} & e' & f' & g' & i' & m' & n' & a' & b' & c' & i' & j' & k' \\
\mathrm{VIII} & f' & e' & h' & j' & i' & p' & b' & a' & d' & m' & i' & l' \\
\mathrm{IX} & g' & h' & e' & h' & l' & i' & c' & d' & a' & n' & p' & i' \\
\mathrm{X} & i' & j' & k' & u' & v' & w' & i' & m' & n' & q' & r' & s' \\
\mathrm{XI} & m' & i' & l' & v' & u' & z' & j' & i' & p' & r' & q' & l' \\
\mathrm{XII} & n' & p' & i' & w' & z' & u' & k' & l' & i' & s' & t' & q' \\
\hline
\end{array} \qquad 12.25
$$

where

$$
\begin{aligned}
a' &= 2(A' + E') & g' &= 2V' + N' + R' & m' &= I' + Q' & t' &= D' \\
b' &= 2B' + F' + I' & h' &= 2Z' + P' + S' & n' &= J' + R' & u' &= T' \\
c' &= 2C' + G' + J' & i' &= E' + M' & p' &= K' + S' & v' &= U' \\
d' &= 2D' + H' + K' & j' &= F' + L' & q' &= A' & w' &= V' \\
e' &= 2(T' + M') & k' &= G' + N' & r' &= B' & z' &= Z' \\
f' &= 2U' + L' + Q' & l' &= H' + P' & s' &= C' & &
\end{aligned}
$$

$$
\begin{aligned}
A' &= 7A+10B+6C+2D-2I-4J-6K-6L-4M-2N \\
B' &= 2B+4C+6D+5E+2F-G-M-2N-3P-4H-3I-2J-K-2Q-R \\
C' &= F+4G+7H+5I+3J+K-B-2C-3D-2E-Q-2R-3S-3T-2U-V \\
D' &= 2B+4C+6D+6E+3F-3H-2I-J-L-2M-3N-4P-3Q-2R-S \\
E' &= T+3U+5V+7W+5X+3Y+Z-I-2J-3K-3L-2M-N-e-2f-3g-3h-2i-j \\
F' &= 2X+4Y+6Z+6a+4b+2c-L-2M-3N-4P-3Q-2R-S-i-2j-3k-2l-m \\
G' &= a+3b+5c+7d+5e+3f+g-Q-2R-3S-3T-2U-V-l-2m-3n-3p-2q-r \\
H' &= 2X+4Y+6Z+6a+4b+2c-M-2N-3P-2Q-R-h-2i-3j-4k-3l-2m-n \\
I' &= 2Q+4R+6S+6T+4U+2V-F-2G-3H-2I-J-a-2b-3c-4d-3e-2f-g \\
J' &= L+3M+5N+7P+5Q+3R+S-B-2C-3D-3E-2F-G-X-2Y-3Z-3a-2b-c \\
K' &= 2Q+4R+6S+6T+4U+2V-E-2F-3G-4H-3I-2J-K-b-2c-3d-2e-f \\
L' &= 2t+4u+6v+6\alpha+4\beta+2\gamma-h-2i-3j-4k-3l-2m-n-\epsilon-2\Omega-3\phi-2\theta-\rho \\
M' &= p+3q+5r+7s+5t+3u+v-e-2f-3g-3h-2i-j-\lambda-2\sigma-3\tau-3\pi-2\rho-\theta \\
N' &= \alpha+3\beta+5\gamma+7\delta+5\lambda+3\sigma+\tau-l-2m-3n-3p-2q-r-\varphi-2\mu-3\omega-3\eta-2\epsilon-\Omega \\
P' &= 2t+4u+6v+6\alpha+4\beta+2\gamma-i-2j-3k-2l-m-\pi-2\rho-3\theta-4\psi-3\Omega-2\epsilon-\eta \\
Q' &= 2l+4m+6n+6p+4q+2r-b-2c-3d-2e-f-\alpha-2\beta-3\gamma-4\delta-3\lambda-2\sigma-\tau \\
R' &= h+3i+5j+7k+5l+3m+n-x-2y-3z-3a-2b-c-t-2u-3v-3\alpha-2\beta-\gamma \\
S' &= 2l+4m+6n+6p+4q+2r-a-2b-3c-4d-3e-2f-g-\beta-2\gamma-3\delta-2\lambda-\sigma \\
T' &= 7\psi+10\varphi+6\mu+2\omega-2\theta-4\rho-6\pi-6\tau-4\sigma-2\lambda \\
U' &= 2\varphi+4\mu+6\omega+5\eta+2\epsilon-\Omega-4\psi-3\theta-2\rho-\pi-\sigma-2\lambda-3\delta-2\gamma-\beta \\
V' &= \epsilon+4\Omega+7\psi+5\theta+3\rho+\pi-\varphi-2\mu-3\omega-2\eta-\gamma-2\beta-3\alpha-3v-2u-t \\
Z' &= 2\varphi+4\mu+6\omega+6\eta+3\epsilon-3\phi-2\theta-\rho-\tau-2\sigma-3\lambda-4\delta-3\gamma-2\beta-\alpha
\end{aligned}
$$

Each of the primed lower-case letters (there are twenty-three of them) represents the self- or mutual impedance of one of the twelve *windings*. Of the 144 possible impedances there are twenty-three different ones.

The individual coil impedances (unprimed letters) may represent *any one* of the harmonic, total, slot-leakage, etc., reactances, calculated by the methods shown. The numerical substitutions into the winding formulas given above can be done on a calculating machine.

The twelve windings themselves may be connected into groups by $\mathbf{C}''$ in various ways during starting. The value of the new $\mathbf{z}'''$ is found from the above $\mathbf{z}''$ by $\mathbf{C}_t'' \cdot \mathbf{z}'' \cdot \mathbf{C}'' = \mathbf{z}'''$, giving the various types of self- and mutual reactances of the interconnected windings.

The calculations from $\mathbf{z}'$ to $\mathbf{z}''$ may be performed by two or even three steps if so desired.

CHAPTER XIII

SPINOR TRANSFORMATIONS

I. THE CONJUGATE OF GEOMETRIC OBJECTS

(*a*) All transformation tensors **C** so far considered contain only *real* numbers (constants) such as 1, -1, n. In many engineering problems *the transformation tensor* **C** *contains also complex numbers (constants) such as* $a + jb$. *In such cases all equations and all transformation formulas assume a more general form.*

It is emphasized that these more general formulas to be given now are only special cases of still more general formulas that are used when the components of the transformation tensor **C** are not real or complex *constants* as 5 or $a + jb$, but are *functions of the variable* x^α.

(*b*) When the n-matrices of a geometric object of any valence contain complex components, an additional geometric object can be built from it. *The "conjugate" of a geometric object is formed by replacing each component* $(a + jb)$ *of each n-matrix by its conjugate imaginary* $(a - jb)$. In direct notation the conjugate of a geometric object **A** is denoted by an asterisk as **A***. (The index notation will be given presently.) For instance, if along some particular reference frame

$$\mathbf{e} = \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \hline a + jb & c & -jd & e - jf \end{array} \tag{13.1}$$

its conjugate is

$$\mathbf{e}^* = \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \hline a - jb & c & jd & e + jf \end{array} \tag{13.2}$$

Or if

$$\mathbf{z} = \begin{array}{c|cccc} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \hline \mathbf{a} & a + jb & -jc & 0 & 0 \\ \mathbf{b} & -jc & d - je & f & 0 \\ \mathbf{c} & -f & 0 & 0 & g + jh \\ \mathbf{d} & 0 & jp & k + jl & m - jn \end{array} \tag{13.3}$$

its conjugate is

$$\mathbf{z}^* = \begin{array}{cc} & \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{array} \\ \begin{array}{c} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{array} & \left[\begin{array}{c|c|c|c} a - jb & jc & 0 & 0 \\ \hline jc & d + je & f & 0 \\ \hline -f & 0 & 0 & g - jh \\ \hline 0 & -jp & k - jl & m + jn \end{array}\right] \end{array} \qquad 13.4$$

(*c*) Taking again the conjugate of a conjugate geometric object, the original is reestablished. That is

$$(\mathbf{A}^*)^* = \mathbf{A} \qquad 13.5$$

The conjugate of a product is taken by taking the conjugate of each geometric object. That is

$$(\mathbf{A} \cdot \mathbf{B})^* = \mathbf{A}^* \cdot \mathbf{B}^* \qquad 13.6$$

The conjugate of an inverse is the same as the inverse of the conjugate. That is

$$(\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1} \qquad 13.7$$

In other words, it is possible to take first the inverse of a tensor of valence two, then its conjugate, or first its conjugate and then its inverse. The final tensor is the same.

II. A MORE GENERAL DEFINITION OF POWER

(*a*) A complex number $\hat{\imath} = a + jb$ is represented in a plane (the time plane) by a line with components a and b (Fig. 13.1). Its product by itself $(\hat{\imath})(\hat{\imath})$ has to be so defined that it should be equal to the square of its absolute value, that is to $a^2 + b^2$. This result can be found only if the product of $\hat{\imath}$ by itself is *arbitrarily* defined as

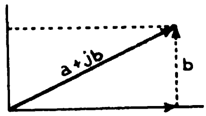


FIG. 13.1.—Complex Number as a Time Vector

$$(\hat{\imath})(\hat{\imath}^*) = (a + jb)(a - jb) = a^2 + b^2 \qquad 13.8$$

that is by taking the conjugate of one of the complex numbers.

(*b*) In general *the product of two vectors (giving a scalar) is defined by taking the conjugate of one of the vectors.* That is, if

$$\mathbf{a} = \begin{array}{c} \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{array} \\ \left[\begin{array}{c|c|c|c} (a + jb) & c + jd & e + jf & g + jh \end{array}\right] \end{array}$$

$$\mathbf{b} = \left[\begin{array}{c|c|c|c} p + jq & r + js & t + ju & v + jz \end{array}\right]$$

then their product is defined as

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b}^* &= (a+jb)(p-jq)+(c+jd)(r-js)\\ &\quad+(e+jf)(t-ju)+(g+jh)(v-jz)\\ &=(ap+bq+cr+ds+et+fu+gv+hz)+\\ &\quad+j(bp-aq+dr-cs+ft-eu+hv-gz)\end{aligned}$$

(*c*) According to the above definition *the power input of an a-c system is defined as*

$$\boxed{P = \mathbf{e}^*\cdot\mathbf{i} = (e_\alpha)^* i^\alpha} \qquad 13.9$$

It is an "apparent" power having a real and an imaginary component. By convention the conjugate of the *voltage* vector will be taken instead of that of the current vector in order that the reactive power due to a lagging current should be negative. This conforms to the practice of assuming the impressed voltage vector as the reference time axis. For instance, if the currents in two circuits **m** and **n** are lagging, that is if

$$\begin{array}{cc} & \mathbf{m} \qquad\qquad \mathbf{n} \\ \mathbf{e} = & \boxed{a+j0 \;\big|\; 0+jb} \end{array} \qquad\qquad \begin{array}{cc} & \mathbf{m} \qquad\qquad \mathbf{n} \\ \mathbf{i} = & \boxed{0-jc \;\big|\; d+j0} \end{array}$$

then the total power input is

$$P = \mathbf{e}^*\cdot\mathbf{i} = (a)(-jc) + (-jb)(d) = -j(ac+bd)$$

representing the reactive power due to the lagging currents as negative. If the conjugate of **e** is not taken, the answer $P = -j(ac - bd)$ is incorrect.

As a special case, when the components are real numbers, the product of two vectors reduces to the usual form.

III. THE TRANSFORMATION FORMULAS

(*a*) Let the current vector **i** of a system be transformed to **i′** by the transformation formula $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ where some of the components of **C** are complex numbers.

In order to investigate the transformation formula of the various tensors **e**, **z**, etc., when **C** contains complex components, let a linear form, the power input, again be assumed invariant

$$\boxed{P = \mathbf{e}^*\cdot\mathbf{i} = \mathbf{e}^{*\prime}\cdot\mathbf{i}' = P'} \qquad 13.10$$

This formula, however, is more general than the analogous formula of equation 4.22.

(*b*) To find the transformation formula of $\mathbf{e}$, again let $\mathbf{i} \equiv \mathbf{C} \cdot \mathbf{i}'$ be substituted into equation 13.10 as

$$\mathbf{e}^* \cdot \mathbf{C} \cdot \mathbf{i}' \equiv \mathbf{e}^{*\prime} \cdot \mathbf{i}'$$

Being an identity, $\mathbf{i}'$ can be dropped, leaving

$$\mathbf{e}^* \cdot \mathbf{C} \equiv \mathbf{e}^{*\prime}$$

Taking the conjugate of both sides

$$\mathbf{e} \cdot \mathbf{C}^* = \mathbf{e}'$$

$$\boxed{\mathbf{e}' = \mathbf{C}_t^* \cdot \mathbf{e}} \qquad 13.11$$

and

$$\boxed{\mathbf{e} = \mathbf{C}_t^{*-1} \cdot \mathbf{e}'} \qquad 13.12$$

Hence *if the transformation tensor* $\mathbf{C}$ *contains complex components, the conjugate of* $\mathbf{C}$ *also occurs in the transformation formulas.*

(*c*) Now let the transformation formula of $\mathbf{z}$ be established by following the reasoning of Section VIII, Chapter IV. The equation of voltage is

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$$

Replacing $\mathbf{i}$ and $\mathbf{e}$ in terms of their primed values

$$\mathbf{C}_t^{*-1} \cdot \mathbf{e}' = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$$

$$\mathbf{e}' = \mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$$

Since by the Second Generalization Postulate the form of the equation of voltage is unchanged in any reference frame

$$\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$$

Hence the transformation formula of $\mathbf{z}$ is

$$\boxed{\mathbf{z}' = \mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}} \qquad 13.13$$

IV. THE "SEQUENCE" TENSOR

(*a*) A simple example of a transformation matrix $\mathbf{C}$ having complex components is the one formed from the three "sequence operators" used in the method of symmetrical components.

Let three unequal coils with unequal mutual impedances between them be given. Their impedance tensor, impressed voltage, and current vectors are

$$\mathbf{z} = \begin{array}{c|ccc} & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{a} & Z_{aa} & Z_{ab} & Z_{ac} \\ \mathrm{b} & Z_{ab} & Z_{bb} & Z_{bc} \\ \mathrm{c} & Z_{ac} & Z_{bc} & Z_{cc} \end{array} \qquad \mathbf{e} = \begin{array}{ccc} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline e_a & e_b & e_c \end{array} \qquad \mathbf{i} = \begin{array}{ccc} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline i_a & i_b & i_c \end{array}$$

(*b*) Let now the actual currents i^a, i^b, and i^c be replaced by another set of *hypothetical* currents i^0, i^1, and i^2 (called zero-, positive- and negative phase-sequence currents respectively), by the following substitutions

$$i^a = \frac{1}{\sqrt{3}}(i^0 + i^1 + i^2)$$

$$i^b = \frac{1}{\sqrt{3}}(i^0 + a^2 i^1 + a i^2) \qquad \mathbf{C} = \frac{1}{\sqrt{3}} \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline \mathrm{a} & 1 & 1 & 1 \\ \mathrm{b} & 1 & a^2 & a \\ \mathrm{c} & 1 & a & a^2 \end{array} \qquad 13.14$$

$$i^c = \frac{1}{\sqrt{3}}(i^0 + a i^1 + a^2 i^2)$$

where the operators
$$a = -\tfrac{1}{2} + j\cdot 866 = \epsilon^{j120^\circ}$$
$$a^2 = -\tfrac{1}{2} - j\cdot 866 = \epsilon^{-j120^\circ}$$

rotate a time vector $(a + jb)$ counterclockwise (or clockwise) 120° respectively. They satisfy the relation

$$1 + a + a^2 = 0$$

The conjugate of a is a^2, and that of a^2 is a. Also $a^3 = 1$ and $a^4 = a$. In the following this particular transformation tensor **C** will be called briefly the "*sequence tensor.*"

The factor $1/\sqrt{3}$ *is introduced here in order that the power* $\mathbf{e}^*\cdot\mathbf{i}$ *should be invariant under the sequence transformation.* (See Section XIX.) *Thereby the sequence transformation tensor can be used in conjunction with other invariant transformation tensors on equal footing.*

The components of the sequence transformation tensor **C** are formed by the coefficients of the new currents. Its inverse conjugate and conjugate inverse (and their transpose) are

$$\mathbf{C}^{-1} = \frac{1}{\sqrt{3}} \begin{array}{cc} & \begin{array}{ccc} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \end{array} \qquad \mathbf{C}_t^{-1} = \frac{1}{\sqrt{3}} \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \end{array}$$

$$\mathbf{C}^{*} = \frac{1}{\sqrt{3}} \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \end{array} \qquad \mathbf{C}_t^{*} = \frac{1}{\sqrt{3}} \begin{array}{cc} & \begin{array}{ccc} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \end{array}$$

$$\mathbf{C}^{*-1} = \frac{1}{\sqrt{3}} \begin{array}{cc} & \begin{array}{ccc} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \end{array} \qquad \mathbf{C}_t^{*-1} = \frac{1}{\sqrt{3}} \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \end{array} \qquad 13.15$$

The seven different tensors shown possess only two different types of matrices, namely

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a^2 & a \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}$$

with the indices arranged differently. The determinant of the first matrix is $\sqrt{3}(a - a^2)$, that of the second matrix $\sqrt{3}(a^2 - a)$.

(*c*) The three rows of the transformation tensor **C** are the three so-called "sequence operators" namely

1. Zero phase-sequence operator $S^0 = (1, 1, 1)$.
2. Positive phase-sequence operator $S^1 = (1, a^2, a)$.
3. Negative phase-sequence operator $S^2 = (1, a, a^2)$.

That is, *the usual method of symmetrical components uses three different transformation matrices* (each matrix being a one-rowed matrix) whereas the present method uses only *one* transformation matrix having three rows, thereby speeding up the calculations and simplifying the method of reasoning.

(*d*) The applied voltage vector is by $\mathbf{e}' = \mathbf{C}_t^{*} \cdot \mathbf{e}$

$$\mathbf{e}' = \frac{1}{\sqrt{3}} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} e_a + e_b + e_c & e_a + ae_b + a^2e_c & e_a + a^2e_b + ae_c \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} e_0 & e_1 & e_2 \end{bmatrix} \end{array} \qquad 13.16$$

The new components of the current vector are by $\mathbf{i}' = \mathbf{C}^{-1}\cdot\mathbf{i}$

$$\mathbf{i}' = \frac{1}{\sqrt{3}} \begin{array}{c} \begin{array}{ccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad \end{array} \\ \begin{bmatrix} i^a + i^b + i^c & i^a + ai^b + a^2i^c & i^a + a^2i^b + ai^c \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} i^0 & i^1 & i^2 \end{bmatrix} \end{array} \qquad 13.17$$

The new voltages and currents are found from the old ones by the same formulae, since $\mathbf{C}_t^* = \mathbf{C}^{-1}$.

V. THE SEQUENCE IMPEDANCE TENSOR

(*a*) Considering the most general form of the impedance tensor of three *unequal* coils

$$\mathbf{z} = \begin{array}{c|ccc} & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{a} & Z_{aa} & Z_{ab} & Z_{ac} \\ \mathrm{b} & Z_{ba} & Z_{bb} & Z_{bc} \\ \mathrm{c} & Z_{ca} & Z_{cb} & Z_{cc} \end{array} \qquad 13.18$$

its components along the sequence axes are found by $\mathbf{C}_t^*\cdot\mathbf{z}\cdot\mathbf{C}$. The first step is

$$\mathbf{z}\cdot\mathbf{C} = \frac{1}{\sqrt{3}} \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline \mathrm{a} & Z_{aa} + Z_{ab} + Z_{ac} & Z_{aa} - a^2Z_{ab} + aZ_{ac} & Z_{aa} + aZ_{ab} + a^2Z_{ac} \\ \mathrm{b} & Z_{ba} + Z_{bb} + Z_{bc} & Z_{ba} + a^2Z_{bb} + aZ_{bc} & Z_{ba} + aZ_{bb} + a^2Z_{bc} \\ \mathrm{c} & Z_{ca} + Z_{cb} + Z_{cc} & Z_{ca} + a^2Z_{cb} + aZ_{cc} & Z_{ca} + aZ_{cb} + a^2Z_{cc} \end{array}$$

This intermediary tensor is usually written as

$$\mathbf{z}\cdot\mathbf{C} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline \mathrm{a} & Z_{a0} & Z_{a1} & Z_{a2} \\ \mathrm{b} & Z_{b0} & a^2Z_{b1} & aZ_{b2} \\ \mathrm{c} & Z_{c0} & aZ_{c1} & a^2Z_{c2} \end{array} \qquad 13.19$$

where

$$\begin{aligned} Z_{a0} &= (1/\sqrt{3})(Z_{aa} + Z_{ab} + Z_{ac}) \\ Z_{a1} &= (1/\sqrt{3})(Z_{aa} + a^2Z_{ab} + aZ_{ac}) \\ Z_{a2} &= (1/\sqrt{3})(Z_{aa} + aZ_{ab} + a^2Z_{ac}) \end{aligned} \qquad 13.20$$

This tensor represents the impedances of each phase due to applied phase-sequence currents by the equation $\mathbf{e} = \mathbf{z}\cdot\mathbf{C}\cdot\mathbf{i}'$, as shown in Section X,. Chapter IV. It is used when the currents are expressed along the *sequence* axes and the voltages along the actual *circuit* axes.

The second step is $\mathbf{C}_t^* \cdot (\mathbf{z} \cdot \mathbf{C}) =$

$$\mathbf{z}' = \frac{1}{3}\;\begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 0 & Z_{a0} + Z_{b0} + Z_{c0} & Z_{a1} + a^2 Z_{b1} + aZ_{c1} & Z_{a2} + aZ_{b2} + a^2 Z_{c2} \\ \hline 1 & Z_{a0} + aZ_{b0} + a^2 Z_{c0} & Z_{a1} + Z_{b1} + Z_{c1} & Z_{a2} + a^2 Z_{b2} + aZ_{c2} \\ \hline 2 & Z_{a0} + a^2 Z_{b0} + aZ_{c0} & Z_{a1} + aZ_{b1} + a^2 Z_{c1} & Z_{a2} + Z_{b2} + Z_{c2} \\ \hline \end{array} \qquad 13.21$$

$$\mathbf{z}' = \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 0 & Z_{00} & Z_{01} & Z_{02} \\ \hline 1 & Z_{10} & Z_{11} & Z_{12} \\ \hline 2 & Z_{20} & Z_{21} & Z_{22} \\ \hline \end{array} \qquad 13.22$$

(*b*) When the mutual inductances of the three unequal coils are zero, the impedance tensors are

$$\mathbf{z} = \begin{array}{c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \hline \mathbf{a} & Z_a & 0 & 0 \\ \hline \mathbf{b} & 0 & Z_b & 0 \\ \hline \mathbf{c} & 0 & 0 & Z_c \\ \hline \end{array} \qquad \mathbf{z}' = \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 0 & Z_0 & Z_2 & Z_1 \\ \hline 1 & Z_1 & Z_0 & Z_2 \\ \hline 2 & Z_1 & Z_1 & Z_0 \\ \hline \end{array} \qquad 13.23$$

where

$$\left.\begin{aligned} Z_0 &= (\tfrac{1}{3})(Z_a + Z_b + Z_c) \\ Z_1 &= (\tfrac{1}{3})(Z_a + aZ_b + a^2 Z_c) \\ Z_2 &= (\tfrac{1}{3})(Z_a + a^2 Z_b + aZ_c) \end{aligned}\right\} \qquad 13.24$$

They are called the zero-, positive-, and negative-sequence reactances, respectively.

It should be noted that *the sequence impedance tensors* $\mathbf{z}'$ *are not symmetrical,* even though they refer to stationary coils. In general a *symmetrical impedance tensor is no longer symmetrical after it is transformed by a* $\mathbf{C}$ *with complex components.*

(*c*) The equation $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$ gives the three sequence voltages when three sequence currents are applied. In the general case, when the three coils are unequal,. each applied sequence current produces all three sequence voltages. For instance, the zero sequence current i^0 produces a negative sequence voltage equal to $i^0 Z_{20} = i^0(Z_{a2} + Z_{b2} + Z_{c2})/3$.

The equation is solved as $\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}'$, giving the three sequence currents due to applied three sequence voltages. In case of *unequal coils* each applied sequence voltage produces all three sequence currents and the form $\mathbf{z}'$ (equations 13.22 or 13.23) has no advantage over the original form $\mathbf{z}$ of equation 13.18.

VI. REDUCTION TO DIAGONAL FORM

(*a*) *When the three coils are equal* the sequence impedance tensor $\mathbf{z}'$ assumes special forms that reduce the amount of calculation.

Let it be assumed that the impedance tensor of the original three coils has the form

$$\mathbf{z} = \begin{array}{c|ccc|} & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{a} & Z & X_1 & X_2 \\ \mathrm{b} & X_2 & Z & X_1 \\ \mathrm{c} & X_1 & X_2 & Z \\ \hline \end{array} \qquad 13.25$$

the three coils having equal self-impedances Z and *two* (not three) different mutual inductances X_1 and X_2. Such a case occurs in three-phase synchronous and induction machines with smooth air gaps. *The sequence impedance tensor of equation* 13.21 *becomes a diagonal 2-tensor*:

$$\mathbf{z}' = \frac{1}{3}\begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z+X_1+X_2 & 0 & 0 \\ 1 & 0 & Z+a^2X_1+aX_2 & 0 \\ 2 & 0 & 0 & Z+aX_1+a^2X_2 \\ \hline \end{array} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z_0 & 0 & 0 \\ 1 & 0 & Z_1 & 0 \\ 2 & 0 & 0 & Z_2 \\ \hline \end{array} \qquad 13.26$$

where Z_0, Z_1, and Z_2 are the zero-, positive-, and negative-sequence reactances respectively.

(*b*) When the two mutual inductances X_1 and X_2 are equal, the sequence impedance tensor becomes (since $1 + a + a^2 = 0$).

$$\mathbf{z}' = \frac{1}{3}\begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z+2X & 0 & 0 \\ 1 & 0 & Z-X & 0 \\ 2 & 0 & 0 & Z-X \\ \hline \end{array} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z_0 & 0 & 0 \\ 1 & 0 & Z_1 & 0 \\ 2 & 0 & 0 & Z_1 \\ \hline \end{array} \qquad 13.27$$

That is, ***with equal mutual inductances the positive- and negative-sequence reactances are equal.***

(*c*) The admittance tensor $\mathbf{y}'$ also has a diagonal form

$$\mathbf{y}' = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & 1/Z_0 & 0 & 0 \\ 1 & 0 & 1/Z_1 & 0 \\ 2 & 0 & 0 & 1/Z_2 \\ \hline \end{array} \qquad 13.28$$

showing that each sequence voltage produces only its own sequence current, and vice versa.

(*d*) *One of the most important problems in the theory of matrices is to find a transformation matrix that reduces a given matrix to a diagonal form. It is interesting that the method of symmetrical components supplies a transformation matrix* **C** (*equation* 13.14) *which reduces certain symmetrical matrices to a diagonal form.* (Of course in the theory of matrices a linear form is not necessarily invariant under the transformation.) Undoubtedly there are other groups of **C**'s that reduce still more complicated **z**'s to diagonal form.

(*e*) If the impedance tensor has n rows and columns, the transformation matrix **C** of the method of symmetrical components is

$$\mathbf{C} = \frac{1}{\sqrt{n}} \quad \begin{array}{c|c|c|c|c|c|} & 0 & 1 & 2 & & n-1 \\ \hline 0 & 1 & 1 & 1 & & 1 \\ \hline 1 & 1 & a^{-1} & a^{-2} & & a^{-1(n-1)} \\ \hline 2 & 1 & a^{-2} & a^{-4} & & a^{-2(n-1)} \\ \hline & & & & & \\ \hline n-1 & & a^{-(n-1)} & a^{-2(n-1)} & & a^{-(n-1)(n-1)} \\ \hline \end{array} \qquad 13.29$$

where a is the nth root of unity. When $n = 3$, this reduces to the **C** of equation 13.14.

(The determinant of **C** is called the "Vandermonde determinant," whose general form is

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & b & c & \cdots & f \\ a^2 & b^2 & c^2 & \cdots & f^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a^{n-1} & b^{n-1} & c^{n-1} & \cdots & f^{n-1} \end{vmatrix} \qquad 13.30$$

Here $a, b, c \cdots f$ are replaced by the n different nth roots of unity.)

VII. ON THE USE OF THE SEQUENCE TENSOR

(*a*) Three-phase networks consist of various types of three-phase apparatus interconnected in any manner, in shunt, or in series, just as single coils are. The component three-phase apparatus may be generators, transformers, transmission lines, loads.

The use of the sequence tensor of the previous section reduces the amount of analysis if the following two conditions are satisfied:

1. If each individual three-phase apparatus is balanced, that is if its impedance tensor has the form of equations 13.25 or 13.27.

2. If each is *interconnected* with the other apparatus in a balanced manner.

(*b*) If one or more of the apparatus or their interconnections are unbalanced then the whole network is divided into two parts:

1. The part containing the several balanced apparatus.
2. The part containing the several unbalanced apparatus.

The analysis of the first part is simplified by the use of the sequence tensor; that of the second is not, in fact, it often becomes more complex.

In most three-phase networks the unbalanced part consists of a fault or an unbalanced load while the balanced part contains several apparatus and the use of the sequence tensor is decidedly worth while.

The analysis of interconnected three-phase apparatus is undertaken in a systematic manner in Chapter XIX. A few simpler examples will be worked out presently.

(*c*) In all problems so far considered the axes of the "*primitive*" network always were the actual coil axes. However, in three-phase apparatus the design constants may be known along the sequence axes instead. Hence the impedance tensor **z** of the primitive network of each three-phase apparatus may be expressed: (1) either along the actual phase axes **a**, **b**, **c**; (2) or along the sequence axes **0**, **1**, **2**.

Similarly the impressed voltages **e** or the currents **i** are expressed along these two types of axes.

However, *the transformation tensor* **C** *showing the manner of interconnection of the coils should be expressed first along the actual circuit axes* **a**, **b**, **c** *and then only should it be changed to the sequence axes* **0**, **1**, **2**.

In finding the various geometric objects of the *resultant* network they also may be expressed either along the phase axes **a**, **b**, **c** or along the sequence axes **0**, **1**, **2**, or along both.

VIII. GENERATOR EXPRESSED ALONG THE SEQUENCE AXES

(*a*) Let a three-phase generator be given whose impedance tensor **z′** and generated voltage **e′** are expressed along the *sequence* axes as

$$\mathbf{z}' = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & Z_0 & & \\ 1 & & Z_1 & \\ 2 & & & Z_2 \end{array} \qquad 13.31 \qquad\qquad \mathbf{e}' = \begin{array}{ccc} 0 & 1 & 2 \\ \hline 0 & e_1 & 0 \end{array} \qquad 13.32$$

Let the terminals of the generator be connected to any type of load by **C**. If the load and **C** are expressed along actual circuit axes *it is advantageous to change the sequence axes of* **z** *and* **e** *of the generator to the circuit axes* **a**, **b**, **c**.

(*b*) From equation 13.13, $\mathbf{z}' = \mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$, the reverse step follows as

$$\boxed{\mathbf{z} = \mathbf{C}_t^{*-1} \cdot \mathbf{z}' \cdot \mathbf{C}^{-1}} \qquad 13.33$$

and from $\mathbf{e}' = \mathbf{C}_t^* \cdot \mathbf{e}$ it follows that

$$\boxed{\mathbf{e} = \mathbf{C}_t^{*-1} \cdot \mathbf{e}'} \qquad 13.34$$

where the primed quantities are expressed along the sequence axes. Hence the components of the geometric objects of the generator along the actual circuit axes are by $\mathbf{C}_t^{*-1} \cdot \mathbf{z}' \cdot \mathbf{C}^{-1}$

$$\mathbf{z} = \frac{1}{3}\; \begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{array} \\ \left|\begin{array}{c|c|c} Z_0 + Z_1 + Z_2 & Z_0 + aZ_1 + a^2Z_2 & Z_0 + a^2Z_1 + aZ_2 \\ \hline Z_0 + a^2Z_1 + aZ_2 & Z_0 + Z_1 + Z_2 & Z_0 + aZ_1 + a^2Z_2 \\ \hline Z_0 + aZ_1 + a^2Z_2 & Z_0 + a^2Z_1 + aZ_2 & Z_0 + Z_1 + Z_2 \end{array}\right| \end{array} \qquad 13.35$$

$$\mathbf{e} = \begin{array}{c} \begin{array}{ccc} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{array} \\ \left|\begin{array}{c|c|c} e_1 & a^2e_1 & ae_1 \end{array}\right| \end{array} \qquad 13.36$$

Using these components of the geometric objects as a starting point, the method of analysis of the generator is the same as that of any other asymmetrical stationary network of three coils.

IX. GENERATOR CONNECTED TO A LOAD

(*a*) Let the generator be connected to a load as shown in Fig. 13.2. There are two meshes, hence two variables are assumed, say i^b and i^g.

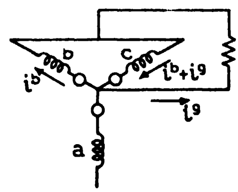


FIG. 13.2.—Unbalanced Load on Generator

The impedance tensor of the primitive network is

$$\mathbf{z} = \frac{1}{3}\; \begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{g} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{g} \end{array} \\ \left|\begin{array}{c|c|c|c} Z_0 + Z_1 + Z_2 & Z_0 + aZ_1 + a^2Z_2 & Z_0 + a^2Z_1 + aZ_2 & 0 \\ \hline Z_0 + a^2Z_1 + aZ_2 & Z_0 + Z_1 + Z_2 & Z_0 + aZ_1 + a^2Z_2 & 0 \\ \hline Z_0 + aZ_1 + a^2Z_2 & Z_0 + a^2Z_1 + aZ_1 & Z_0 + Z_1 + Z_2 & 0 \\ \hline 0 & 0 & 0 & 3Z_g \end{array}\right| \end{array}$$

Equating the old and the new currents flowing in each coil

$$i^a = 0$$
$$i^b = i^{b'}$$
$$i^c = -i^{b'} - i^{g'}$$
$$i^g = i^{g'}$$

$$\mathbf{C} = \begin{array}{c|cc} & b' & g' \\ \hline a & & \\ b & 1 & \\ c & -1 & -1 \\ g & & 1 \end{array} \qquad 13.37$$

The coefficients of the new currents give C.

Hence by $\mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$ and by $\mathbf{C}_t^* \cdot \mathbf{e}$

$$\mathbf{z}' = \begin{array}{c|cc} & b' & g' \\ \hline b' & Z_1 + Z_2 & [Z_1(1-a) + Z_2(1-a^2)]/3 \\ g' & [Z_1(1-a^2) + Z_2(1-a)]/3 & (Z_0 + Z_1 + Z_2 + 3Z_g)/3 \end{array} \qquad 13.38$$

$$\mathbf{e}' = \begin{array}{cc} b' & g' \\ \hline (a^2 - a)e & -ae \end{array} \qquad 13.39$$

$$\mathbf{y}' = \begin{array}{c|cc} & b' & g' \\ \hline b' & (Z_0 + Z_1 + Z_2 + 3Z_g)/3D & [Z_1(a-1) + Z_2(a^2-1)]/3D \\ g' & [Z_1(a^2-1) + Z_2(a-1)]/3D & (Z_1 + Z_2)/D \end{array} \qquad 13.40$$

where $$D = [Z_1Z_2 + (Z_1 + Z_2)(Z_0 + 3Z_g)]/3$$

$$\mathbf{i}' = \begin{array}{c|c} b' & 3e[(a^2 - a)(Z_0 + 3Z_g) + (a^2 - 1)Z_2]/D \\ g' & 3e[2aZ_1 + (1 - 2a^2)Z_2]/D \end{array} \qquad 13.41$$

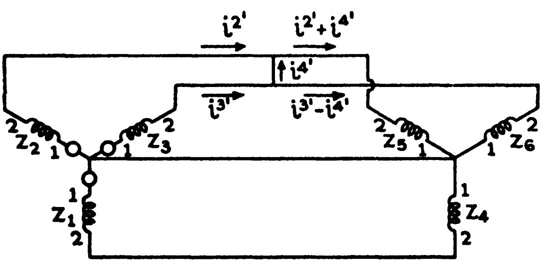


FIG. 13.3.—Short-circuit on Generator

(b) As a second example consider the short circuit of a loaded generator as shown in Fig. 13.3. The geometric objects of the primitive network are

$$\mathbf{z} = \begin{array}{c|cc} & g & 1 \\ \hline g & z_g & \\ 1 & & z_l \end{array} \qquad \mathbf{e} = \begin{array}{cc} g & 1 \\ \hline e_g & 0 \end{array} \qquad 13.42$$

where $\mathbf{z}_g$ is given in equation 13.55 and $\mathbf{z}_l$ in equation 13.22 or 13.23.

The transformer tensor is

$$
\begin{aligned}
i^1 &= i^{1'} \\
i^2 &= i^{2'} \\
i^3 &= i^{3'} \\
i^4 &= -i^{1'} \\
i^5 &= -i^{2'} - i^{4'} \\
i^6 &= -i^{3'} + i^{4'}
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c|cccc|}
 & 1' & 2' & 3' & 4' \\
\hline
1 & 1 & & & \\
2 & & 1 & & \\
3 & & & 1 & \\
4 & -1 & & & \\
5 & & -1 & & -1 \\
6 & & & -1 & -1 \\
\hline
\end{array}
= \begin{array}{c|cc|}
 & g' & 1' \\
\hline
g & \mathbf{I} & \\
1 & -\mathbf{I} & \mathbf{A} \\
\hline
\end{array}
\qquad 13.43
$$

The geometric objects of the actual network are by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ and by $\mathbf{C}_t \cdot \mathbf{e}$

$$
\mathbf{z}' = \begin{array}{c|cc|}
 & g' & 1' \\
\hline
g' & \mathbf{z}_g + \mathbf{z}_l & -\mathbf{z}_l \cdot \mathbf{A} \\
1' & -\mathbf{A}_t \cdot \mathbf{z}_l & \mathbf{A}_t \cdot \mathbf{z}_l \cdot \mathbf{A} \\
\hline
\end{array}
\qquad
\mathbf{e}' = \begin{array}{|cc|}
g' & 1' \\
\hline
\mathbf{e}_g & 0 \\
\hline
\end{array}
\qquad 13.44
$$

X. THE GROUP OF "SYMMETRICAL COMPONENTS" G_{sc}

(*a*) The transformation matrices that occur in the method of symmetrical components, namely $\mathbf{C}$ and $\mathbf{C}^{-1}$, together with two other matrices defined as $\mathbf{C} \cdot \mathbf{C}^{-1} = \mathbf{I}$ and $\mathbf{C} \cdot \mathbf{C} = \mathbf{S}$, form a "group" with four elements

$$
\mathbf{C} = \frac{1}{\sqrt{3}} \begin{array}{c|ccc|}
 & 0 & 1 & 2 \\
\hline
a & 1 & 1 & 1 \\
b & 1 & a^2 & a \\
c & 1 & a & a^2 \\
\hline
\end{array}
\qquad
\mathbf{C}^{-1} = \frac{1}{\sqrt{3}} \begin{array}{c|ccc|}
 & a & b & c \\
\hline
0 & 1 & 1 & 1 \\
1 & 1 & a & a^2 \\
2 & 1 & a^2 & a \\
\hline
\end{array}
$$

$$
\mathbf{S} = \begin{array}{c|ccc|}
 & a & b & c \\
\hline
a & 1 & 0 & 0 \\
b & 0 & 0 & 1 \\
c & 0 & 1 & 0 \\
\hline
\end{array}
\qquad
\mathbf{I} = \begin{array}{c|ccc|}
 & a & b & c \\
\hline
a & 1 & 0 & 0 \\
b & 0 & 1 & 0 \\
c & 0 & 0 & 1 \\
\hline
\end{array}
\qquad 13.45
$$

These four matrices satisfy the four conditions as elements of a group, given in Section III, Chapter XI. In particular:

1. The product of any two elements is an element of the group. This can be seen by forming a "*multiplication table*" in which each

component is the product of two border elements of the corresponding row and column

$$\begin{array}{c|c|c|c|c|}
 & \mathrm{C} & \mathrm{S} & \mathrm{C}^{-1} & \mathrm{I} \\
\hline
\mathrm{C} & \mathrm{S} & \mathrm{C}^{-1} & \mathrm{I} & \mathrm{C} \\
\hline
\mathrm{S} & \mathrm{C}^{-1} & \mathrm{I} & \mathrm{C} & \mathrm{S} \\
\hline
\mathrm{C}^{-1} & \mathrm{I} & \mathrm{C} & \mathrm{S} & \mathrm{C}^{-1} \\
\hline
\mathrm{I} & \mathrm{C} & \mathrm{S} & \mathrm{C}^{-1} & \mathrm{I} \\
\hline
\end{array} \qquad 13.46$$

2. The associative law is satisfied, that is, $\mathrm{C}\cdot(\mathrm{S}\cdot\mathrm{C}^{-1}) = (\mathrm{C}\cdot\mathrm{S})\cdot\mathrm{C}^{-1}$
3. Each element has its inverse. (The inverse of S is S.)
4. The unit element I exists.

It is emphasized that this group of four elements is a subgroup of numerous other groups having more than four elements. In particular G_{se} is a subgroup of the group of linear transformations with complex components.

(*b*) *These four matrices* C, S, C^{-1}, *and* I *have properties analogous to those of the four numbers* j, -1, $-j$, *and* 1 *that also form a group. That is, the abstract properties of* C *are analogous to that of the complex operator* j, C^{-1} is analogous to $-j$, S to -1, and I to 1. The similarity of the properties of the two groups is shown by the similarity of their multiplication table,

$$\begin{array}{c|c|c|c|c|}
 & j & -1 & -j & 1 \\
\hline
j & -1 & -j & 1 & j \\
\hline
-1 & -j & 1 & j & -1 \\
\hline
-j & 1 & j & -1 & -j \\
\hline
1 & j & -1 & -j & 1 \\
\hline
\end{array} \qquad 13.47$$

In the two tables analogous elements occur in the same order.

Since the matrix S represents a *reversal of the phase rotation* from **a**-**b**-**c** to **a**-**c**-**b** (in analogy to -1 that represents a reversal of direction), consequently *the sequence matrix* C *may be assumed to represent a change of phase rotation through an imaginary angle* (in analogy to j, that represents a rotation of direction through an imaginary angle).

It will be shown in Section XIV, Chapter XIX, that the transformation matrix S is used in the zigzag connection of multiwinding transformers, when the three phases are interconnected. Hence the sequence matrix represents also an *imaginary interconnection* of the three phases.

XI. PHASE-SHIFT TRANSFORMERS

(*a*) As another example of transformations where the transformation tensor contains complex components, let *balanced, three-phase, multiwinding transformers* be considered where an electrical phase involves several magnetic phases.

Let three identical multiwinding transformers be given, and on each core let, say, four different windings exist. In terms of leakage reactances X_{a-b} the impedance tensor of the first transformer is

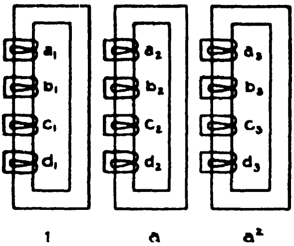


FIG. 13.4.—Three-phase Four-winding transformers

$$\mathbf{z}_1 = \begin{array}{c|cccc|} & \mathbf{a}_1 & \mathbf{b}_1 & \mathbf{c}_1 & \mathbf{d}_1 \\ \hline \mathbf{a}_1 & 0 & X_{a-b} & X_{a-c} & X_{a-d} \\ \mathbf{b}_1 & X_{a-b} & 0 & X_{b-c} & X_{b-d} \\ \mathbf{c}_1 & X_{a-c} & X_{b-c} & 0 & X_{c-d} \\ \mathbf{d}_1 & X_{a-d} & X_{b-d} & X_{c-d} & 0 \\ \hline \end{array} \qquad 13.48$$

The impedance tensor of each transformer has the same matrix, except that the axes **a**, **b**, **c**, **d** have different subscripts.

When the coils of the three transformers are interconnected in any manner forming *balanced* three-phase networks and are connected to balanced impedances, in each electrical phase the same current flows through as in the other two phases, but shifted in time by 120° or 240°. Similarly in the windings of the second transformer the same currents flow as in the corresponding windings of the first, but shifted by 120° in time, that is, by $a = \varepsilon^{j120°}$. In the windings of the third transformer the currents are shifted from those of the first by $a^2 = \varepsilon^{j240°}$. (Of course, in each winding $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$, and $\mathbf{d}_1$ of the first transformer, the currents are out of phase in time.) Hence: (1) If in the first transformer i^a, i^b, i^c, i^d flow, (2) then in the second transformer ai^a, ai^b, ai^c, ai^d flow, (3) and in the third transformer a^2i^a, a^2i^b, a^2i^c, a^2i^d flow.

It is customary to represent each winding by a straight line, in particular the windings of the first transformer, say by vertical lines (or by horizontal), those of the second transformer by lines at 120° from the vertical, and those of the third by 240°, as shown in Fig. 13.5. When any two of the windings are connected in series, the connections are represented as in Fig. 13.6.

Since in a balanced system there is no neutral current, the currents flowing in a star and in a delta connection are to be assumed as in

Fig. 13.7. A star may be assumed to have a neutral wire in establishing the number of meshes.

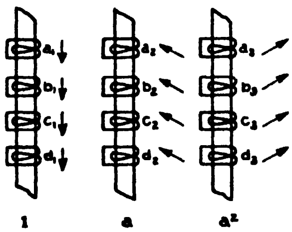


FIG. 13.5.—Representation of Phases

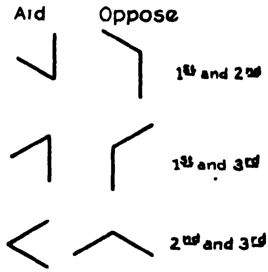


FIG. 13.6.—Series Connections

It should be noted that *each time the current is rotated* 120° *clockwise it is multiplied by a.*

(*b*) Now let the twelve windings be interconnected into a three-phase winding *in any manner.* For instance, let them be connected in the following manner:

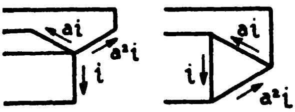
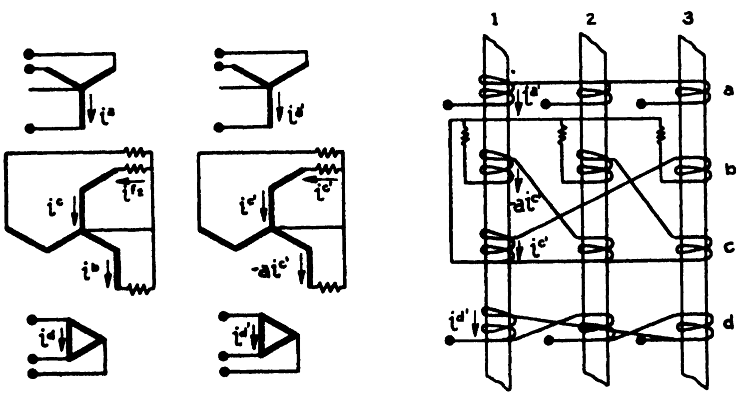


FIG. 13.7

1. The *a* windings into star.
2. The *d* windings into delta.
3. The *c* windings into star.
4. The *b* windings in opposing series with the *c* windings; in particular the *b* winding of the *third* transformer in opposing series with



(*a*) Old currents (*b*) New currents (*c*) Actual connections

FIG. 13.8.—Phase-shift Zig-zag Transformer

the *c* winding of the *first* transformer as shown in Fig. 13.8. (Such a connection is called a "zigzag" connection.)

5. The *b* windings are connected in series with a balanced star load.

XII. ESTABLISHMENT OF THE NEW CURRENTS

(*a*) *The number of new currents to be considered is one-third of the number of meshes,* since the remaining currents are the same but multiplied by a and a^2. Hence *it will be sufficient to set up a relation between the old and the new currents flowing only in one-third of the windings, say in the windings of the first transformer and in one of the load impedances. Similarly it will be sufficient to neglect the magnetizing current in the first transformer only.*

(*b*) In Figs. 13.8*a* and 13.8*b* are shown the old and the new currents flowing in one-third of the windings of the wye-delta zigzag transformer. Since the number of meshes is nine, the number of new currents assumed is $9/3 = 3$, namely, $i^{a'}$, $i^{c'}$, and $i^{d'}$. *They are assumed to flow, say, in three of the vertical windings.* In the fourth vertical winding Z_b flows $ai^{c'}$, at 120° from $i^{c'}$. It flows through Z_b in the opposite direction from the original i^b through one of the loads flows from $i^{c'}$. It makes no difference which load impedance is considered.

In order to determine the new currents in all the other vertical lines, it is often necessary to determine the new currents in some of the other windings *as an intermediary step,* as shown in Tables 13.1 and 13.2 with dotted arrows.

In general, the new currents should be established in all lines having the same direction (either in all vertical, or in all horizontal, etc., lines) *and in one-third of the loads.*

XIII. THE TRANSFORMATION TENSOR

(*a*) In setting up the transformation tensor the same steps are repeated as in an unbalanced transformer analysis, namely:

1. The manner of interconnection is represented by $\mathbf{C}_1$.
2. The ignoration of the magnetizing current is represented by $\mathbf{C}_2$.
3. The resultant transformation tensor $\mathbf{C}$ is $\mathbf{C}_1 \cdot \mathbf{C}_2$.

The difference between the balanced case considered here and the unbalanced case considered in Chapter XI is that only one-third of the windings and one-third of the equations are used in the balanced case.

(*b*) Considering the old and the new currents flowing, say, in the four *vertical* lines (in the windings of the first transformer) of Figs. 13.8*a* and *b* and in one (any one) of the loads

$$\begin{aligned} i^a &= i^{a'} \\ i^b &= \quad -a i^{c'} \\ i^c &= \quad i^{c'} \\ i^d &= \quad\quad i^{d'} \\ i^f &= \quad i^{c'} \end{aligned} \qquad \mathbf{C}_1 = \begin{array}{c|c|c|c|} & \mathbf{a}' & \mathbf{c}' & \mathbf{d}' \\ \hline \mathbf{a}_1 & 1 & 0 & 0 \\ \hline \mathbf{b}_1 & 0 & -a & 0 \\ \hline \mathbf{c}_1 & 0 & 1 & 0 \\ \hline \mathbf{d}_1 & 0 & 0 & 1 \\ \hline \mathbf{f}_2 & 0 & 1 & 0 \\ \hline \end{array} \tag{13.49}$$

The winding interconnection reduces the number of variables per phase from five to three.

Instead of the four windings of the first transformer, of course, any other four (but differently lettered) windings could have been considered, for instance c_2 instead of c_1, etc.

(*c*) The equation of constraint, neglecting the magnetizing current in the first transformer, is, before interconnection

$$n_a i^a + n_b i^b + n_c i^c + n_d i^d = 0 \tag{13.50}$$

Replacing the old currents by the new currents, it becomes

$$n_a i^{a'} - n_b a i^{c'} + n_c i^{c'} + n_d i^{d'} = 0$$

Assuming, say, $i^{d'}$ in the delta as the magnetizing current to be neglected, the equation of constraint may be replaced by the set of equations defining a transformation tensor

$$\begin{aligned} i^{a'} &= i^{a''} \\ i^{c'} &= \qquad\qquad\qquad\qquad i^{c''} \\ i^{d'} &= -\frac{n_a}{n_d} i^{a''} + \left(\frac{n_b a}{n_d} - \frac{n_c}{n_d}\right) i^{c''} \end{aligned} \qquad \mathbf{C}_2 = \begin{array}{c|c|c|} & \mathbf{a}'' & \mathbf{c}'' \\ \hline \mathbf{a}' & 1 & 0 \\ \hline \mathbf{c}' & 0 & 1 \\ \hline \mathbf{d}' & N_1 & N_2 \\ \hline \end{array} \tag{13.51}$$

(*d*) Hence the resultant transformation tensor is $\mathbf{C}_1 \cdot \mathbf{C}_2 =$

$$\mathbf{C} = \begin{array}{c|c|c|} & \mathbf{a}'' & \mathbf{c}'' \\ \hline \mathbf{a}_1 & 1 & 0 \\ \hline \mathbf{b}_1 & 0 & -a \\ \hline \mathbf{c}_1 & 0 & 1 \\ \hline \mathbf{d}_1 & N_1 & N_2 \\ \hline \mathbf{f}_2 & 0 & 1 \\ \hline \end{array} \qquad \mathbf{C}_t^* = \begin{array}{c|c|c|c|c|c|} & \mathbf{a}_1 & \mathbf{b}_1 & \mathbf{c}_1 & \mathbf{d}_1 & \mathbf{f}_2 \\ \hline \mathbf{a}'' & 1 & 0 & 0 & N_1^* & 0 \\ \hline \mathbf{c}'' & 0 & -a^2 & 1 & N_2^* & 1 \\ \hline \end{array} \tag{13.52}$$

XIV. THE IMPEDANCE TENSOR

Once the transformation tensor $\mathbf{C}$ has been established, the remaining work is the same as in any other mesh network.

(*a*) The impedance tensor of the first four-winding transformer and one of the loads *before interconnection* is

$$\mathbf{z} = \begin{array}{c|c|c|c|c|c|} & \mathbf{a}_1 & \mathbf{b}_1 & \mathbf{c}_1 & \mathbf{d}_1 & \mathbf{f}_2 \\ \hline \mathbf{a}_1 & & Z_{a-b} & Z_{a-c} & Z_{a-d} & \\ \hline \mathbf{b}_1 & Z_{a-b} & & Z_{b-c} & Z_{b-d} & \\ \hline \mathbf{c}_1 & Z_{a-c} & Z_{b-c} & & Z_{c-d} & \\ \hline \mathbf{d}_1 & Z_{a-d} & Z_{b-d} & Z_{c-d} & & \\ \hline \mathbf{f}_2 & & & & & Z \\ \hline \end{array} \qquad 13.53$$

The impressed voltage vector is

$$\mathbf{e} = \begin{array}{|c|c|c|c|c|} \mathbf{a}_1 & \mathbf{b}_1 & \mathbf{c}_1 & \mathbf{d}_1 & \mathbf{f}_2 \\ \hline e_a & 0 & 0 & e_d & 0 \\ \hline \end{array} \qquad 13.54$$

where e_a is the star-to-neutral voltage.

(*b*) After interconnection and ignoration of the magnetizing current the new components of the impedance tensor are found by $\mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$ as

$$\mathbf{z}' = \begin{array}{c|c|c|} & \mathbf{a}'' & \mathbf{c}'' \\ \hline \mathbf{a}'' & Z_{a-d}(N_1 + N_1^*) & \begin{array}{l} -aZ_{a-b} + Z_{a-c} + N_2 Z_{a-d} \\ -aN_1^* Z_{b-d} + N_1^* Z_{c-d} \end{array} \\ \hline \mathbf{c}'' & \begin{array}{l} -a^2 Z_{a-b} + Z_{a-c} + N_2^* Z_{a-d} \\ -a^2 N_1 Z_{b-d} + N_1 Z_{c-d} \end{array} & \begin{array}{l} Z_{b-c} - Z_{b-d}(N_2 a^2 + N_2^* a) \\ +Z_{c-d}(N_2 + N_2^*) + Z \end{array} \\ \hline \end{array} \qquad 13.55$$

(*c*) The new currents (that is, the currents in windings *a* and *c*) are found by $\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}'$, where

$$\mathbf{e}' = \mathbf{C}_t^* \cdot \mathbf{e} = \begin{array}{|c|c|} \mathbf{a}'' & \mathbf{b}'' \\ \hline e_a + N_1^* e_d & N_2^* e_d \\ \hline \end{array} \qquad 13.56$$

(*d*) If $\mathbf{i}'$ has been calculated, then:

1. The currents flowing in the individual windings of the first transformer are found by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$, that is

$$\mathbf{i} = \begin{array}{|c|c|c|c|c|} \mathbf{a}_1 & \mathbf{b}_1 & \mathbf{c}_1 & \mathbf{d}_1 & \mathbf{f}_2 \\ \hline i^{a''} & -ai^{c''} & i^{c''} & N_1 i^{a''} + N_2 i^{c''} & i^{c''} \\ \hline \end{array} \qquad 13.57$$

2. The voltages induced in the individual windings of the first transformer are found by $\mathbf{e} = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$ (where $\mathbf{z} \cdot \mathbf{C}$ has already been calculated in finding $\mathbf{z}'$)

$$
\mathbf{z}\cdot\mathbf{C}\cdot\mathbf{i}' = \begin{array}{c|c|}
\cline{2-2}
\mathbf{a}_1 & N_1 Z_{a-d} i^{a''} + (-aZ_{a-b} + Z_{a-c} + N_2 Z_{a-d}) i^{c''} \\
\cline{2-2}
\mathbf{b}_1 & (Z_{a-b} + N_1 Z_{b-d}) i^{a''} + (Z_{b-c} + N_2 Z_{b-d}) i^{c''} \\
\cline{2-2}
\mathbf{c}_1 & (Z_{a-c} + N_1 Z_{c-d}) i^{a''} + (-aZ_{b-c} + N_2 Z_{c-d}) i^{c''} \\
\cline{2-2}
\mathbf{d}_1 & Z_{a-d} i^{a''} + (-aZ_{b-d} + Z_{c-d}) i^{c''} \\
\cline{2-2}
\mathbf{f}_2 & Z i^{c''} \\
\cline{2-2}
\end{array} \qquad 13.58
$$

XV. HEXAGON PHASE-SHIFT AUTOTRANSFORMER

As another example consider the auto-transformer connection of Fig. 13.9.

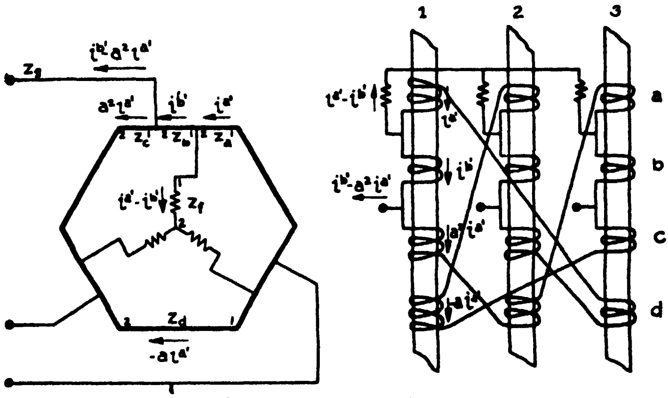


FIG. 13.9.—Hexagon Phase-shift Autotransformer

To take care of the impressed voltage across the leads it will be assumed that an impedance Z_g of zero value is in series with it. The twelve windings, the loads, and the leads form six meshes, hence it is sufficient to assume $6/3 = 2$ new currents, say $i^{a'}$ and $i^{b'}$. They are assumed to flow in two of the horizontal windings. The new currents in the remaining horizontal lines and in one load and lead (forming one-third of the windings) are also shown. The relation between the old and the new currents in the windings of the first transformer (horizontal lines) is

$$
\begin{aligned}
i^a &= i^{a'} \\
i^b &= \qquad\quad i^{b'} \\
i^c &= a^2 i^{a'} \\
i^d &= -a i^{a'} \\
i^f &= i^{a'} - i^{b'} \\
i^g &= -a^2 i^{a'} + i^{b'}
\end{aligned}
\qquad
\mathbf{C}_1 = \begin{array}{c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathbf{a}'} & \multicolumn{1}{c}{\mathbf{b}'} \\
\cline{2-3}
\mathbf{a} & 1 & 0 \\
\cline{2-3}
\mathbf{b} & 0 & 1 \\
\cline{2-3}
\mathbf{c} & a^2 & 0 \\
\cline{2-3}
\mathbf{d} & -a & 0 \\
\cline{2-3}
\mathbf{f} & 1 & -1 \\
\cline{2-3}
\mathbf{g} & -a^2 & 1 \\
\cline{2-3}
\end{array} \qquad 13.59
$$

The equation of constraint for the first transformer before interconnection is

$$n_a i^a + n_b i^b + n_c i^c + n_d i^d = 0 \qquad 13.60$$

In terms of the new currents

$$n_a i^{a'} + n_b i^{b'} + n_c a^2 i^{a'} - n_d a i^{a'} = 0$$

or

$$(n_a + a^2 n_c - a n_d) i^{a'} + n_b i^{b'} = 0$$

Neglecting, say, $i^{b'}$, the equations of transformation are

$$\begin{aligned} i^{a'} &= -n_b/(n_a + a^2 n_c - a n_d)\, i^{b''} \\ i^{b'} &= \qquad\qquad\qquad\qquad\quad i^{b''} \end{aligned} \qquad \mathbf{C}_2 = \begin{array}{c|c|} & \mathbf{b}'' \\ \hline \mathbf{a}' & N \\ \hline \mathbf{b}' & 1 \\ \hline \end{array} \qquad 13.61$$

The resultant transformation tensor is

$$\mathbf{C}_1 \cdot \mathbf{C}_2 = \mathbf{C} = \begin{array}{c|c|} & \mathbf{b}'' \\ \hline \mathbf{a} & N \\ \hline \mathbf{b} & 1 \\ \hline \mathbf{c} & Na^2 \\ \hline \mathbf{d} & -Na \\ \hline \mathbf{f} & N-1 \\ \hline \mathbf{g} & -a^2N+1 \\ \hline \end{array} \qquad 13.62$$

reducing the four-winding transformer to *one* equivalent coil.

The impedance tensor $\mathbf{z}$ of the primitive transformer and the load is given in equation 13.53 with an extra row and column $\mathbf{g}$ containing all zeros. The remaining work consists of calculating $\mathbf{C}^* \cdot \mathbf{z} \cdot \mathbf{C} = \mathbf{z}'$, finding $\mathbf{i}'$, etc.

A few examples of $\mathbf{C}$ of balanced three-winding transformers is given in Tables 13.1 and 13.2, and that of a nine-winding transformer is shown in Table 13.3.

XVI. "BARRED" AND "UNBARRED" INDICES

(*a*) It was shown that when the transformation tensor contains complex components the transformation formulas of the various geometric objects are more complicated. In particular the transformation formulas contain in addition to $\mathbf{C}$ and its inverse $\mathbf{C}^{-1}$, also their conjugate, namely, $\mathbf{C}^*$ and $\mathbf{C}^{-1*}$.

In index notation a convention is introduced to denote whether **C** or its conjugate is to be used. *An index that is to be transformed by the*

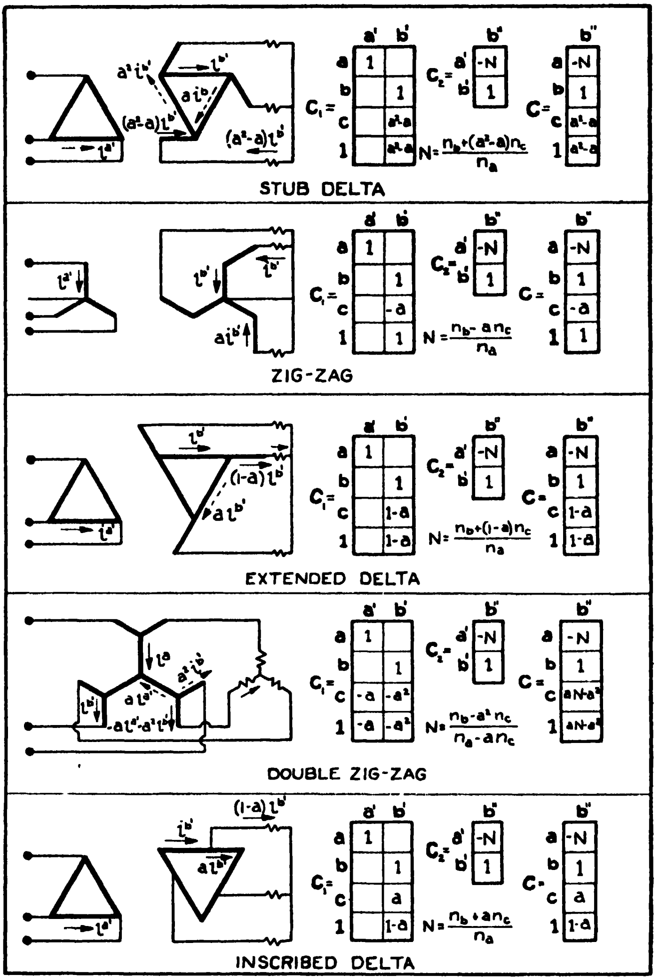


TABLE 13.1.—Balanced Three-phase Multiwinding Transformers and their Transformation Matrices

First column—C_1 showing interconnection of coils
Second column—C_2 neglecting magnetizing currents
Third column—C representing their resultant

conjugate of $C^{\alpha}_{\alpha'}$ or $C^{\alpha'}_{\alpha}$ will be written with a bar over it as $e_{\bar{\alpha}}$ and will be called a "barred" index. Otherwise an index is left unchanged and will

be called an "*unbarred*" index as i^{α}. Hence, according to their transformation formulae 13.13, $z_{\alpha\beta}$ is to be written as $z_{\bar{\alpha}\beta}$ and $y^{\alpha\beta}$ as $y^{\bar{\alpha}\beta}$.

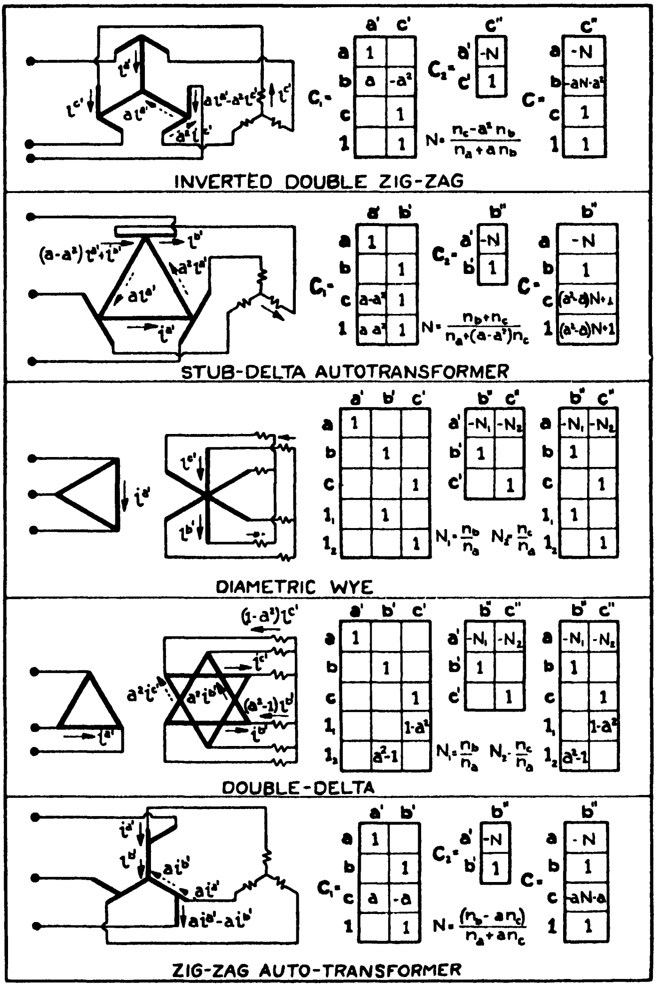


Table 13.2.—Balanced Three-phase Multiwinding Transformers and Their Transformation Matrices

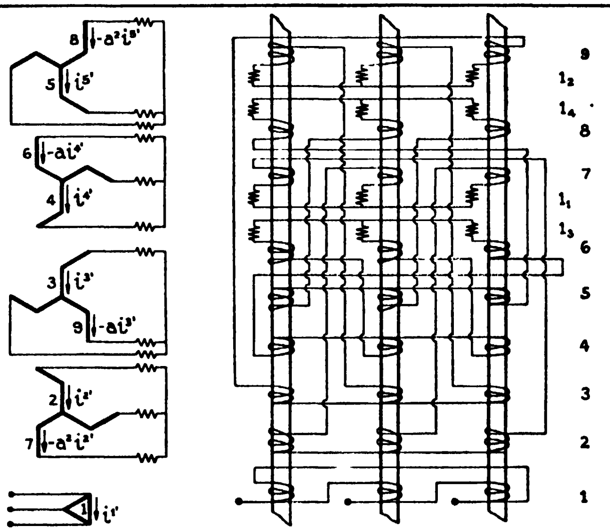
First column—C_1 showing interconnection of coils
Second column—C_2 neglecting magnetizing currents
Third column—C representing their resultant

(It is customary to use a "dot" in place of a bar over an index as $e_{\dot{\alpha}}$, but for printing and writing exigencies the bar will be preferred.)

(*b*) *When the conjugate of a geometric object is taken, all barred indices become unbarred, and vice versa.* For instance

$$(C^{\alpha}_{\alpha'})^* = C^{\bar{\alpha}}_{\bar{\alpha}'} \quad \text{or} \quad (e_{\bar{\alpha}})^* = e_{\alpha} \tag{13.63}$$

$$(z_{\bar{\alpha}\beta})^* = z_{\alpha\bar{\beta}} \quad \text{or} \quad (y^{\alpha\bar{\beta}})^* = y^{\bar{\alpha}\beta} \tag{13.64}$$



$C_1 =$

	1'	2'	3'	4'	5'
1	1				
2		1			
3			1		
4				1	
5					1
6				-a	
7		$-a^2$			
8					$-a^2$
9			-a		
1_1		1			
1_2			1		
1_3				1	
1_4					1

$C_2 =$

	2"	3"	4"	5"
1'	$-N_1$	$-N_2$	$-N_3$	$-N_4$
2'	1			
3'		1		
4'			1	
5'				1

$$N_1 = \frac{n_2 - a^2 n_7}{n_1} \qquad N_2 = \frac{n_3 - a n_9}{n_1} \qquad N_3 = \frac{n_4 - a n_6}{n_1} \qquad N_4 = \frac{n_5 - a^2 n_8}{n_1}$$

$C =$

	2"	3"	4"	5"
1	$-N_1$	$-N_2$	$-N_3$	$-N_4$
2	1			
3		1		
4			1	
5				1
6			-a	
7	$-a^2$			
8				$-a^2$
9		-a		
1_1	1			
1_2		1		
1_3			1	
1_4				1

TABLE 13.3.—Quadruple Zig-zag Transformer and Its Transformation Matrix C

(*c*) That is, in index notation not only *upper* and *lower* indices, but also *barred* and *unbarred* indices, have to be distinguished in order to indicate the formulas of transformation of the various geometric

objects. Hence *the transformation formulas of some of the tensors hitherto introduced*, if the transformation tensor has complex components, are

$$
\begin{array}{ll|ll}
\mathbf{i} & = \mathbf{C}\cdot\mathbf{i}' & i^{\alpha} & = C^{\alpha}_{\alpha'}\, i^{\alpha'} \\
\mathbf{e} & = \mathbf{C}_t^{*}\cdot\mathbf{e}' & e_{\bar{\alpha}} & = C^{\bar{\alpha}'}_{\bar{\alpha}}\, e_{\bar{\alpha}'} \\
\mathbf{z}' & = \mathbf{C}_t^{*}\cdot\mathbf{z}\cdot\mathbf{C} & z_{\alpha'\beta'} & = z_{\alpha\beta}\, C^{\alpha}_{\bar{\alpha}'}\, C^{\beta}_{\beta'} \\
\mathbf{y}' & = \mathbf{C}^{-1}\cdot\mathbf{y}\cdot\mathbf{C}_t^{-1*} & y^{\alpha'\bar{\beta}'} & = y^{\alpha\bar{\beta}}\, C^{\alpha'}_{\alpha}\, C^{\bar{\beta}'}_{\bar{\beta}} \\
\mathbf{C}' & = \mathbf{C}_1^{-1}\cdot\mathbf{C}\cdot\mathbf{C}_2 & C^{\alpha''}_{\alpha'''} & = C^{\alpha}_{\alpha'}\, C^{\alpha''}_{\alpha}\, C^{\alpha'}_{\alpha'''}
\end{array}
\qquad 13.65
$$

It should be noted that the transformation formulas of tensors follow automatically from the indices. It may also be noted that, *whenever* $\mathbf{C}_t$ *occurs, it occurs as a conjugate* $\mathbf{C}_t^{*}$.

These formulas reduce to the formulas given in equation 6.17 when $C^{\alpha}_{\alpha'}$ does not contain complex components. The transformation formulas of i^{α} and $C^{\alpha}_{\alpha'}$ are the same as before, since they contain no barred indices.

(*d*) *It should be expressly noted that in each term of every equation not only the upper and lower indices but also the barred and unbarred indices are balanced.* That is:

1. The free index in each term is either each barred, or each unbarred.

2. The two dummy indices are either both barred or both unbarred.

An *invariant transformation* in which some of the components of the transformation tensor are complex numbers, will be called a "*spinor transformation.*"

XVII. TENSOR AND SPIN INDICES

(*a*) Tensors in which both barred and unbarred indices may occur are also called "*Hermitian tensors*" (after the mathematician Hermite). Also they are called "*spinors.*" The indices themselves (that may be either barred or unbarred) are called "*spin indices.*"

There are geometric objects in which an index is independent of whether the transformation tensor has complex components or not. Such indices are called "tensor indices." *The same geometric object may have both tensor and spin indices.* All indices used in this chapter are *spin* indices; those used in previous chapters are *tensor* indices. When both types of indices occur together in the same tensor, the spin indices may be denoted by capital letters, the tensor indices by lower-case letters.

(*b*) Hence hitherto the following types of indices have been introduced:

1. Fixed and variable.
2. Free and dummy.
3. Covariant and contravariant.
4. Open and closed.
5. Individual and compound.
6. Barred and unbarred.
7. Tensor and spin.

Each of these indices attracts a different type of transformation tensor.

(*c*) The equations of performance (in terms of spin indices) of mesh junction and orthogonal networks are respectively

$$\boxed{e_{\bar{\alpha}} = z_{\bar{\alpha}\beta} i^{\beta}} \tag{13.66}$$

$$\boxed{I^{\alpha} = Y^{\alpha\bar{\beta}} E_{\bar{\beta}}} \tag{13.67}$$

$$\boxed{E_{\bar{\alpha}} + e_{\bar{\alpha}} = z_{\bar{\alpha}\beta}(i^{\beta} + i^{\beta})} \tag{13.68}$$

$$\boxed{i^{\alpha} + I^{\alpha} = Y^{\alpha\beta}(E_{\dot{\beta}} + e_{\bar{\beta}})} \tag{13.69}$$

When it is not intended to use transformation tensors with complex components then the spin indices are replaced by tensor indices (by simply leaving out the bars).

(*d*) The equation of power in terms of spin indices is

$$P = e_{\bar{\alpha}} i^{\alpha} = \boxed{P = e_{\alpha} i^{\alpha}} \quad \text{or} \quad \boxed{P = E_{\alpha} I^{\alpha}} \tag{13.70}$$

It should be remembered that $e_{\bar{\alpha}}$ and $E_{\bar{\alpha}}$ originally have barred indices. Hence wherever they appear without bars it means that *their conjugate is taken*. Hence if the indices in the last equation are spin indices, they show that in calculating power the conjugate of $e_{\bar{\alpha}}$ or $E_{\bar{\alpha}}$ has to be taken.

XVIII. WEIGHTED TENSORS *

In the previous sections a tensor has been defined as a geometric object whose transformation formula contains only $C^{\alpha}_{\alpha'}$ or its inverse $C^{\alpha'}_{\alpha}$, one for each index.

When the transformation formula of a tensor contains in addition to

* The next two sections may be left out at the first reading.

$C^{\alpha}_{\alpha'}$ *and* $C^{\alpha'}_{\alpha}$ *also a scalar* (say a number or a function) *then it is called a "pseudo-tensor."* For instance, if

$$z_{\alpha'\beta'} = z_{\alpha\beta} C^{\alpha}_{\alpha'} C^{\beta}_{\beta'} k \qquad 13.71$$

where k may be 5 or $\log x$, then $z_{\alpha\beta}$ is not a tensor but a pseudo-tensor.

A special case of pseudo-tensors is of importance. *If the scalar k is equal to the determinant c of $C^{\alpha}_{\alpha'}$ raised to the p-th power, then the pseudo-tensor is called a "weighted tensor," or a "tensor of weight p."* In physical problems the "tensors of weight *one*," also called "*tensor densities*," are of special importance. Hence when:

$$\begin{array}{lll} k = \text{any function} & \longrightarrow & \text{pseudo-tensor} \\ k = \quad c^p & \longrightarrow & \text{tensor of weight } p \\ k = \quad c & \longrightarrow & \text{tensor density} \end{array}$$

Weighted spinors contain in their transformation formula also c^* raised to the q-th power.

In electrical engineering problems *weighted tensors occur for instance when a change is made from three-phase to single-phase or to two-phase quantities*, in general whenever the attention *temporarily* is restricted to a part of the system under consideration.

XIX. WEIGHTED SPINORS

(*a*) Weighted spinors are introduced when the original definition is used for the method of symmetrical components, that is, when *the new sequence variables i^0, i^1, and i^2 represent phase currents and not $\sqrt{3}$ times their value.* Such a definition is often convenient. In that case

$$\mathbf{C} = \begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & a^2 & a \\ \hline 1 & a & a^2 \\ \hline \end{array} \end{array} \qquad \mathbf{C}^{-1} = \frac{1}{3} \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{array} \\ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & a & a^2 \\ \hline 1 & a^2 & a \\ \hline \end{array} \end{array} \qquad 13.72$$

that is, the factor $1/\sqrt{3}$ is missing in the definition of $\mathbf{C}$.

Owing to this difference in the definition of $\mathbf{C}$, the original expression for power in terms of sequence quantities

$$P' = e_0 i^0 + e_1 i^1 + e_2 i^2 \qquad 13.73$$

represents the *power input into one phase* and not into the whole system. On the other hand, the power in terms of phase quantities

$$P = e_a i^a + e_b i^b + e_c i^c \qquad 13.74$$

represents the power input into *all three phases*. Hence in the original definition the equation for the invariance of power is

$$\boxed{P = e_\alpha i^\alpha \equiv 3e_{\alpha'} i^{\alpha'} = e_{\alpha'} i^{\alpha'} k = P'} \qquad 13.75$$

containing the scalar $k = 3$.

Since the linear form for power contains a scalar, the transformation formulas of the various spinors will contain this scalar in some form.

(*b*) Originally it is assumed that the variable i^α is transformed by the given $C^\alpha_{\alpha'}$ as $i^\alpha = C^\alpha_{\alpha'} i^{\alpha'}$ without containing a scalar. Substituting it into equation 13.75.

$$e_\alpha C^\alpha_{\alpha'} i^{\alpha'} \equiv e_{\alpha'} i^{\alpha'} k \qquad \Big| \qquad \mathbf{e}^* \cdot \mathbf{C} \cdot \mathbf{i}' = \mathbf{e}'^* \cdot \mathbf{i}' k$$

Cancelling $i^{\alpha'}$

$$e_\alpha C^\alpha_{\alpha'} = e_{\alpha'} k \qquad \Big| \qquad \mathbf{e}^* \cdot \mathbf{C} = \mathbf{e}'^* k$$

Multiplying both sides by the inverse of $C^\alpha_{\alpha'}$

$$e_\alpha = e_{\alpha'} C^{\alpha'}_\alpha k \qquad \Big| \qquad \mathbf{e}^* = \mathbf{C}_t^{-1} \cdot \mathbf{e}'^* k$$

Taking the conjugate of both sides

$$\boxed{e_{\bar\alpha} = k e_{\bar\alpha'} C^{\bar\alpha'}_{\bar\alpha}} \qquad \Big| \qquad \boxed{\mathbf{e} = k \mathbf{C}_t^{-1*} \cdot \mathbf{e}'} \qquad 13.76$$

$$e_{\bar\alpha} = 3 e_{\bar\alpha'} C^{\bar\alpha'}_{\bar\alpha} \qquad \Big| \qquad \mathbf{e} = 3\mathbf{C}_t^{-1*} \cdot \mathbf{e}'$$

Hence $e_{\bar\alpha}$ is a weighted spinor.

The determinant of the transformation matrix $C^{\bar\alpha'}_{\bar\alpha} = \mathbf{C}^{-1*}$ is $a - a^2 = j\sqrt{3} = c$. Hence *the scalar $k = 3$ is equal to cc^*.*

It should be noted that the value of $3\mathbf{C}_t^{-1*}$ is the same as that of $\mathbf{C}$ hence also in the original definition *the voltage vector is transformed the same way as the current vector.*

(*c*) To find the transformation formula of $z_{\bar\alpha\beta}$ as a weighted spinor, the reasoning of Section IIIb is used. Given

$$e_{\bar\alpha} = z_{\bar\alpha\beta} i^\beta$$

Replacing i^β by $C^\beta_{\beta'} i^{\beta'}$ and $e_{\bar\alpha}$ by $e_{\bar\alpha'} C^{\bar\alpha'}_{\bar\alpha} k^*$

$$e_{\bar\alpha'} C^{\bar\alpha'}_{\bar\alpha} k^* = z_{\bar\alpha\beta} C^\beta_{\beta'} i^{\beta'}$$

$$e_{\bar\alpha'} = z_{\bar\alpha\beta} C^{\bar\alpha}_{\bar\alpha'} C^\beta_{\beta'} k^{*-1} i^{\beta'}$$

Since in the new reference frame

$$e_{\bar{\alpha}'} = z_{\alpha'\beta'}i^{\beta'}$$

the transformation formula of $z_{\bar{\alpha}\beta}$ is

$$\boxed{z_{\bar{\alpha}'\beta'} = z_{\bar{\alpha}\beta}C^{\bar{\alpha}}_{\bar{\alpha}'}C^{\beta}_{\beta'}k^{*-1}} \qquad 13.78$$

showing that $z_{\bar{\alpha}\beta}$ *is a* weighted spinor.

(*d*) Hence, *if the original definitions of the method of symmetrical components are retained, then all spinors become weighted spinors, since in a three-phase problem the attention is restricted to one of the phases only.*

CHAPTER XIV

JUNCTION NETWORKS

I. DUALISM OF PHYSICAL PROBLEMS

(*a*) In the analysis of physical phenomena it is found that many (if not all) measurable quantities, laws, and methods of reasonings occur *in pairs* that are in a certain *reciprocal* relation to each other. Electric field and magnetic field, waves and particles, matter and energy, etc., are all manifestations of the dualistic side of nature. In electrical engineering problems such dualistic relations exist, for instance, between voltage and current, impedance and admittance, shunt and series connections, etc.

The advantage of the recognition of such dual properties is that *all equations, all reasonings can be duplicated in terms of dual quantities without going through all over again the same proof* and that *new relations can be easily discovered by simple analogy.*

(*b*) *It will be found, that, whatever reasonings and equations have been established for mesh networks, all can be repeated in a dual form for junction networks.* The following dual expressions will have to be interchanged in the reasonings and equations:

1. Mesh and junction-pair (not junction).
2. Coil and junction.
3. Short circuit and open circuit.
4. Series and shunt.
5. Voltage and current.
6. Impedance and admittance.
7. Covariance and contravariance.

(*c*) Among the geometric objects hitherto introduced the following dual quantities exist:

1. The dual of $\mathbf{e}$ is $\mathbf{I}$ $\quad (e_{\bar{\alpha}} \rightarrow I^{\alpha})$
2. The dual of $\mathbf{i}$ is $\mathbf{E}$ $\quad (i^{\alpha} \rightarrow E_{\bar{\alpha}})$
3. The dual of $\mathbf{z}$ is $\mathbf{Y}$ $\quad (z_{\bar{\alpha}\beta} \rightarrow Y^{\alpha\bar{\beta}})$
4. The dual of $\mathbf{C}$ is $\mathbf{C}_t^{*-1}$ $\quad (C^{\alpha}_{\alpha'} \rightarrow C^{\bar{\alpha}'}_{\bar{\alpha}})$

II. ALL-JUNCTION NETWORKS

(*a*) First the case of *n* coils arranged in *n* junction-pairs will be analyzed in detail. The analysis of *n* coils arranged in *less than n junction-pairs* (of far greater practical importance) is a *special case* of that of an all-junction network.

Again *the only limitation put upon the physical nature of the coils connected together is that they are linear and that* **Y** *is not a function of* **I** *or* **E**. The limitation is put upon the nature of their *interconnections* that at the instant under consideration they are rigid. That is, *the components of the transformation tensor* **C** *are constant and not functions of time.*

(*b*) Fig. 14.1 shows seven different ways of arranging six coils to form a network with six junction-pairs. It should be remembered that the number of junction-pairs is equal to the number of junctions minus the number of independent sub-networks.

A junction-pair consists of any two junctions selected in an arbitrary order on the same sub-network. It is assumed that a difference of potential E' appears between these two junctions and that a current I' enters the network through the first junction and the same current I' leaves the network at the second junction of the junction-pair.

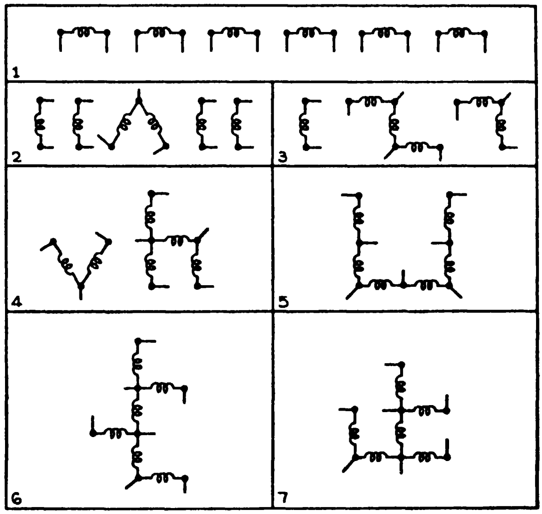


FIG. 14.1.—Various Types of Interconnections of Six Coils into Six Junction-pairs

(*c*) In the study of mesh networks it was shown that a "branch current" may be replaced by a hypothetical "mesh current" by assuming the branch currents to continue their flow in a closed circuit. This replacement *does not change the equations*, only the physical interpretation of the current i.

A similar change in the *physical interpretation* of **I** may be made in junction networks, where the "junction currents" entering the network at the junction-pairs may be assumed to continue their flow

through the network. *The hypothetical circuit traced through the network from one junction of a junction-pair to the other is called an "open mesh."* Just as the hypothetical "mesh current" has a physical existence only as it flows through its respective branch, similarly the hypothetical "open-mesh current" has a physical existence only in the leads of its respective junction-pair.

The manner of determination of the open mesh belonging to a junction-pair will be shown in the study of orthogonal networks.

(*d*) In setting up the equations it will be assumed that known *currents* **I** are impressed across the individual coils whose self- and mutual *admittances* **Y** are also known. The differences of potential **E** which appear across the coils due to the impressed currents **I** will be investigated. In order to do that *the equation of current* $\mathbf{I}' = \mathbf{Y}' \cdot \mathbf{E}'$ expressed along the junction-pairs *will have to be set up*; then that may be solved for $\mathbf{E}'$ (if no other manipulations are needed). That is, *now the voltages across the junction-pairs* $\mathbf{E}'$ *are the "variables" instead of the mesh currents* $\mathbf{i}'$.

Although there are n coils and n known impressed currents across them, still the n unknown differences of potentials appearing across the n coils are not independent of one another. It is sufficient to set up and solve the equations for as many differences of potentials as there are junction-pairs, and the remaining ones can be immediately determined from these. The situation is analogous to that existing in mesh networks where there are n coils and n known impressed voltages and where the n unknown currents in the n coils may be found by *setting up and solving the equations for only as many currents as there are meshes.*

(*e*) Instead of assuming **I** as an impressed current and **E** as the difference of potential due to it, the problem may also be stated that **I** is a current flowing into the two junctions of the various coils due to an applied voltage **E** across each coil. Or it may be stated that **I** *is a current flowing into some outside loads* $\mathbf{Z}_L$ *or* $\mathbf{Y}_L$ (*which, however, do not appear in the equation or on the diagram*) and **E** *is the difference of potential appearing across the loads.* The first method of statement will be preferred in setting up the equations to emphasize the analogy with the mesh analysis, where **e** is considered as the impressed and **i** as the response quantity. Hence in junction networks **I** is the impressed and **E** is the response quantity. Of course, in the *manipulation* of the equations any variety of impressed and response quantities may occur.

III. THE INVARIANCE OF POWER

(*a*) It should be noted that, if the same currents **I** are impressed into the two junctions of each corresponding coil of the various networks of Fig. 14.1, then *the same differences of potential* **E** *appear across corresponding coils* since through each coil only the current impressed to its junctions may flow, the remaining coils being open-circuited. Hence *the power input* **I**·**E**, *a linear form, remains "invariant" under the various interconnections of n coils into n junction-pairs*, that is

$$\boxed{\mathbf{I}\cdot\mathbf{E} = \mathbf{I}'\cdot\mathbf{E}'} \qquad \Bigg| \qquad \boxed{I^{u}E_{u} = I^{u'}E_{u'}} \qquad 14.1$$

(*b*) Also just as in all-mesh networks *all the different types of networks of Fig.* 14.1 *have the same equation of performance* **I** = **Y**·**E** *if the differences of potential E_u appearing across each coil are assumed as the variables.* But as soon as different E_u are assumed as variables the equations become different for each network.

In other words, the key to the possibility of transforming the equation of performance of any network of Fig. 14.1 (say that of the first one) to that of any other all-junction network is supplied by the fact that each network has one and only one reference frame in which the equation of the superimposed electromagnetic quantities is identical.

That is, *all the networks are different, but the equations of the superimposed electro-magnetic phenomena are identical for the different networks, if one particular reference frame is selected on each network. These particular reference frames (the junction-pairs across the individual coils) serve as "bridges" across which it is possible to pass from one network to any other network.*

IV. THE ADMITTANCE TENSOR Y′

(*a*) Let the all-junction network of Fig. 14.2(*a*) be given in which a different set of currents **I** is impressed across each coil.

Its primitive junction network is shown in Fig. 14.2(*b*) having *asymmetrical* mutual admittances between, say, Y^{bb}, Y^{dd} and Y^{cc}, Y^{ee}.

The method of analysis of junction networks follows *step by step* the analysis of mesh networks with the difference that the dual quantities (shown in Section I) are interchanged.

(*b*) The first step is to set up the geometric objects and the equation of performance of the primitive junction network. Its equation of current is

$$\boxed{\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}} \qquad \Bigg| \qquad \boxed{I^{u} = Y^{uv}E_{v}} \qquad 14.2$$

where

$$\mathbf{I} = \begin{array}{cccccc} \text{a} & \text{b} & \text{c} & \text{d} & \text{f} & \text{g} \\ \hline I^a & I^b & I^c & I^d & I^f & I^g \end{array} \qquad 14.3$$

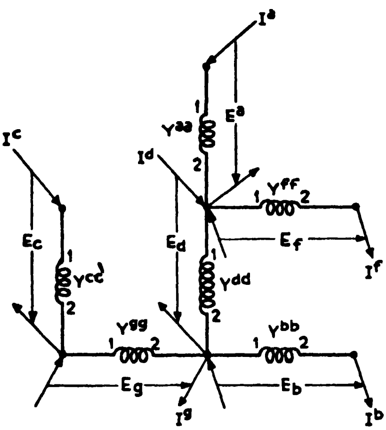
$$\mathbf{E} = \begin{array}{cccccc} \text{a} & \text{b} & \text{c} & \text{d} & \text{f} & \text{g} \\ \hline E_a & E_b & E_c & E_d & E_f & E_g \end{array} \qquad 14.4$$

Y = (14.5)

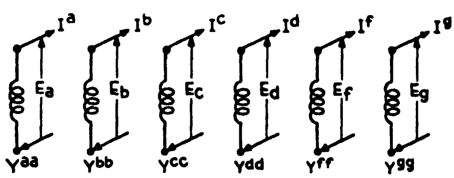
	a	b	c	d	f	g
a	Y^{aa}					
b		Y^{bb}		Y^{bd}		
c			Y^{cc}			Y^{cg}
d		Y^{db}		Y^{dd}		
f					Y^{ff}	
g			Y^{gc}			Y^{gg}

(c) *The effect of the interconnection of the coils is to introduce additional junction-pairs into which currents may be impressed or across which loads may be connected.* It should be noted that in the primitive network of Fig. 14.2*b* at no other points can currents be impressed except at those shown.

In the new network of Fig. 14.2 there is a large variety of ways in



(*a*) Given all-junction network



(*b*) Its primitive network

FIG. 14.2

which the six new junction-pairs may be selected across which the new currents $\mathbf{I}'$ are impressed. The selection depends on the requirement of the problem. Let the six *independent* junction-pairs shown in Fig. 14.3 be selected arbitrarily. It should be noted that each junction is covered at least once.

It should be remembered that, if the two junctions of each coil are selected as the junction-pairs of the *new* network, then the equation of

current of the primitive network $\mathbf{I} = \mathbf{Y} \cdot \mathbf{E}$ can be used for the new network *without any change.*

(*d*) To set up a relation between the old and the new voltages $\mathbf{E}$ and $\mathbf{E}'$, first *the differences of potentials appearing across each individual coil in terms of the new impressed voltages have to be found.*

For this step Kirchhoff's second law is used, that: "*The sum of the voltages around a closed mesh is zero.*"

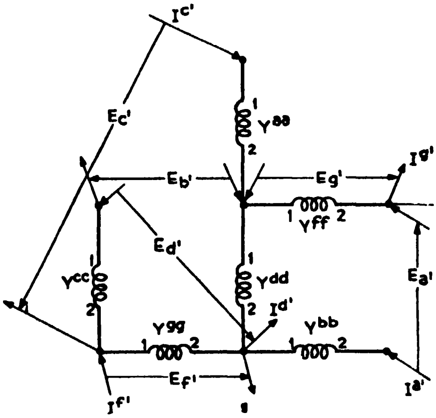


FIG. 14.3.—Assumed Junction-pairs

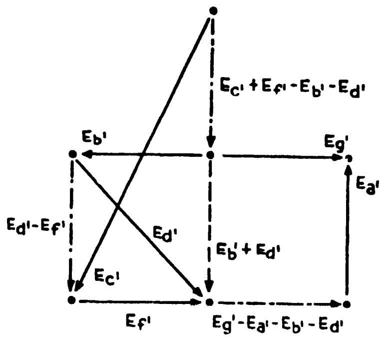


FIG. 14.4.—Differences of Potential Across Individual Coils

The differences of potentials appearing across each coil are calculated in Fig. 14.4 by:

1. First drawing between the junction-pairs the assumed new voltages $\mathbf{E}'$ with their proper signs.
2. Then finding the voltages between the two ends of each coil in terms of the new voltages E_a, E_b, etc., by assuming that *the sum of the differences of potential in any closed circuit is zero.* The closed circuit need not follow the coils. (That is, the closed circuits do not form meshes.)

(*e*) Equating the differences of potentials appearing across each coil *before* and *after* interconnection, the following equations may be set up

$$
\begin{aligned}
E_a &= \quad\quad -E_{b'} + E_{c'} - E_{d'} + E_{f'} \\
E_b &= -E_{a'} - E_{b'} \quad - E_{d'} \quad + E_{g'} \\
E_c &= \quad\quad E_{d'} - E_{f'} \\
E_d &= \quad E_{b'} \quad + E_{d'} \\
E_f &= \quad\quad E_{g'} \\
E_g &= \quad\quad E_{f'}
\end{aligned}
\qquad
\mathbf{C}_t^{-1} = \begin{array}{c|cccccc|}
 & a' & b' & c' & d' & f' & g' \\ \hline
a & & -1 & 1 & -1 & 1 & \\
b & -1 & -1 & & -1 & & 1 \\
c & & & & 1 & -1 & \\
d & & 1 & & 1 & & \\
f & & & & & & 1 \\
g & & & & & 1 & \\ \hline
\end{array}
\qquad (14.6)
$$

representing the equation of transformation of the variables

$$\boxed{\mathbf{E} = \mathbf{C}_t^{-1} \cdot \mathbf{E}'} \qquad \Big| \qquad \boxed{E_u = C_u^{u'} E_{u'}} \tag{14.7}$$

Hence the coefficients of the new variables represent the transposed inverse transformation tensor, $\mathbf{C}_t^{-1}$. The determinant of its matrix is not zero.

(*f*) Once the inverse transformation tensor $\mathbf{C}^{-1}$ has been established, the new components of the geometric objects of the new network $\mathbf{Y}'$ and $\mathbf{I}'$ are set up.

The new components of the admittance tensor are found by

$$\boxed{\mathbf{Y}' = \mathbf{C}^{-1} \cdot \mathbf{Y} \cdot \mathbf{C}_t^{-1}} \qquad \Big| \qquad \boxed{Y^{u'v'} = Y^{uv} C_u^{u'} C_v^{v'}} \tag{14.8}$$

$\mathbf{Y}' =$

	a′	b′	c′	d′	f′	g′
a′	Y^{bb}	$Y^{bb} - Y^{bd}$	0	$Y^{bb} - Y^{bd}$	0	$-Y^{bb}$
b′	$Y^{bb} - Y^{db}$	$Y^{aa} + Y^{bb} - Y^{bd} - Y^{db} + Y^{dd}$	$-Y^{aa}$	$Y^{aa} + Y^{bb} - Y^{bd} - Y^{db} + Y^{dd}$	$-Y^{aa}$	$-Y^{bb} + Y^{db}$
c′		$-Y^{aa}$	Y^{aa}	$-Y^{aa}$	Y^{aa}	0
d′	$Y^{bb} - Y^{db}$	$Y^{aa} + Y^{bb} - Y^{bd} - Y^{db} + Y^{dd}$	$-Y^{aa}$	$Y^{aa} + Y^{bb} - Y^{bd} - Y^{cc} + Y^{db} + Y^{dd}$	$-Y^{aa} - Y^{cc} + Y^{cg}$	$-Y^{bb} + Y^{db}$
f′	0	$-Y^{aa}$	Y^{aa}	$-Y^{aa} - Y^{cc} + Y^{gc}$	$Y^{aa} + Y^{cc} - Y^{cg} - Y^{gc} + Y^{gg}$	0
g′	$-Y^{bb}$	$-Y^{bb} + Y^{bd}$	0	$-Y^{bb} + Y^{bd}$	0	$Y^{bb} + Y^{ff}$

14.9

If the mutual admittances are the same in both directions, this matrix is symmetrical.

V. THE CURRENT VECTOR I′

(*a*) The new components of the current vector $\mathbf{I}'$ are found by

$$\boxed{\mathbf{I}' = \mathbf{C}^{-1} \cdot \mathbf{I}} \qquad \Big| \qquad \boxed{I^{u'} = C_u^{u'} \dot{I}^u} \tag{14.10}$$

$\mathbf{I}' =$

a′	b′	c′	d′	f′	g′
$-I^b$	$-I^a - I^b + I^d$	I^a	$-I^a - I^b + I^c + I^d$	$I^a - I^c + I^g$	$I^b + I^f$

14.11

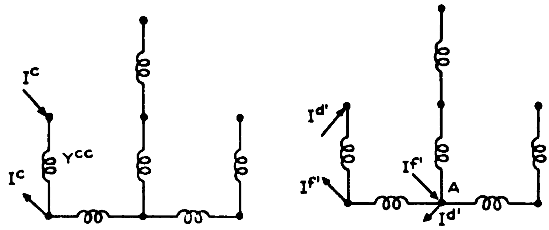
$\mathbf{I}' =$

a′	b′	c′	d′	f′	g′
$I^{a'}$	$I^{b'}$	$I^{c'}$	$I^{d'}$	$I^{f'}$	$I^{g'}$

14.12

That is, if across the individual coils is impressed I^a, I^b, $\cdots$, then through the various junction-pairs (in at one junction and out at the other junction) flow $-I^b$, $-I^a - I^b + I^d$, etc.

The admittance of these coils, whose currents meet at a junction to build a junction-current, forms the diagonal components of the admittance tensor $\mathbf{Y}'$. For instance, by equation 14.12, $I^{f'}$ is built up from I^a, $-I^c$, and I^g; hence the three admittances Y^{aa}, Y^{cc}, and Y^{gg} (and any mutual admittance between them) form the $\mathbf{f}'\mathbf{f}'$ diagonal component of $\mathbf{Y}'$. (In mesh networks the impedances of those coils whose impressed voltage add up to the mesh voltage form the diagonal components of the impedance tensor $\mathbf{z}'$.)



(*a*) Current impressed across a coil (*b*) Coil current in terms of junction-pair currents

FIG. 14.5

(*b*) *This last step of calculating* $\mathbf{I}'$ *from* $\mathbf{I}$ *is used whenever a current is impressed* (*or a load is connected*) *not across a junction-pair, but across a coil.* For instance, it may be possible that the only current impressed in the present example is I^c as shown in Fig. 14.5*a*. However, the two junctions of coil Y^{cc} are not assumed as a junction-pair whose voltage is chosen as a variable, hence I^c cannot be assumed to be a component of $\mathbf{I}'$. The value of $\mathbf{I}'$ with I^c alone impressed is by $\mathbf{C}^{-1}\cdot\mathbf{I}$

$$\mathbf{I}' = \begin{array}{cccccc} \mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \mathbf{d}' & \mathbf{f}' & \mathbf{g}' \\ \hline 0 & 0 & 0 & I^c & -I^c & 0 \end{array} \qquad 14.13$$

That is, I^c forms part of two different impressed junction-currents, analogously to mesh networks where one impressed voltage in series with a coil may form part of several mesh voltages. Fig. 14.5*b* shows how I^c forms the impressed currents on two junction-pairs $\mathbf{d}'$ and $\mathbf{f}'$. Through the other junction A the *resultant* current is of course zero.

(*c*) If the two junctions through which the impressed current flows do not form a junction-pair and are not connected together by a coil then *it is assumed that a coil with zero admittance connects the two junctions* so that the primitive junction network has one additional coil. This assumption is analogous to that used in mesh networks where a coil with zero impedance is assumed in series with an impressed voltage in the absence of an actual coil. (Two junctions that are not connected by a coil, but have currents impressed on them, may be assumed to form an "admittanceless branch" or an "apparent coil").

(*d*) The equation of current is established as

$$\boxed{\mathbf{I}' = \mathbf{Y}' \cdot \mathbf{E}'} \qquad \Bigg| \qquad \boxed{I^{u'} = Y^{u'v'} E_{v'}} \qquad 14.14$$

representing six ordinary equations.

In many practical problems this equation is subdivided into at least two invariant equations.

(*e*) When the currents $\mathbf{I}'$ flowing through the junction-pairs are known, the differences of potential $\mathbf{E}'$ appearing across the junction-pairs are found by

$$\boxed{\mathbf{E}' = \mathbf{Z}' \cdot \mathbf{I}'} \qquad \Bigg| \qquad \boxed{E_{v'} = Z_{v'u'} i^{u'}} \qquad 14.15$$

where $\mathbf{Z}'$ is the inverse of $\mathbf{Y}'$.

Once the unknown differences of potentials at the junction-pairs $\mathbf{E}'$ have been found, the quantities existing in each individual coil may be calculated with their aid.

(*f*) The differences of potential appearing across each individual coil are found by the equation of transformation of $\mathbf{E}$

$$\mathbf{E} = \mathbf{C}_t^{-1} \cdot \mathbf{E}' \qquad \Big| \qquad E_u = C_u^{u'} E_{u'} \qquad 14.16$$

already given in equation 14.6.

The currents $\mathbf{I}$ flowing in the individual coils are found from $\mathbf{I} = \mathbf{Y} \cdot \mathbf{E}$ by replacing $\mathbf{I}$ by $\mathbf{I}_c$ and $\mathbf{E}$ by $\mathbf{C}_t^{-1} \cdot \mathbf{E}$, giving

$$\mathbf{I}_c = \mathbf{Y} \cdot \mathbf{C}_t^{-1} \cdot \mathbf{E}' \qquad \Big| \qquad I^u = Y^{u'v'} C_{u'}^{u} E_{v'} \qquad 14.17$$

where the matrix $\mathbf{Y} \cdot \mathbf{C}_t^{-1}$ has already been calculated in equation 14.9. (In an all-junction network $\mathbf{I}$ and $\mathbf{I}_c$ are numerically equal). Hence

$$\mathbf{I}_c = \begin{array}{c|c|} & \\ \mathbf{a} & Y^{aa}(-E_{b'} + E_{c'} - E_{d'} + E_{f'}) \\ \mathbf{b} & Y^{bb}(-E_{a'} - E_{b'} - E_{d'} + E_{g'}) + Y^{bd}(E_{b'} + E_{d'}) \\ \mathbf{c} & Y^{cc}(E_{d'} - E_{f'}) + Y^{cg} E_{f'} \\ \mathbf{d} & Y^{dd}(E_{b'} + E_{d'}) + Y^{db}(-E_{a'} - E_{b'} - E_{d'} + E_{g'}) \\ \mathbf{f} & Y^{ff} E_{g'} \\ \mathbf{g} & Y^{gg} E_{f'} + Y^{gc}(E_{d'} - E_{f'}) \end{array} \qquad 14.18$$

(*g*) If the transformation tensor $\mathbf{C}_t^{-1}$ is singular, the self- and mutual impedances of the individual coils $\mathbf{Z}$ are found from those of the junction-pairs $\mathbf{Z}'$ by

$$\boxed{\mathbf{Z} = \mathbf{C}_t^{-1} \cdot \mathbf{Z}' \cdot \mathbf{C}^{-1}} \qquad \Bigg| \qquad \boxed{Z_{uv} = Z_{u'v'} C_u^{u'} C_v^{v'}} \qquad 14.19$$

VI. SUMMARY OF STEPS

The steps of setting up the equation of current of any *junction* network parallel the steps of setting up the equation of voltage of a mesh network *except that each concept is replaced by its "dual" concept.*

(*a*) The primitive junction network and its geometric objects **Y**, **I**, and **E** are first established:

1. The primitive network consists of all coils of the given network open-circuited.
2. The admittance tensor **Y** of the primitive network is set up containing the self- and mutual admittances of the individual coils.
3. The current vector **I** contains the currents entering and leaving each coil.
4. The voltage vector **E** contains the various differences of potentials (known or unknown) appearing across each coil.

(*b*) The next step is to set up the transposed inverse transformation tensor $\mathbf{C}_t^{-1}$ changing the primitive network into the actual network.

1. Assume as many independent differences of potentials $\mathbf{E}'$ between the various junctions as there are junction-pairs in the system. The two end points of each voltage represent a junction-pair.
2. *Write along each coil the differences of potential appearing across them*, expressed in terms of the assumed new $\mathbf{E}'$, by using Kirchhoff's Second Law.
3. Equate the old and the new voltages appearing across each individual coil. There are as many equations as there are coils. The left-hand side contains the old voltages, the right-hand side the new voltages.
4. The coefficients of the new voltages $\mathbf{E}'$ form the transposed inverse transformation tensor $\mathbf{C}_t^{-1}$, representing mathematically Kirchhoff's Second Law.

(*c*) The next step is to find the new components of the geometric objects $\mathbf{Y}'$ and $\mathbf{I}'$ and the equation of current of the new system.

1. The admittance tensor $\mathbf{Y}'$ is found by the transformation formula,

$$\mathbf{Y}' = \mathbf{C}^{-1}\cdot\mathbf{Y}\cdot\mathbf{C}_t^{-1} \quad \Big| \quad Y^{u'v'} = Y^{uv}C_u^{u'}C_v^{v'} \qquad 14.20$$

2. The currents impressed across the new junction-pairs are found by the transformation formula

$$\mathbf{I}' = \mathbf{C}^{-1}\cdot\mathbf{I} \quad \Big| \quad I^{u'} = C_u^{u'}I^u \qquad 14.21$$

3. The equation of current of the new network is established as

$$\mathbf{I}' = \mathbf{Y}'\cdot\mathbf{E}' \quad \Big| \quad I^{u'} = Y^{u'v'}E_{v'} \qquad 14.22$$

(*d*) Once the equation of performance of a system has been established it is subjected to all types of manipulations depending on the problems at hand.

When the equation of current is treated as one unit without subdivision, the unknown differences of potentials appearing across the assumed junction-pairs are found in two steps:

1. The inverse of the admittance tensor is calculated, giving the impedance tensor

$$\mathbf{Z}' = \mathbf{Y}'^{-1} \quad \Big| \quad Z_{v'u'} = (Y^{u'v'})^{-1} \qquad 14.23$$

2. The unknown differences of potential are found by

$$\mathbf{E}' = \mathbf{Z}'\cdot\mathbf{I}' \quad \Big| \quad E_{v'} = Z_{v'u'}I^{u'} \qquad 14.24$$

The unknown quantities of the individual coils may be found with the aid of $\mathbf{E}'$ as follows:

3. The differences of potential across the individual coils $\mathbf{E}$ are found by

$$\mathbf{E} = \mathbf{C}_t^{-1}\cdot\mathbf{E}' \quad \Big| \quad E_v = C_v^{v'}E_{v'} \qquad 14.25$$

4. The currents flowing in the individual coils $\mathbf{I}_c$ are found by

$$\mathbf{I}_c = \mathbf{Y}\cdot\mathbf{C}_t^{-1}\cdot\mathbf{E}' \quad \Big| \quad I^u = Y^{u'v'}C_{u'}^{u}E_{v'} \qquad 14.26$$

where $\mathbf{Y}\cdot\mathbf{C}_t^{-1}$ has already been calculated in finding $\mathbf{Y}'$.

5. The self- and mutual impedances of the individual coils are

$$\mathbf{Z}_c = \mathbf{C}_t^{-1}\cdot\mathbf{Z}'\cdot\mathbf{C}^{-1} \quad \Big| \quad Z_{uv} = Z_{u'v'}C_u^{u'}C_v^{v'} \qquad 14.27$$

(*e*) *In direct notation it is advantageous to replace* $\mathbf{C}_t^{*-1}$ *in all equations by a new symbol, say* $\mathbf{A}$. In terms of $\mathbf{A}$ all transformation formulas of mesh networks may be used for the corresponding *dual* quantities of junction networks if $\mathbf{C}_t^{*-1}$ everywhere is replaced by $\mathbf{A}$, also $\mathbf{C}^{-1}$ by $\mathbf{A}_t^*$ and so on. For instance:

$$\mathbf{z}' = \mathbf{C}_t^*\cdot\mathbf{z}\cdot\mathbf{C} \rightarrow \mathbf{Y}' = \mathbf{A}_t^*\cdot\mathbf{Y}\cdot\mathbf{A} \qquad 14.28$$

$$\mathbf{y}_c = \mathbf{C}\cdot\mathbf{y}'\cdot\mathbf{C}_t^* \rightarrow \mathbf{Z}_c = \mathbf{A}\cdot\mathbf{Z}'\cdot\mathbf{A}_t^* \qquad 14.29$$

$$\mathbf{e}' = \mathbf{C}_t^*\cdot\mathbf{e} \rightarrow \mathbf{I}' = \mathbf{A}_t^*\cdot\mathbf{I} \qquad 14.30$$

$$\mathbf{i} = \mathbf{C}\cdot\mathbf{i}' \rightarrow \mathbf{E} = \mathbf{A}\cdot\mathbf{E}' \qquad 14.31$$

VII. THE EQUATIONS OF CONSTRAINTS

(*a*) Let it be assumed now that in the previous network of Fig. 14.3 *three of the junction-pairs are short-circuited* as shown in Fig. 14.6

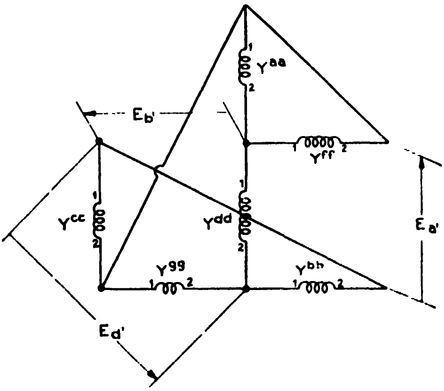


FIG. 14.6.—Short-circuiting Three of the Junction-pairs

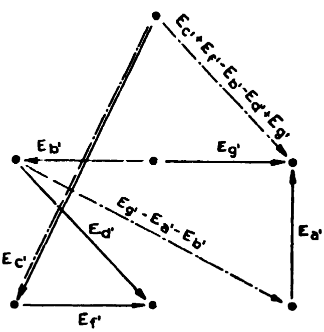


FIG. 14.7.—Differences of Potential Before the Short-circuit

so that *three junction-pairs exist in place of six.* Short-circuiting the junction-pairs is equivalent of making the differences of potential appearing across them equal to zero. That is, *the short-circuiting of the three junction-pairs is equivalent to introducing the following three equations of constraint that must exist between the six new junction-pairs* (See Fig. 14.7.)

$$
\begin{aligned}
E_{c'} &= 0 \\
E_{c'} + E_{f'} - E_{b'} - E_{d'} + E_{g'} &= 0 \qquad\qquad 14.32 \\
E_{g'} - E_{a'} - E_{b'} &= 0
\end{aligned}
$$

With the aid of the three equations of constraint three of the variables may be eliminated, say $E_{c'}$ $E_{g'}$ and $E_{f'}$ as

$$
\begin{aligned}
E_{c'} &= 0 \\
E_{g'} &= E_{a'} + E_{b'} \qquad\qquad 14.33 \\
E_{f'} &= E_{d'} - E_{a'}
\end{aligned}
$$

leaving as new variables $E_{a'}$, $E_{b'}$, and $E_{d'}$. Substituting these equations into the equations of transformation 14.6, the relation between the old and the new variables becomes

$$
\begin{aligned}
E_a &= -E_{a'} - E_{b'} \\
E_b &= \qquad\qquad - E_{d'} \\
E_c &= \quad E_{a'} \\
E_d &= \qquad\quad E_{b'} + E_{d'} \\
E_f &= \quad E_{a'} + E_{b'} \\
E_g &= -E_{a'} \qquad\quad E_{d'}
\end{aligned}
\qquad
C_t^{-1} = \begin{array}{c|c|c|c|}
 & a' & b' & d' \\
\hline
a & -1 & -1 & \\
b & & & -1 \\
c & 1 & & \\
d & & 1 & 1 \\
f & 1 & 1 & \\
g & -1 & & 1 \\
\hline
\end{array}
\qquad 14.34
$$

The coefficients of the new variables give a singular inverse transformation tensor having as many columns as there are new variables, namely three.

The method of analysis from now on is the same as in case of a non-singular (square) inverse transformation tensor. The new admittance tensor $\mathbf{Y}'$, however, will have *three* rows and columns instead of six, etc.

(*b*) The new components of the admittance tensor are found from those of the primitive network, equation 14.5, by $\mathbf{C}^{-1}\cdot\mathbf{Y}\cdot\mathbf{C}_t^{-1}$ as

$$
\mathbf{Y}' = \begin{array}{c|c|c|c|}
 & a' & b' & d' \\
\hline
a' & \begin{matrix} Y^{aa} + Y^{cc} - Y^{cg} + Y^{ff} \\ - Y^{gc} + Y^{gg} \end{matrix} & Y^{aa} + Y^{ff} & Y^{cg} - Y^{gg} \\
\hline
b' & Y^{aa} + Y^{ff} & Y^{aa} + Y^{dd} + Y^{ff} & - Y^{db} + Y^{dd} \\
\hline
d' & Y^{gc} - Y^{gg} & - Y^{bd} + Y^{dd} & \begin{matrix} Y^{bb} - Y^{bd} - Y^{db} \\ Y^{dd} + Y^{gg} \end{matrix} \\
\hline
\end{array}
\qquad 14.35
$$

The new components of the current vector, entering and leaving the junction-pairs, are $\mathbf{C}^{-1}\cdot\mathbf{I} =$

$$
\mathbf{I}' = \begin{array}{|c|c|c|}
a' & b' & d' \\
\hline
-I^a+I^c+I^f-I^g & -I^a+I^d+I^f & -I^b+I^d+I^g \\
\hline
\end{array}
= \begin{array}{|c|c|c|}
a' & b' & d' \\
\hline
I^{a'} & I^{b'} & I^{d'} \\
\hline
\end{array}
\qquad 14.36
$$

The equation of current of the new network is $\mathbf{I}' = \mathbf{Y}'\cdot\mathbf{E}'$, and so on.

VIII. SINGULAR TRANSFORMATION TENSOR

(*a*) Again *instead of setting up equations of constraint to represent the short-circuiting of junction-pairs, it is possible to assume right at the start as many new variables as there are junction-pairs* and follow the analysis of Section IV as if the inverse transformation tensor $\mathbf{C}_t^{-1}$ were non-singular.

It should be remembered that the number of junction-pairs of any network is equal to the number of coils minus the number of closed meshes.

Similarly, *any formula developed in terms of* $\mathbf{C}^{-1}$ *is valid for both singular and non-singular* $\mathbf{C}^{-1}$. However, formulas in which the *inverse* of a singular $\mathbf{C}^{-1}$ is needed should not be employed.

(*b*) As an example let fifteen coils be interconnected into the network of Fig. 14.8 having five junction-pairs and ten meshes. Assuming no voltage impressed *in series* with the coils, the network is primarily a junction network, since it has far fewer junction-pairs than meshes.

Let it be assumed that *two currents are impressed across the network.* The first, I^7, enters the network at junction A and leaves it at junction C. The second current I^5 enters the network at junction E and leaves it at junction F. In other words, let it be assumed that the two junctions of coil Y^5 *are, say, the input terminals and the two junctions of coil* Y^7 *are the output terminals of the network.* The question is, what are the differences of potentials appearing across the input and output terminals?

(*c*) The steps in setting up its equation of current are the following:

1. Its primitive network consists of fifteen open-circuited coils.
2. The admittance tensor of the primitive network is

$$\mathbf{Y} = \begin{array}{c|ccccccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\hline
1 & Y^1 & & & & & & & & & & & & & & \\
2 & & Y^2 & & & & & & & & & & & & & \\
3 & & & Y^3 & & & & & & & & & & & & \\
4 & & & & Y^4 & & & Y^{47} & & & & & & & & \\
5 & & & & & Y^5 & & & & Y^{59} & & & & & & \\
6 & & & & & & Y^6 & & & & & & & & & \\
7 & & & & Y^{74} & & & Y^7 & & & & & & & & \\
8 & & & & & & & & Y^8 & & & & & & & \\
9 & & & & & Y^{95} & & & & Y^9 & & & & & & \\
10 & & & & & & & & & & Y^{10} & & & & & \\
11 & & & & & & & & & & & Y^{11} & & & & \\
12 & & & & & & & & & & & & Y^{12} & & & \\
13 & & & & & & & & & & & & & Y^{13} & & \\
14 & & & & & & & & & & & & & & Y^{14} & \\
15 & & & & & & & & & & & & & & & Y^{15} \\
\hline
\end{array} \qquad 14.37$$

assuming asymmetrical mutual admittances, say between Y^4 and Y^7 and between Y^5 and Y^9. This tensor is not symmetrical.

3. The current vector impressed on the primitive network is

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
=					I^5		$-I^7$								

14.38

(*d*) To set up the transformation tensor the following steps are taken:

1. The five junction-pairs to be assumed are shown in Fig. 14.9.

2. The differences of potentials appearing *across each coil* are also shown in Fig. 14.9.

3. Equating the old and the new differences of potential appearing across each coil, the following fifteen equations may be written (by comparing Figs. 14.8 and 14.9):

$$
\begin{aligned}
E_1 &= E_{1'} \\
E_2 &= E_{2'} \\
E_3 &= E_{3'} \\
E_4 &= E_{4'} \\
E_5 &= E_{5'} \\
E_6 &= E_{1'}+E_{2'}+E_{3'}+E_{4'}+E_{5'} \\
E_7 &= E_{1'}+E_{2'} \\
E_8 &= E_{1'}+E_{2'}+E_{3'} \\
E_9 &= E_{1'}+E_{2'}+E_{3'}+E_{4'} \\
E_{10} &= E_{2'}+E_{3'} \\
E_{11} &= E_{2'}+E_{3'}+E_{4'} \\
E_{12} &= E_{2'}+E_{3'}+E_{4'}+E_{5'} \\
E_{13} &= E_{3'}+E_{4'} \\
E_{14} &= E_{3'}+E_{4'}+E_{5'} \\
E_{15} &= E_{4'}+E_{5'}
\end{aligned}
$$

$C_t^{-1} =$

	1′	2′	3′	4′	5′
1	1				
2		1			
3			1		
4				1	
5					1
6	1	1	1	1	1
7	1	1			
8	1	1	1		
9	1	1	1	1	
10		1	1		
11		1	1	1	
12		1	1	1	1
13			1	1	
14			1	1	1
15				1	1

14.39

4. The coefficients of the new variables give the transpose inverse transformation tensor C_t^{-1}.

(*e*) The next step is to find the equation of current.

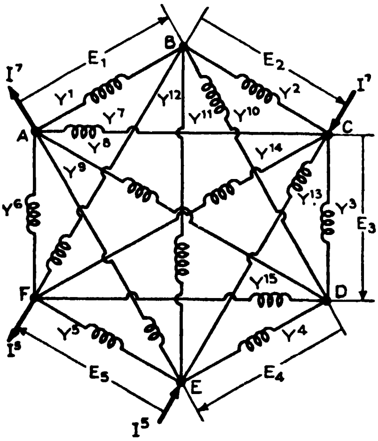


FIG. 14.8.—Junction Network

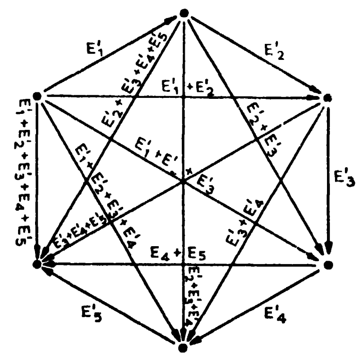


FIG. 14.9.—Differences of Potentials Across Individual Coils

1. The admittance tensor is found by $\mathbf{C}^{-1}\cdot\mathbf{Y}\cdot\mathbf{C}_t^{-1}$ or by $\mathbf{A}_t\cdot\mathbf{Y}\cdot\mathbf{A}$ as

	1′	2′	3′	4′	5′
1′	$Y^1+Y^6+Y^7+Y^8+Y^9$	$Y^6+Y^7+Y^8+Y^9$	$Y^6+Y^8+Y^9$	$Y^6+Y^{74}+Y^9$	Y^6+Y^{95}
2′	$Y^6+Y^7+Y^8+Y^9$	$Y^2+Y^6+Y^7+Y^8+Y^9+Y^{10}+Y^{11}+Y^{12}$	$Y^6+Y^8+Y^9+Y^{10}+Y^{11}+Y^{12}$	$Y^6+Y^{74}+Y^9+Y^{11}+Y^{12}$	$Y^6+Y^{95}+Y^{12}$
$\mathbf{Y}'=$ 3′	$Y^6+Y^8+Y^9$	$Y^6+Y^8+Y^9+Y^{10}+Y^{11}+Y^{12}$	$Y^3+Y^6+Y^8+Y^9+Y^{10}+Y^{11}+Y^{12}+Y^{13}+Y^{14}$	$Y^6+Y^9+Y^{11}+Y^{12}+Y^{13}+Y^{14}$	$Y^6+Y^{95}+Y^{12}+Y^{14}$
4′	$Y^{47}+Y^6+Y^9$	$Y^{47}+Y^6+Y^9+Y^{11}+Y^{12}$	$Y^6+Y^9+Y^{11}+Y^{12}+Y^{13}+Y^{14}$	$Y^4+Y^6+Y^9+Y^{11}+Y^{12}+Y^{13}+Y^{14}+Y^{15}$	$Y^6+Y^{95}+Y^{12}+Y^{14}+Y^{15}$
5′	$Y^{59}+Y^6$	$Y^{59}+Y^6+Y^{12}$	$Y^{59}+Y^6+Y^{12}+Y^{14}$	$Y^{59}+Y^6+Y^{12}+Y^{14}+Y^{15}$	$Y^5+Y^6+Y^{12}+Y^{14}+Y^{15}$

14.40

2. The new components of the impressed current vector, entering and leaving the junction-pairs, are by $\mathbf{C}^{-1}\cdot\mathbf{I} = \mathbf{A}_t\cdot\mathbf{I} =$

$$\mathbf{I}' = \begin{array}{ccccc} 1' & 2' & 3' & 4' & 5' \\ \hline -I^7 & -I^7 & & & I^5 \end{array} \qquad 14.41$$

The current impressed across coil Y^7 is part of two impressed

junction-pair currents $I^{1'}$ and $I^{2'}$. The resultant current through junction B (belonging to both junction-pairs) is zero.

3. The equation of current is $\mathbf{I}' = \mathbf{Y}' \cdot \mathbf{E}'$ representing five equations with five variables $\mathbf{E}'$.

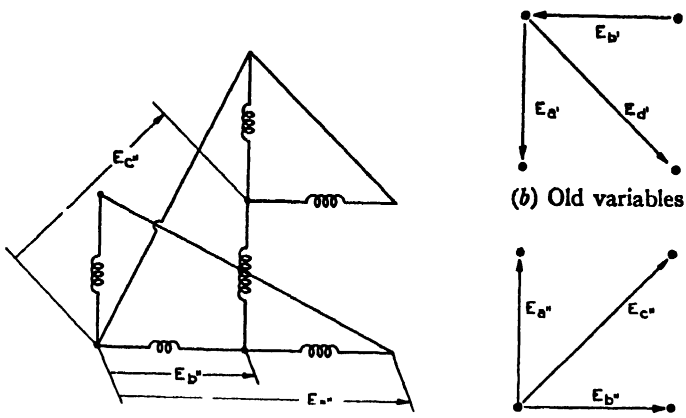
(*f*) If the currents flowing into the junction-pairs $\mathbf{I}'$ are given, the voltages to be applied at the junction-pairs $\mathbf{E}'$ to produce $\mathbf{I}'$ are found by $\mathbf{E}' = \mathbf{Y}^{-1} \cdot \mathbf{I}'$, that is, by calculating the inverse of $\mathbf{Y}'$.

In most problems the invariant equation $\mathbf{I}' = \mathbf{Y}' \cdot \mathbf{E}'$ is subdivided into several invariant equations for further manipulation.

IX. THE TRANSFORMATION OF JUNCTION-PAIRS

(*a*) Just as for mesh networks, when the equation of performance of a junction-network had been expressed along some assumed junction-pairs, ***a new set of junction-pairs may be introduced*** by setting up a new transformation tensor $\mathbf{C}_t^{-1}$.

The necessity of selecting a new set of variables $\mathbf{E}''$ may arise quite often. When a network is used in several different types of problems,



(*a*) Given network (*b*) Old variables (*c*) New variables

FIG. 14.10.—The Transformation of Junction-pairs

the various geometric objects calculated for one type of problem may be used again in several other problems by simply introducing a new transformation tensor, instead of starting each time all over again with the primitive network.

(*b*) As an example of such a transformation let it be assumed in Fig. 14.6 that ***one of the junctions is at ground potential*** and let the difference of potential between the ground and the other junctions be assumed as variables. That is, let the three junction-pairs $E_{a'}$, $E_{b'}$,

and $E_{d'}$ of Fig. 14.6 be replaced by another set of three junction-pairs $E_{a''}$, $E_{b''}$, and $E_{d''}$ as shown in Fig. 14.10*a*.

Since several coils are interconnected by admittanceless branches, there are only four junction-points shown on Figs. 14.10 *b* and *c* between which differences of potentials can be assumed to exist. From these simplified diagrams the relation between the old and the new differences of potentials can be set up as

$$
\begin{aligned}
E_{a'} &= -E_{a''} \\
E_{b'} &= E_{a''} \qquad\qquad - E_{c''} \\
E_{d'} &= -E_{a''} + E_{b''}
\end{aligned}
\qquad
\mathbf{C}_t'^{-1} = \begin{array}{c|ccc} & a'' & b'' & c'' \\ \hline a' & -1 & 0 & 0 \\ b' & 1 & 0 & -1 \\ d' & -1 & 1 & 0 \end{array}
\qquad 14.42
$$

The coefficients of the new variables give $\mathbf{C}_t'^{-1}$.

(*c*) If the admittance tensor $\mathbf{Y}''$ of the network along this second reference frame is desired, it may be found by $\mathbf{C}'^{-1}\cdot\mathbf{Y}'\cdot\mathbf{C}_t'^{-1}$, where $\mathbf{Y}'$ is given in equation 14.33

$$
\mathbf{Y}'' = \begin{array}{c|ccc} & a'' & b'' & c'' \\ \hline a'' & Y^{bb} + Y^{cc} & -Y^{cg} - Y^{bb} + Y^{bd} & -Y^{bd} \\ b'' & -Y^{gc} - Y^{bb} + Y^{db} & Y^{bb} - Y^{bd} - Y^{db} + Y^{dd} + Y^{gg} & Y^{bd} - Y^{dd} \\ c'' & -Y^{db} & Y^{db} - Y^{dd} & Y^{aa} + Y^{dd} + Y^{ff} \end{array}
\qquad 14.43
$$

(*d*) The new current vector is by $\mathbf{C}'^{-1}\cdot\mathbf{I}'$ =

$$
= \mathbf{I}'' = \begin{array}{ccc} a'' & b'' & c'' \\ \hline I^b - I^c & -I^b + I^d + I^g & I^a - I^d - I^f \end{array}
\qquad 14.44
$$

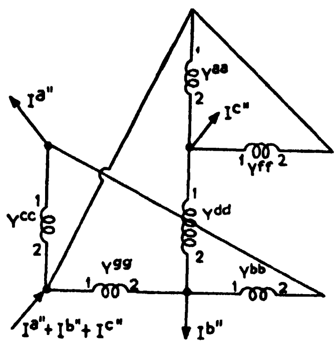


FIG. 14.11.—Impressed Junction-pair Currents

representing the currents flowing out of the various junction-pairs and flowing into the common ground as shown in Fig. 14.11.

If the currents $\mathbf{I}''$ flowing into the junctions are known, the differences of potential appearing between the ground and the other junctions are found by $\mathbf{E}'' = \mathbf{Y}''^{-1}\cdot\mathbf{I}''$ by calculating the inverse of $\mathbf{Y}''$.

X. NUMERICAL EXAMPLE

(*a*) As a numerical example let the admittances be pure conductances $Y^{aa} = 2$, $Y^{bb} = 5$, $Y^{cc} = 6$, $Y^{dd} = 7$, $Y^{ff} = 8$, $Y^{gg} = 9$, and let the three junction-pair currents be

$$\mathbf{I}'' = \begin{array}{ccc} \mathrm{a}'' & \mathrm{b}'' & \mathrm{c}'' \\ \hline 1 & 3 & 4 \end{array} \tag{14.45}$$

Then the admittance tensor $\mathbf{Y}''$ and its inverse are, by substituting into equation 14.43,

$$\mathbf{Y}'' = \begin{array}{c|ccc} & \mathrm{a}'' & \mathrm{b}'' & \mathrm{c}'' \\ \hline \mathrm{a}'' & 11 & -5 & 0 \\ \mathrm{b}'' & -5 & 21 & -7 \\ \mathrm{c}'' & 0 & -7 & 17 \end{array} \quad 14.46 \qquad \mathbf{Y}''^{-1} = \begin{array}{c|ccc} & \mathrm{a}'' & \mathrm{b}'' & \mathrm{c}'' \\ \hline \mathrm{a}'' & 0.1094 & 0.02868 & 0.01181 \\ \mathrm{b}'' & 0.02868 & 0.06311 & 0.026 \\ \mathrm{c}'' & 0.01181 & 0.026 & 0.06952 \end{array} \quad 14.47$$

The differences of potential above the ground are, by $\mathbf{Y}''^{-1}\cdot\mathbf{I}'' =$

$$\mathbf{E}'' = \begin{array}{ccc} \mathrm{a}'' & \mathrm{b}'' & \mathrm{c}'' \\ \hline 0.23725 & 0.32137 & 0.36987 \end{array} \tag{14.48}$$

(*b*) The differences of potential across the *individual coils* cannot be found by $\mathbf{E}' = \mathbf{C}_t'^{-1}\cdot\mathbf{E}''$, since not $\mathbf{E}'$ but $\mathbf{E}$ is the difference of potential across the individual coils. However, $\mathbf{E}$ may be found by $\mathbf{E} = \mathbf{C}_t^{-1}\cdot\mathbf{C}_t'^{-1}\cdot\mathbf{E}''$ since $\mathbf{E} = \mathbf{C}_t^{-1}\cdot\mathbf{E}'$, where $\mathbf{C}_t^{-1}$ is given in equation 14.34. The product of the two transformation tensors of equations 14.34 and 14.42 is

$$\mathbf{C}_t''^{-1} = \mathbf{C}_t^{-1}\cdot\mathbf{C}_t'^{-1} = \begin{array}{c|ccc} & \mathrm{a}'' & \mathrm{b}'' & \mathrm{c}'' \\ \hline \mathrm{a} & 0 & 0 & 1 \\ \mathrm{b} & 1 & -1 & 0 \\ \mathrm{c} & -1 & 0 & 0 \\ \mathrm{d} & 0 & 1 & -1 \\ \mathrm{f} & 0 & 0 & -1 \\ \mathrm{g} & 0 & 1 & 0 \end{array} \tag{14.49}$$

giving the differences of potential across the individual coils by

$$\mathbf{E} = \mathbf{C}_t''^{-1} \cdot \mathbf{E}''$$

	a	b	c	d	f	g	
E =	0.36787	−0.08471	−0.23725	−0.04590	−0.36787	0.32197	14.50

The currents $\mathbf{I}_c$ flowing in each individual coil are by $\mathbf{I}_c = \mathbf{Y} \cdot \mathbf{E} = \mathbf{Y} \cdot \mathbf{C}_t''^{-1} \cdot \mathbf{E}''$

	a	b	c	d	f	g	
$\mathbf{I}_c$ =	0.73574	−0.42356	−1.42356	−0.321295	−2.94296	2.89774	14.51

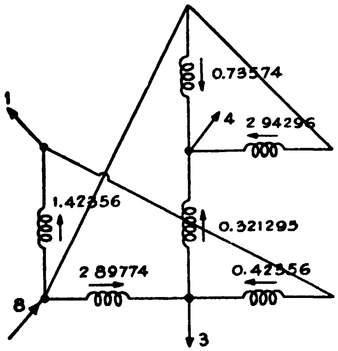


FIG. 14.12.—Currents in Individual Coils

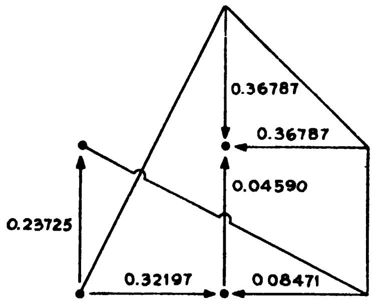


FIG. 14.13.—Voltage-drops across Individual Coils

(*c*) The correctness of **E** and $\mathbf{I}_c$ may be checked by plotting along each coil of the network the newly found currents and differences of potentials, as shown in Figs. 14.12 and 14.13. They satisfy Kirchhoff's laws.

XI. PHYSICAL INTERPRETATION OF THE INVERSE TRANSFORMATION TENSOR

(*a*) It was shown in Section XVII, Chapter VI, that when a network is interconnected into another network the components of **C** correlate the *meshes* of the two networks. *The inverse transformation tensor* $\mathbf{C}_t^{-1}$ *(containing only integers) similarly correlates the junction-pairs of the two networks.* In particular:

1. The *columns* of $\mathbf{C}_t^{-1}$ enumerate the old junction-pairs whose junctions build up the new junction-pairs.

2. The *rows* of $\mathbf{C}_t^{-1}$ enumerate the new junction-pairs whose junctions build up the old junction-pairs.

(*b*) When the old network is the primitive network, then:

1. *The columns of* $\mathbf{C}_t^{-1}$ *enumerate the coils whose two junctions build up the new junction-pairs.*

2. The *rows* of $\mathbf{C}_t^{-1}$ enumerate the new junction pairs to which the junctions of each coil belong.

These physical interpretations play an important part in network synthesis where $\mathbf{C}_t^{-1}$ is given and the network is to be established.

(*c*) *Considering the electromagnetic quantities*:

1. The *columns* of $\mathbf{C}_t^{-1}$ enumerate the old junction pairs whose currents **I** add up to form the new junction-pair currents **I**′.

2. The *rows* of $\mathbf{C}_t^{-1}$ enumerate the new junction-pairs whose voltages **E**′ build up the old junction-pair voltages **E**.

(*d*) When the old network is the primitive network, then:

1. The *columns* of $\mathbf{C}_t^{-1}$ enumerate the coils whose currents **I** build up the new junction-pair currents **I**′.

2. The rows of $\mathbf{C}_t^{-1}$ enumerate the new junction-pairs whose voltages **E**′ add up to form the voltage **E** across the individual coils.

The physical interpretation for each group of transformation matrices is different. The present interpretation is valid only for the group G_t (Section VI*b*, Chapter XI) representing *interconnection* of coils and *containing only integers.*

It should be remembered that, in setting up the transformation tensors **C** or $\mathbf{C}_t^{-1}$ containing integers, only Kirchhoff's First or Second Law is used.

XII. INTERCONNECTION OF NETWORKS

(*a*) Instead of interconnecting individual coils into a system, *it is possible to take several independent junction networks and interconnect them into one resultant junction network with the aid of a* $\mathbf{C}_t^{-1}$. Again the individual systems to be interconnected may be stationary or moving, electrical, mechanical, acoustical, etc., systems. The interconnection of *junction* networks consists of short-circuiting junction-pairs (A-A') of each network with the other *in pairs* as shown in Fig. 14.14. The interconnection changes two junction-pairs into one.

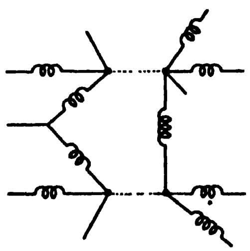


FIG. 14.14.—Interconnection of Junction-pairs

Instead of interconnecting several independent networks, *one system may be divided into several component systems by separating junction-pairs*, and each component network may be analyzed separately, then recombined into one system by a $\mathbf{C}_t^{-1}$.

(*b*) *In interconnecting entire systems again the same steps are followed as in interconnecting individual coils*, namely:

1. Find the geometric objects **I**, **Y**, etc., of the primitive system consisting of the individual systems.

The admittance tensor of the primitive system is

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \qquad 14.52$$

The resultant **Y** has as many axes as the sum of the axes of the component system.

The current vector of the primitive system is

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \cdots \qquad 14.53$$

2. Find the transformation tensor $\mathbf{C}_t^{-1} \doteq C_\alpha^{\alpha'}$, showing the manner of interconnection of the component systems.

3. Find the new components of the geometric objects of the resultant system by

$$\boxed{\mathbf{Y}' = \mathbf{C}^{-1}\cdot(\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \cdots)\cdot\mathbf{C}_t^{-1}} \qquad 14.54$$

$$\boxed{\mathbf{I}' = \mathbf{C}^{-1}\cdot(\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \cdots)} \qquad 14.55$$

The equation of current of the resultant system is $\mathbf{I}' = \mathbf{Y}'\cdot\mathbf{E}'$.

If the axes of some of the component systems $\mathbf{Y}_1$ or $\mathbf{Y}_2$, etc., do not contain some of the junction-pairs that are interconnected, a new set of variables is introduced first that includes the needed junction-pairs.

XIII. EXAMPLE OF INTERCONNECTION OF TWO NETWORKS

(*a*) Let the two junction networks of Figs. 14.6 and 14.8 be interconnected as shown in Fig. 14.15. The junction-pairs that are inter-

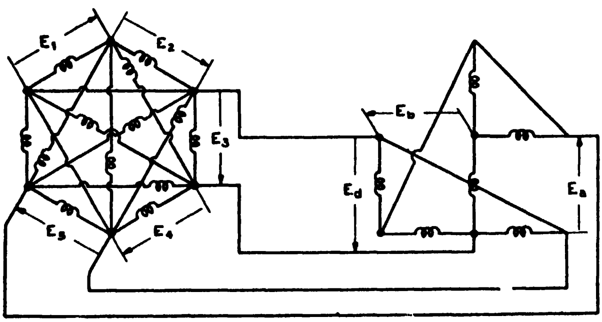


Fig. 14.15.—Interconnected Junction-networks

connected are $E_{5'}$ to $E_{a'}$ and $E_{3'}$ to $E_{d'}$. The other junction-pairs remain unconnected.

The admittance tensors of the component systems are given in equations 14.35 and 14.40. Their sum $\mathbf{Y}_1 + \mathbf{Y}_2$ is the admittance tensor of the primitive system (replacing each component by a single symbol)

$$\mathbf{Y} = \begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & a & b & d \\ \hline 1 & Y^{11} & Y^{12} & Y^{13} & Y^{14} & Y^{15} & & & \\ 2 & Y^{21} & Y^{22} & Y^{23} & Y^{24} & Y^{25} & & & \\ 3 & Y^{31} & Y^{32} & Y^{33} & Y^{34} & Y^{35} & & & \\ 4 & Y^{41} & Y^{42} & Y^{43} & Y^{44} & Y^{45} & & & \\ 5 & Y^{51} & Y^{52} & Y^{53} & Y^{54} & Y^{55} & & & \\ a & & & & & & Y^{aa} & Y^{ab} & Y^{ad} \\ b & & & & & & Y^{ba} & Y^{bb} & Y^{bd} \\ d & & & & & & Y^{da} & Y^{db} & Y^{dd} \end{array} \qquad 14.56$$

The current vector along the primitive system is

$$\mathbf{I} = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & a & b & d \\ \hline I^1 & I^2 & I^3 & I^4 & I^5 & I^a & I^b & I^d \end{array} \qquad 14.57$$

(*b*) The transformation tensor may be set up by noting that the interconnection of the two networks consists of:

1. Changing E_5 and E_a into one junction-pair, say $E_{m'}$, also changing E_3 and E_d into one junction-pair, say $E_{n'}$.

2. Leaving the other junction-pairs of both networks unchanged.

Setting up a relation between the old and the new junction-pairs,

$$\begin{aligned} E_1 &= E_1' \\ E_2 &= E_2' \\ E_3 &= E_{n'} \\ E_4 &= E_4' \\ E_5 &= E_{m'} \\ E_a &= E_{m'} \\ E_b &= E_{b'} \\ E_d &= E_{n'} \end{aligned} \qquad \mathbf{C}_t^{-1} = \begin{array}{c|cccccc} & 1' & 2' & 4' & b' & m' & n' \\ \hline 1 & 1 & & & & & \\ 2 & & 1 & & & & \\ 3 & & & & & & 1 \\ 4 & & & 1 & & & \\ 5 & & & & & 1 & \\ a & & & & & 1 & \\ b & & & & 1 & & \\ d & & & & & & 1 \end{array} \qquad 14.58$$

The coefficients of the new variables represent $\mathbf{C}_t^{-1}$.

(*c*) The admittance tensor of the resultant system, by $\mathbf{C}\ {}^{1}\cdot\mathbf{Y}\cdot\mathbf{C}_t^{-1} = \mathbf{A}_t\cdot\mathbf{Y}\cdot\mathbf{A}$, is

$$\mathbf{Y}' = \begin{array}{c|c|c|c|c|c|c|}
 & 1' & 2' & 4' & b' & m' & n' \\
\hline
1' & y^{11} & y^{12} & y^{14} & & y^{15} & y^{13} \\
\hline
2' & y^{21} & y^{22} & y^{24} & & y^{25} & y^{23} \\
\hline
4' & y^{41} & y^{42} & y^{44} & & y^{45} & y^{43} \\
\hline
b' & & & & y^{bb} & y^{ba} & y^{bd} \\
\hline
m' & y^{51} & y^{52} & y^{54} & y^{ab} & y^{55} + y^{aa} & y^{53} + y^{ad} \\
\hline
n' & y^{31} & y^{32} & y^{34} & y^{db} & y^{35} + y^{da} & y^{33} + y^{dd} \\
\hline
\end{array} \qquad 14.59$$

The current vector is by $\mathbf{I}' = \mathbf{C}^{-1}\cdot\mathbf{I} = \mathbf{A}_t\cdot\mathbf{I} =$

$$\mathbf{I}' = \begin{array}{|c|c|c|c|c|c|}
1' & 2' & 4' & m' & m' & n' \\
\hline
I^1 & I^2 & I^4 & I^b & I^5 + I^a & I^3 + I^d \\
\hline
\end{array} \qquad 14.60$$

(*d*) When the component systems are expressed along the axes that are interconnected (as in the present example), then the step of setting up $\mathbf{C}_t^{-1}$ may be dispensed with and *the resultant* $\mathbf{Y}'$ *may be found as* $\mathbf{Y}_1 + \mathbf{Y}_2$ *if the axes to be interconnected are denoted by the same fixed indices in* $\mathbf{Y}_1$ *and in* $\mathbf{Y}_2$.

XIV. DUAL FORM OF THE REDUCTION FORMULAS

(*a*) Analogously to mesh networks where the single tensor equation $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ can be replaced by a set of n tensor equations, similarly in junction networks the single tensor equation $\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}$ can be replaced by a set of n tensor equations

$$\begin{aligned}
\mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 + \mathbf{Y}^{12}\cdot\mathbf{E}_2 + \mathbf{Y}^{13}\cdot\mathbf{E}_3 + \cdots \mathbf{Y}^{1n}\cdot\mathbf{E}_n \\
\mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 + \mathbf{Y}^{22}\cdot\mathbf{E}_2 + \mathbf{Y}^{23}\cdot\mathbf{E}_3 + \cdots \mathbf{Y}^{2n}\cdot\mathbf{E}_n \\
\mathbf{I}^3 &= \mathbf{Y}^{31}\cdot\mathbf{E}_1 + \mathbf{Y}^{32}\cdot\mathbf{E}_2 + \mathbf{Y}^{33}\cdot\mathbf{E}_3 + \cdots \mathbf{Y}^{3n}\cdot\mathbf{E}_n \qquad 14.61 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbf{I}^n &= \mathbf{Y}^{n1}\cdot\mathbf{E}_1 + \mathbf{Y}^{n2}\cdot\mathbf{E}_2 + \mathbf{Y}^{n3}\cdot\mathbf{E}_3 + \cdots \mathbf{Y}^{nn}\cdot\mathbf{E}
\end{aligned}$$

The manipulation of these equations is shown in Chapters IX and X.

All equations, theorems, etc., developed in Chapter X for the set of linear equations $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ *are equally valid for the set of linear equations* $\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}$ *by simply interchanging the dual quantities,* namely $\mathbf{e}$ by $\mathbf{I}$, $\mathbf{i}$ by $\mathbf{E}$, $\mathbf{z}$ by $\mathbf{Y}$, etc.

(*b*) For instance, if some of the junction-pairs have no currents

flowing through them, the corresponding terminal voltages $\mathbf{E}_2$ may be eliminated from the two tensor equations

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^1 \cdot \mathbf{E}_1 + \mathbf{Y}^2 \cdot \mathbf{E}_2 \\ 0 &= \mathbf{Y}^3 \cdot \mathbf{E}_1 + \mathbf{Y}^4 \cdot \mathbf{E}_2 \end{aligned} \qquad 14.62$$

by replacing the short-circuit self-admittance $\mathbf{Y}^1$ of the first group of junction-pairs by the "open-circuit self-admittance"

$$\boxed{\mathbf{Y}^{1'} = \mathbf{Y}^1 - \mathbf{Y}^2 . \mathbf{Y}^{4-1} . \mathbf{Y}^3} \qquad 14.63$$

(*c*) If the second group of junction-pairs have currents $\mathbf{I}^2$ flowing through them, their presence can still be ignored by replacing the current $\mathbf{I}^1$ entering the first group of junction-pairs by an equivalent current $\mathbf{I}^{1'}$ where

$$\boxed{\mathbf{I}^{1'} = \mathbf{I}^1 - \mathbf{Y}^2 . \mathbf{Y}^{4-1} . \mathbf{I}^2} \qquad 14.64$$

The additional part of this current $-\mathbf{Y}^2 \cdot \mathbf{Y}^{4-1} \cdot \mathbf{I}^2$ is the *short-circuit current* flowing in the first group due to the presence of $\mathbf{I}_2$ in the second group. (Since $\mathbf{Y}^{4-1} \cdot \mathbf{I}^2$ are the junction-pair voltages of the second group with the junction-pairs of the first group short-circuited, and $\mathbf{Y}^2$ times this voltage is the current induced in the first group.) This equation represents the generalization of the *dual* form of Thévenin's theorem. The difference of potential across the eliminated junction-pairs is

$$\mathbf{E}_2' = -\mathbf{Y}^{4-1} \cdot \mathbf{Y}^3 \cdot \mathbf{E}_1 \qquad 14.65$$

(*d*) Whenever the reduction formulas are used to eliminate several covariant variables E_u *at one step* from $\mathbf{I} = \mathbf{Y} \cdot \mathbf{E}$, *the elimination of variables is physically equivalent to eliminating several junction-pairs by mesh-star transformations.* The elimination is possible even if mutual admittances exist between all junction-pairs and if impressed currents exist across the eliminated ones.

(*e*) *The admittance of a network between any two* (or more) *points* may be found by assuming these two points as one of the junction-pairs, then setting up $\mathbf{I} = \mathbf{Y} \cdot \mathbf{E}$, and finally eliminating all axes except those that are needed.

CHAPTER XV

MULTIELECTRODE-TUBE CIRCUITS

I. NON-LINEAR SYSTEMS

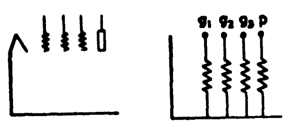
In the various networks so far considered it has been assumed that a *linear* relation exists between the impressed and the response quantities, that is, their equation of performance was considered as having the form $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ or $\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}$, where the proportionality factor $\mathbf{z}$ or $\mathbf{Y}$ was not a function of $\mathbf{i}$ or $\mathbf{E}$.

However, in numerous types of networks the relation between the impressed and response quantities is *non-linear*; for instance, it may have the form $e = zi + wi^2$ in each coil, so that no such linear relation can be established as $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ for the network and *the equations of performance previously considered have to be extended.*

If the variation of the electromagnetic quantities in the non-linear system is sufficiently small, the relation between the impressed and response quantities (both being very small) *is linear* and the hitherto developed equations are still valid. Such a case occurs often, for instance in multielectrode-tube circuits when they act as amplifiers or oscillators and the voltages and currents vary through a small range only.

II. THE n-ELECTRODE TUBE

(*a*) An n-electrode tube may be considered as an all-mesh network or as an all-junction network. Similarly the connected circuit may be analyzed as a mesh or a junction or an orthogonal network. *In most practical applications the tube circuit has fewer junction-pairs than meshes,* hence only the junction point of view will be analyzed in detail.



(*a*) Pentode (*b*) Equivalent Junction Network

FIG. 15.1.

(*b*) Considering a tube as an all-junction network, *it has n junctions, each electrode representing a junction* (Fig. 15.1). The filament acts as a common junction from which the other electrodes branch out, so that each "coil" of the all-junction network consists of the electronic path between the filament and an electrode (the filament serving as a source of electrons).

One of the electrodes is the plate, the others are the grids; but from an analytical point of view there is no difference between a plate and a grid. They all act as junctions. The filament-heating current does not appear in the network, since its value is not pertinent to the analysis to follow. The filament appears only as a lead at the common junction-point.

(c) The all-junction network has $n - 1$ junction-pairs (number of junctions minus number of sub-networks). *The two junctions of each of the $n - 1$ electronic paths ("coils") will be considered as a junction-pair* so that **I** represents the instantaneous currents flowing from the filament to the various electrodes and **E** represents the instantaneous differences of potential appearing between the various electrodes and the filament.

(d) When a d-c. voltage E_a is applied to one of the junction-pairs (electrodes), d-c. currents I^a, I^b, $I^c \cdots$ appear across each of the junction-pairs. As E_a varies, the currents I^a, $I^b \cdots$ also vary, but *not proportionately*. Their variation depends on the value of the d-c. voltages E_b, $E_c \cdots$ existing across the other electrodes, or on the tube itself, etc. The curves $I^a = f^a(E_a)$, $I^b = f^b(E_a)$, $I^c = f^c(E_a)$, etc., are called the "*static characteristic curves*" of the tube.

For each electrode it is possible to draw n such curves for every given filament current and for every given d-c. voltage across the other electrodes. It will be assumed in the following that all these curves $I = f(E)$ are known.

(e) The equations of a five-electrode tube (pentode) will be developed in the following, representing an n-electrode tube.

III. EQUATIONS OF THE TUBE

(a) On a pentode let four d-c. voltages be applied across the four junction-pairs

$$\mathbf{E}_u \doteq \begin{array}{c|c|c|c} a & b & c & p \\ \hline E_a & E_b & E_c & E_p \end{array} \qquad 15.1$$

The four d-c. currents flowing into the junction-pairs are

$$I^u = \begin{array}{c|c|c|c} a & b & c & p \\ \hline I^a & I^b & I^c & I^p \end{array} \qquad 15.2$$

Assume now that one of the terminal voltages, say E_a, varies by a small amount from E_a to $E_a + \Delta E_a$. Then all four currents will also

vary. One of the currents, say I^a, changes from I^a to $I^a + \Delta I^a$. If the curvature of the curve $I^a = f(E_a)$ in the neighborhood of the given d-c. values of E_a and I^a is neglected, then ΔI^a is found from the given value of ΔE_a by the formula

$$\Delta I^a = \frac{\partial I^a}{\partial E_a} \Delta E_a \qquad 15.3$$

where $\partial I^a/\partial E_a$ represents the slope of the curve $I^a = f(E_a)$ at the given d-c. values of I^a and E_a. This slope also is assumed to be known (Fig. 15.2).

Similarly, with the variation of E_a, the other currents also vary by an amount depending on ΔE_a and the slope of the particular $I^a = f(E_a)$ curve. The amount of their change is

$$\Delta I^b = \frac{\partial I^b}{\partial E_a} \Delta E_a$$

$$\Delta I^c = \frac{\partial I^c}{\partial E_a} \Delta E_a$$

$$\Delta I^p = \frac{\partial I^p}{\partial E_a} \Delta E_a$$

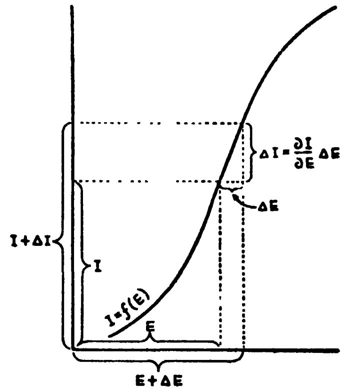


FIG. 15.2.—Static Characteristic Curve

(b) If all four voltages vary, the change in I^a is

$$\Delta I^a = \frac{\partial I^a}{\partial E_a} \Delta E_a + \frac{\partial I^a}{\partial E_b} \Delta E_b + \frac{\partial I^a}{\partial E_c} \Delta E_c + \frac{\partial I^a}{\partial E_p} \Delta E_p$$

because each small ΔE produces its own ΔI irrespective of the presence of other small voltage changes.

(c) Similar equations apply for other current changes, so that the following four linear equations can be set up, representing the change in the currents due to changes in the voltages

$$\begin{aligned}
\Delta I^a &= \frac{\partial I^a}{\partial E_a} \Delta E_a + \frac{\partial I^a}{\partial E_b} \Delta E_b + \frac{\partial I^a}{\partial E_c} \Delta E_c + \frac{\partial I^a}{\partial E_p} \Delta E_p \\
\Delta I^b &= \frac{\partial I^b}{\partial E_a} \Delta E_a + \frac{\partial I^b}{\partial E_b} \Delta E_b + \frac{\partial I^b}{\partial E_c} \Delta E_c + \frac{\partial I^b}{\partial E_p} \Delta E_p \\
\Delta I^c &= \frac{\partial I^c}{\partial E_a} \Delta E_a + \frac{\partial I^c}{\partial E_b} \Delta E_b + \frac{\partial I^c}{\partial E_c} \Delta E_c + \frac{\partial I^c}{\partial E_p} \Delta E_p \\
\Delta I^p &= \frac{\partial I^p}{\partial E_a} \Delta E_a + \frac{\partial I^p}{\partial E_b} \Delta E_b + \frac{\partial I^p}{\partial E_c} \Delta E_c + \frac{\partial I^p}{\partial E_p} \Delta E_p
\end{aligned} \qquad 15.4$$

IV. "BASIC" AND "DERIVED" TENSORS

(*a*) The appearance of derivatives in the above basic equations of tubes requires a more detailed discussion of the differentiation of geometric objects.

All geometric objects considered in the previous chapters, namely **e**, **i**, **z**, and **I**, **E**, **Y**, have been established *by definition*. That is, they were not *derived* from other geometric objects but they themselves served as the minimum number of *basic geometric objects* necessary for the definition of the electromagnetic phenomenon appearing on linear networks.

When the minimum number of *basic* geometric objects of a physical system have once been established, it is possible to derive numerous other geometric objects, called "*derived geometric objects*," from the basic geometric objects by various means. *One of the purposes of tensor analysis is to establish routine procedures for discovering new "derived tensors" from given "basic tensors."*

(*b*) In physical systems where the group of transformation matrices **C** contains only *constant* components as in networks, *one way of finding new tensors is to differentiate the various basic tensors with respect to the variables*, if they are functions of the variables.

It is emphasized, however, that, if the components of the transformation tensors **C** are not constant, the method of *ordinary* differentiation does not yield new tensors. (For such cases tensor analysis has developed a new type of differentiation, called "*absolute differentiation*" or "*covariant differentiation*," that does yield a new derived tensor from a given basic tensor. Since in this volume all transformation tensors **C** have constant components, no need will arise to introduce absolute differentiation.)

The basic tensors that so far have been introduced in non-linear tube circuits are **E** and **I** of equation 15.1 and 15.2, where $\mathbf{E} = E_u$ is the covariant variable. *Since* **I** *is a function of* **E**, *new tensors may be derived by differentiating* **I** *with respect to* **E** *several times in succession.*

V. BUILDING NEW TENSORS BY DIFFERENTIATION

It should be recalled from Section XVII, Chapter I, that:

1. *An n-tensor is differentiated with respect to a vector by differentiating each component of the n-tensor with respect to each component of the vector.*

For instance, if the 2-tensor has n^2 components, each of its components is differentiated n times (if the vector has n components) giving a new tensor with n^3 components.

2. *Whenever a tensor of valence n is differentiated with respect to a vector, the valence of the resultant is $n + 1$.*

The following statement should be added:

3. *If the vector is a covariant vector, that is, having a lower index, the resultant has one additional contravariant (upper) index, and vice versa.*

For instance

$$\frac{\partial Y^{uv}}{\partial E_w} = M^{uvw} \tag{15.5}$$

This last statement can be proved as follows: Let a vector E_u be differentiated with respect to I^v. If u, v, w represent the indices of the old axes and u', v', w' of the new axes, then

$$\frac{\partial E_u}{\partial I^v} = \frac{\partial E_u}{\partial I^{v'}} \frac{\partial I^{v'}}{\partial I^v}$$

However, by equation 6.12

$$\frac{\partial I^{v'}}{\partial I^v} = C_v^{v'} \tag{15.6}$$

therefore

$$\frac{\partial E_u}{\partial I^v} = \frac{\partial E_u}{\partial I^{v'}} C_v^{v'} = Z_{uv'} C_v^{v'} = Z_{uv} \tag{15.7}$$

Hence the new tensor formed from E_u by differentiation with respect to I^v has $1 + 1 = 2$ indices, the second extra index being a covariant (lower) index v. Its other index u is the same as that of the original vector.

The following statement may also be added:

4. *If a tensor is differentiated with respect to a vector, in general the resultant is not a tensor.*

For instance, let a tensor E_u be differentiated. That E_u is a tensor is expressed by writing its formula of transformation and performing all manipulations on that formula. Let

$$E_u = E_{u'} C_u^{u'} \tag{15.8}$$

Differentiating both sides of the equation with respect to I^v

$$\frac{\partial E_u}{\partial I^v} = \frac{\partial E_{u'}}{\partial I^v} C_u^{u'} + E_{u'} \frac{\partial C_u^{u'}}{\partial I^v} \tag{15.9}$$

for a product of geometric objects is differentiated by differentiating each geometric object separately.

$$\frac{\partial E_u}{\partial I^v} = \frac{\partial E_{u'}}{\partial I^{v'}} \frac{\partial I_{v'}}{\partial I^v} C_u^{u'} + E_{u'} \frac{\partial C_u^{u'}}{\partial I^v}$$

By equation 15.6

$$\frac{\partial E_u}{\partial I^v} = \frac{\partial E_{u'}}{\partial I^{v'}} C_v^{v'} C_u^{u'} + E_{u'} \frac{\partial C_u^{u'}}{\partial I^v}$$

and by equation 15.7

$$Z_{uv} = Z_{u'v'} C_v^{v'} C_u^{u'} + E_{u'} \frac{\partial C_u^{u'}}{\partial I^v} \qquad 15.10$$

Hence the transformation formula of $\partial E_u/\partial I^v = Z_{uv}$ is not that of a tensor of valence two, owing to the presence of the last term.

For the present purpose it may be stated that:

5. *Where the transformation is linear, that is, where the transformation tensor has only constant components, the derivatives of all tensors are tensors,* because all additional expressions in their transformation formulas contain $\partial C_u^{u'}/\partial I^v$, which are all zero.

The last two statements are valid not only for *derivatives* of tensors, but also for *differentials* of tensors. For instance, the differential of a vector (1-tensor) in general is not a tensor (shown in Section X*a*, Chapter VII) so that

$$\Delta I^u = \Delta I^{u'} C_{u'}^u + \frac{\partial C_{u'}^u}{\partial I^{v'}} I^{u'} \Delta I^{v'} \qquad 15.11$$

In the present analysis the last term drops out, for $\partial C_{u'}^u/\partial I^{v'} = 0$, showing that ΔI^u is transformed here as a contravariant tensor of valence one.

VI. THE ADMITTANCE TENSOR

(*a*) The transformations to be considered have only constant components; therefore, in the four linear equations 15.4 the various ΔI can be considered as the components of a contravariant vector $\Delta\mathbf{I} = \Delta I^u$, that is,

$$\Delta I^u = \begin{array}{c} u \\ \begin{array}{|c|c|c|c|} \hline a & b & c & p \\ \hline \Delta I^a & \Delta I^b & \Delta I^c & \Delta I^p \\ \hline \end{array} \end{array} \qquad 15.12$$

The various ΔE can be considered as the components of a covariant vector $\Delta\mathbf{E} = \Delta E_u$, that is

$$\Delta E_u = \begin{array}{c} u \\ \begin{array}{|c|c|c|c|} \hline a & b & c & p \\ \hline \Delta E_a & \Delta E_b & \Delta E_c & \Delta E_p \\ \hline \end{array} \end{array} \qquad 15.13$$

The coefficients of E_u form the components of a twice contravariant

tensor $\mathbf{Y} = Y^{uv}$, called the "admittance tensor," since they are the derivatives of a vector I^u with respect to the vector E_v

$$Y^{uv} = \begin{array}{c|cccc} u \backslash v & a & b & c & p \\ \hline a & \frac{\partial I^a}{\partial E_a} & \frac{\partial I^a}{\partial E_b} & \frac{\partial I^a}{\partial E_c} & \frac{\partial I^a}{\partial E_p} \\ b & \frac{\partial I^b}{\partial E_a} & \frac{\partial I^b}{\partial E_b} & \frac{\partial I^b}{\partial E_c} & \frac{\partial I^b}{\partial E_p} \\ c & \frac{\partial I^c}{\partial E_a} & \frac{\partial I^c}{\partial E_b} & \frac{\partial I^c}{\partial E_c} & \frac{\partial I^c}{\partial E_p} \\ p & \frac{\partial I^p}{\partial E_a} & \frac{\partial I^p}{\partial E_b} & \frac{\partial I^p}{\partial E_c} & \frac{\partial I^p}{\partial E_p} \end{array} \qquad 15.14$$

so that the equation 15.4 can be written in index notation as

$$\boxed{\Delta I^u = Y^{uv} \Delta E_v} \qquad 15.15$$

(*b*) Y^{uv} can also be written as

$$\boxed{Y^{uv} = \frac{\partial I^u}{\partial E_v}} \qquad 15.16$$

so that equation 15.15 can be written

$$\boxed{\Delta I^u = \frac{\partial I^u}{\partial E_v} \Delta E_v} \qquad 15.17$$

This tensor equation, representing the n linear equations 15.4 with n variables, is similar to equation 15.3 representing one equation with one variable.

(*c*) Each term in the admittance tensor represents the slope of an $I = f(E)$ curve at the operating point, and each term is a real, and not a complex, number. These constants are called self- and mutual conductances (reflex and transconductances, respectively) of the various coils and are denoted by G^{mn}. The admittance tensor 15.14 is written in terms of the conductances

$$Y^{uv} = \begin{array}{c|cccc} u \backslash v & a & b & c & p \\ \hline a & G^{aa} & G^{ab} & G^{ac} & G^{ap} \\ b & G^{ba} & G^{bb} & G^{bc} & G^{bp} \\ c & G^{ca} & G^{cb} & G^{cc} & G^{cp} \\ p & G^{pa} & G^{pb} & G^{pc} & G^{pp} \end{array} \qquad 15.18$$

Many components may be zero, of course.

(*d*) It is customary to define the inverse of a self-conductance as a resistance; that is, $G^{mm} = 1/r_{mm}$, and the ratio of a mutual conductance to a self-conductance is defined as the amplification factor

$$\mu_a^b = \frac{G^{ab}}{G^{aa}} = \frac{\dfrac{\partial I^a}{\partial E_b}}{\dfrac{\partial I^a}{\partial E_a}} = -\frac{dE_a}{dE_b} \qquad 15.19$$

and

$$\mu_b^a = \frac{G^{ba}}{G^{bb}} = -\frac{dE_b}{dE_a} \qquad 15.20$$

so that

$$\boxed{G^{ba} = \frac{\mu_b^a}{r_{bb}}} \qquad 15.21$$

Hence, in terms of the amplification factors and resistances, the admittance tensor of a pentode is

$$Y^{uv} = \begin{array}{c|cccc} {}_u\diagdown^v & a & b & c & p \\ \hline a & \dfrac{1}{r_{aa}} & \dfrac{\mu_a^b}{r_{aa}} & \dfrac{\mu_a^c}{r_{aa}} & \dfrac{\mu_a^p}{r_{aa}} \\ b & \dfrac{\mu_b^a}{r_{bb}} & \dfrac{1}{r_{bb}} & \dfrac{\mu_b^c}{r_{bb}} & \dfrac{\mu_b^p}{r_{bb}} \\ c & \dfrac{\mu_c^a}{r_{cc}} & \dfrac{\mu_c^b}{r_{cc}} & \dfrac{1}{r_{cc}} & \dfrac{\mu_c^p}{r_{cc}} \\ d & \dfrac{\mu_p^a}{r_{pp}} & \dfrac{\mu_p^b}{r_{pp}} & \dfrac{\mu_p^c}{r_{pp}} & \dfrac{1}{r_{pp}} \end{array} \qquad 15.22$$

The admittance tensor of a tetrode is found by omitting the row and column of *c*. If the customary notation is used,

$$Y^{uv} = \begin{array}{c|ccc} {}_u\diagdown^v & a & b & p \\ \hline a & \dfrac{1}{r_a} & \dfrac{\eta_a}{r_a} & \dfrac{\nu_a}{r_a} \\ b & \dfrac{\eta_b}{r_b} & \dfrac{1}{r_b} & \dfrac{\nu_b}{r_b} \\ p & \dfrac{\mu_a}{r_p} & \dfrac{\mu_b}{r_p} & \dfrac{1}{r_p} \end{array} = \begin{array}{c|ccc} {}_u\diagdown^v & a & b & p \\ \hline a & G^{aa} & G^{ab} & G^{ap} \\ b & G^{ba} & G^{bb} & G^{bp} \\ p & G^{pa} & G^{pb} & G^{pp} \end{array} \qquad 15.23$$

where μ is the plate amplification factor.
ν is the grid amplification factor.
η is the cross amplification factor.

The admittance tensor of a triode is found by omitting the rows and columns of b and c

$$Y^{uv} = \begin{array}{c|cc} u \backslash v & g & p \\ \hline g & \dfrac{1}{r_g} & \dfrac{\mu_g}{r_g} \\ p & \dfrac{\mu_p}{r_p} & \dfrac{1}{r_p} \end{array} = \begin{array}{c|cc} u \backslash v & g & p \\ \hline g & G^{gg} & G^{gp} \\ p & G^{pg} & G^{pp} \end{array} \qquad 15.24$$

When the multielectrode tubes act as amplifiers or oscillators in most problems they are treated as triodes. The role of the extra grids is to change the triode conductances to special values.

(*e*) *If in the analysis of triodes the grid current is zero,* r_g is infinite so that *the admittance tensor of such a triode is*

$$Y^{uv} = \begin{array}{c|cc} u \backslash v & g & p \\ \hline g & 0 & 0 \\ p & \dfrac{\mu_p}{r_p} & \dfrac{1}{r_p} \end{array} = \begin{array}{c|cc} u \backslash v & g & p \\ \hline g & 0 & 0 \\ p & G^{pg} & G^{pp} \end{array} \qquad 15.25$$

The admittance tensor of a *tetrode* in the absence of grid currents is

$$Y^{uv} = \begin{array}{c|ccc} u \backslash v & a & b & p \\ \hline a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ p & \dfrac{\mu_a}{r_p} & \dfrac{\mu_b}{r_p} & \dfrac{1}{r_p} \end{array} = \begin{array}{c|ccc} u \backslash v & a & b & p \\ \hline a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ p & G^{pa} & G^{pb} & G^{pp} \end{array} \qquad 15.26$$

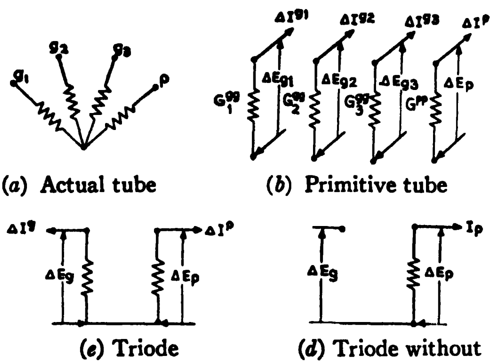
VII. MULTIELECTRODE TUBES AND MULTIWINDING TRANSFORMERS

Since the admittance tensor **Y** is not symmetrical an *n-electrode tube is represented by* $n - 1$ *coils with unilateral mutual conductances between them.* The coils are joined at a common junction representing the filament as shown in Fig. 15.3*a*.

The triode is shown in Fig. 15.3*c* with, and in Fig. 15.3*d* without, a grid coil. *It is interesting to note that an isolated junction-point forms part of the network.*

The representation of n-electrode tubes in a junction network by $n - 1$ coils with mutual conductances between them is analogous to the representation of n-winding transformers in mesh networks by n coils with mutual inductances. However, there are several differences in their analytical form. For instance:

1. In a multiwinding transformer, both the impedance and admittance tensors are symmetrical; that is, the rows and columns can be interchanged. In a thermionic tube the mutual conductances are different in the two directions; in other words, G^{ab} is different from G^{ba}, and the effect of a change in the plate voltage upon a grid current is different from the same change in the grid voltage upon the plate current.



(a) Actual tube (b) Primitive tube

(c) Triode (d) Triode without grid current

FIG. 15.3

2. In a transformer each component in the tensor is a real or a complex number during steady-state conditions. In a tube each component is a real number representing conductances. (It may be mentioned that in a high-frequency tube, where the time of flight of the electrons between electrodes must be considered, the real numbers in the admittance tensor are replaced by complex numbers.)

3. In transformers the **z** or **y** tensors are equally valid for any applied terminal voltage or for a change in the applied voltage, but in a tube or in other non-linear systems they are valid only for small changes in the voltages or currents.

VIII. EQUIVALENT JUNCTION NETWORKS OF TUBES

(a) The equation of current of a triode $\Delta\mathbf{I} = \mathbf{Y}\cdot\Delta\mathbf{E}$ is from equation 15.25

$$\begin{aligned} \Delta I^g &= \frac{1}{r_g}\,\Delta E_g + \frac{\mu_g}{r_g}\,\Delta E_p = G^{gg}\Delta E_g + G^{gp}\Delta E_p \\ \Delta I^p &= \frac{\mu_p}{r_p}\,\Delta E_g + \frac{1}{r_p}\,\Delta E_p = G^{pg}\Delta E_g + G^{pp}\Delta E_p \end{aligned} \qquad 15.27$$

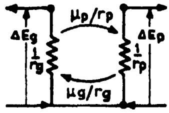
That is, a grid current flows not only if a potential appears on the grid,

but also when a potential ΔE_p appears on the plate (and vice versa) because of the existence of mutual admittance between them.

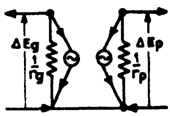
(*b*) Instead of representing the grid and the plate by two coils with asymmetrical mutual admittances μ_g/r_g and μ_p/r_p, with self-admittances $1/r_g$ and $1/r_p$, it is customary to represent them by the same two coils having no mutual admittances. *The role of the asymmetrical mutual admittances is replaced by two current generators* injecting across the grid the current $(\mu_g/r_g)\Delta E_p$ and across the plate $(\mu_p/r_p)\Delta E_g$ as given by the equations and shown in Fig. 15.4*b*.

In the language of junction networks *the unilateral mutual admittance between the two junction-pairs is replaced by currents impressed across the two junction-pairs* as shown in Fig. 15.4*c*. Of course, across the same junction-pairs are also impressed ΔE_g and ΔE_p, respectively.

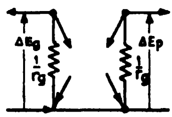
In an n-electrode tube the mutual admittances between the $n - 1$ coils of Fig. 15.3*a* may be replaced by impressing $n - 2$ different sets of currents across each coil.



(*a*) Triode with grid current

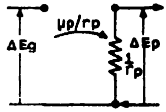


(*b*) Mutual conductance as a current generator

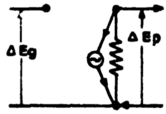


(*c*) Mutual conductance as impressed current

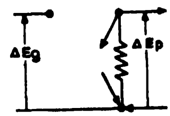
FIG. 15.4



(*a*) Triode without grid current



(*b*) Mutual conductance as a current generator



(*c*) Mutual conductance as impressed current

FIG. 15.5

(*c*) When the grid resistance r_g is made infinite, no grid current flows, $\Delta I_g = 0$, and the equivalent junction network of the *triode* is shown in Fig. 15.5.

IX. THE INTERCONNECTION OF TUBES WITH NETWORKS

(*a*) When a network contains tubes, then *the presence of each tube is represented by two (or more) additional coils, having the self- and mutual admittances of equations* 15.22 *to* 15.26. That is, the primitive junction network contains all the coils of the outside network plus two (or more) coils for each tube. *The transformation tensor is set up and the whole analysis is performed as for any junction network studied in the previous chapter.*

(*b*) Sometimes it is advantageous to divide the whole system first into two component systems comprising the tubes and static networks, then to analyze each of them separately, and finally to recombine them

into the original network by equation 14.59. That is, if the admittance tensor of the tube (or tubes) is $\mathbf{Y}_2$ and that of the static network (or networks) is $\mathbf{Y}_1$, also if the two systems are interconnected by $\mathbf{C}_t^{-1} = \mathbf{A}$, then *the admittance tensor of the resultant network is*

$$\mathbf{Y}' = \mathbf{A}_t \cdot (\mathbf{Y}_1 + \mathbf{Y}_2) \cdot \mathbf{A} \qquad 15.28$$

They are interconnected by placing junction-pairs in parallel.

The static network itself may be decomposed into several parts, then recombined again. For instance, *the feedback couplings between the grids and plate may be analyzed separately* since the analysis of the remaining part of the static network is usually simple. Similarly the tube system itself may be subdivided into several parts depending on the manner of their interconnection. *In the static network many of the coils may be replaced by a single equivalent coil.*

(*c*) Since *each* junction-pair of the tube is connected to corresponding junction-pairs of the outside network, *the effect of the transformation tensor* $\mathbf{C}_t^{-1}$ *is simply to change the* $\mathbf{g}_2$ *and* $\mathbf{p}_2$ *axes of the tube and the* $\mathbf{g}_1$ *and* $\mathbf{p}_1$ *axes of the network to* $\mathbf{g}$ *and* $\mathbf{p}$. *This change is equivalent to simply adding the tensors* $\mathbf{Y}_1$ *and* $\mathbf{Y}_2$.

That is, *if the junction-pairs of the network include the grid and plate axes, the resultant* $\mathbf{Y}'$ *is found as* $\mathbf{Y}_1 + \mathbf{Y}_2$ without going through the process of transformation. To express it in another way, the effect of the transformation is to drop the subscripts of the $\mathbf{g}$ and $\mathbf{p}$ axes.

(*d*) Once the resultant $\mathbf{Y}'$ and thereby the equation $\mathbf{I}' = \mathbf{Y}' \cdot \mathbf{E}'$ has been established, the equation may be subjected to all types of manipulations, some of them to be shown later in Chapter XXI.

In many problems the impressed current $\mathbf{I}'$ is known across *one* junction-pair and the difference of potential $\mathbf{E}'$ is to be found across *another* junction-pair. In such cases *all the inactive rows and columns should be eliminated from* $\mathbf{Y}'$ *by the reduction formulas* of Chapter X, leaving only two rows and columns along the input and output axes. (The elimination of *one* row and column at a time is equivalent to the usual simplification by a mesh-star transformation.)

The resultant two equations are

$$\begin{aligned} \Delta I^a &= Y^{aa}\Delta E_a + Y^{ab}\Delta E_b \\ 0 &= Y^{ba}\Delta E_a + Y^{bb}\Delta E_b \end{aligned} \qquad 15.29$$

where ΔI^a is known and ΔE_b is unknown. Eliminating ΔE_a from the second equation

$$\Delta E_a = -\,(Y^{ba})^{-1}\,Y^{bb}\Delta E_b$$

$$\Delta I^a = \left(\frac{Y^{ab}Y^{ba} - Y^{aa}Y^{bb}}{Y^{ba}}\right)\Delta E_b$$

Hence *the output voltage in terms of the input current is*

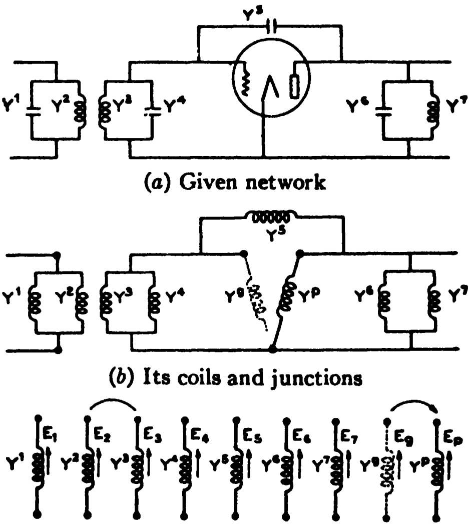
$$\boxed{\Delta E_b = \frac{Y^{ba}}{Y^{ab}Y^{ba} - Y^{aa}Y^{bb}}\Delta I^a} \qquad 15.30$$

In other problems *the ratio of the output to input voltage* is required. In such cases from the second of equation 15.29

$$\boxed{\frac{\Delta E_b}{\Delta E_a} = -\frac{Y^{ba}}{Y^{bb}}} \qquad 15.31$$

X. INTERMEDIARY-FREQUENCY AMPLIFIER

(*a*) Let the circuit of Fig. 15.6*a* be analyzed, representing one stage in an intermediary-frequency amplifier of a receiver, in which the effect of the grid-plate capacity feedback Y^5 is to be investigated.



(*a*) Given network

(*b*) Its coils and junctions

(*c*) Its primitive network

FIG. 15.6.—Intermediary-frequency Amplifier

When the tube is replaced by two coils, the resultant junction network is shown in Fig. 15.6*b*, having nine coils. There are five junctions and two sub-networks, consequently there are $5 - 2 = 3$ junction-pairs and $9 - 3 = 6$ meshes. Hence the network should be analyzed as a junction-network.

(*b*) To find the admittance tensor **Y** of the primitive network it is

necessary to find the admittances of the coils. However, usually only their impedances are known as

$$\mathbf{Z} = \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & Z_1 & & & & & & \\ 2 & & Z_2 & Z_{23} & & & & \\ 3 & & Z_{23} & Z_3 & & & & \\ 4 & & & & Z_4 & & & \\ 5 & & & & & Z_5 & & \\ 6 & & & & & & Z_6 & \\ 7 & & & & & & & Z_7 \\ \hline \end{array}$$

The admittance tensor is found by calculating the inverse of $\mathbf{Z}$. Because of its diagonal form it may be considered as a compound diagonal matrix, in which each matrix has a single component except one that has four components, representing the transformer of coils 2 and 3. *The inverse of a diagonal compound matrix is found by calculating the inverse of each of its components separately* (equation 10.46). Hence the admittance tensor of the outside network is

$$\mathbf{Z}^{-1} = \mathbf{Y} = \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1/Z_1 & & & & & & \\ 2 & & Z_3/D & -Z_{23}/D & & & & \\ 3 & & -Z_{23}/D & Z_2/D & & & & \\ 4 & & & & 1/Z_4 & & & \\ 5 & & & & & 1/Z_5 & & \\ 6 & & & & & & 1/Z_6 & \\ 7 & & & & & & & 1/Z_7 \\ \hline \end{array}$$

where $D = Z_2Z_3 - (Z_{23})^2$.

The admittance tensor of the primitive network of Fig. 15.6*c* has nine rows and columns, as

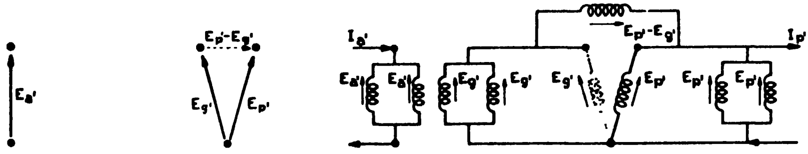
$$\mathbf{Y} = \begin{array}{c|ccccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & g & p \\ \hline 1 & Y^1 & & & & & & & & \\ 2 & & Y^2 & Y^{23} & & & & & & \\ 3 & & Y^{23} & Y^3 & & & & & & \\ 4 & & & & Y^4 & & & & & \\ 5 & & & & & Y^5 & & & & \\ 6 & & & & & & Y^6 & & & \\ 7 & & & & & & & Y^7 & & \\ g & & & & & & & & & \\ p & & & & & & & & G^{pg} & G^{pp} \\ \hline \end{array} \qquad 15.32$$

It should be noted that by equation 15.25 *the row of* **g** *contains only zero components, but not its column.*

The impressed current vector is

$$
\Delta\mathbf{I} = \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & p & g \\ \hline \multicolumn{1}{|c|}{\Delta I^1} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline \end{array}
$$

(*c*) In assuming the three new junction-pairs, they should include, if possible, the input and output terminals, also the filament-grid and



(*a*) The new junction-pairs (*b*) Voltages across individual coils

FIG. 15.7

the filament-plate coils. Assuming the three junction-pairs of Fig. 15.7*a*, the differences of potentials appearing across all coils are shown in Fig. 15.7*b*.

The transformation tensor is set up by equating the old and the new voltages across each coil as (*using E instead of ΔE*)

$$
\begin{aligned}
E_1 &= E_{a'} \\
E_2 &= E_{a'} \\
E_3 &= \quad E_{g'} \\
E_4 &= \quad E_{g'} \\
E_5 &= \quad -E_{g'} + E_{p'} \\
E_6 &= \qquad\qquad E_{p'} \\
E_7 &= \qquad\qquad E_{p'} \\
E_g &= \quad E_{g'} \\
E_p &= \qquad\qquad E_{g'}
\end{aligned}
\qquad
\mathbf{C}_t^{-1} = \mathbf{A} = \begin{array}{c|c|c|c|} & a' & g' & p' \\ \hline 1 & 1 & & \\ \hline 2 & 1 & & \\ \hline 3 & & 1 & \\ \hline 4 & & 1 & \\ \hline 5 & & -1 & \\ \hline 6 & & & 1 \\ \hline 7 & & & 1 \\ \hline g & & 1 & \\ \hline p & & & 1 \\ \hline \end{array} \qquad 15.33
$$

The coefficients of the *new* voltages give the transpose inverse transformation tensor $\mathbf{C}_t^{-1}$.

(*d*) The resultant admittance tensor of the network is by $\mathbf{A}_t \cdot \mathbf{Y} \cdot \mathbf{A}$ =

$$
\mathbf{Y}' = \begin{array}{c|c|c|c|} & a' & g' & p' \\ \hline a' & Y^1 + Y^2 & Y^{23} & 0 \\ \hline g' & Y^{23} & Y^3 + Y^4 + Y^5 & -Y^5 \\ \hline p' & 0 & -Y^5 + G^{pg} & Y^5 + Y^6 + Y^7 + G^{pp} \\ \hline \end{array} \qquad 15.34
$$

The impressed current vector is by $\mathbf{C}^{-1}\cdot\Delta\mathbf{I} = \mathbf{A}_t\cdot\Delta\mathbf{I} =$

$$\Delta\mathbf{I}' = \begin{array}{c|c|c} \mathbf{a}' & \mathbf{g}' & \mathbf{p}' \\ \hline \Delta I^a & 0 & 0 \end{array} \qquad 15.35$$

The differences of potentials (the covariant variables) are

$$\Delta\mathbf{E}' = \begin{array}{c|c|c} \mathbf{a}' & \mathbf{g}' & \mathbf{p}' \\ \hline \Delta E'_a & \Delta E'_g & \Delta E'_p \end{array} \qquad 15.36$$

and the equation of current is $\Delta\mathbf{I}' = \mathbf{Y}'\cdot\Delta\mathbf{E}'$.

(*e*) If the difference of potential across the output $\Delta E'_p$ is to be found, then *axis* $\mathbf{g}'$ *is inactive and the corresponding row and column may be eliminated by the reduction formula.* Arranging axis $\mathbf{g}'$ as the last row

$$\mathbf{Y}' = \begin{array}{c|c|c|c} & \mathbf{a}' & \mathbf{p}' & \mathbf{g}' \\ \hline \mathbf{a}' & Y^1 + Y^2 & 0 & Y^{23} \\ \hline \mathbf{p}' & 0 & Y^5 + Y^6 + Y^7 + G^{pp} & -Y^5 + G^{pg} \\ \hline \mathbf{g}' & Y^{23} & -Y^5 & Y^3 + Y^4 + Y^5 \end{array} \qquad 15.37$$

Eliminating the row of $\mathbf{g}'$ by the admittance reduction formula

$$\mathbf{Y}^{1\prime} = \mathbf{Y}^1 - \mathbf{Y}^2.\mathbf{Y}^{4-1}.\mathbf{Y}^3 \qquad 15.38$$

$$\mathbf{Y}^{1\prime} = \mathbf{Y}^1 - \begin{array}{|c|} \hline Y^{23} \\ \hline -Y^5 + G^{pg} \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 1/(Y^3 + Y^4 + Y^5) \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline Y^{23} & -Y^5 \\ \hline \end{array}$$

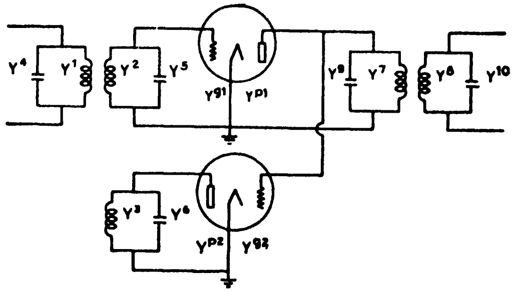
$$= \mathbf{Y}^1 - \begin{array}{|c|c|} \hline \dfrac{Y^{23}Y^{23}}{Y^3 + Y^4 + Y^5} & -\dfrac{Y^{23}Y^5}{Y^3 + Y^4 + Y^5} \\ \hline \dfrac{(-Y^5 + G^{pg})Y^{23}}{Y^3 + Y^4 + Y^5} & -\dfrac{(-Y^5 + G^{pg})Y^5}{Y^3 + Y^4 + Y^5} \\ \hline \end{array}$$

$$\mathbf{Y}^1 = \begin{array}{c|c|c} & \mathbf{a}' & \mathbf{p}' \\ \hline \mathbf{a}' & Y^1 + Y^2 - \dfrac{(Y^{23})^2}{Y^3 + Y^4 + Y^5} & \dfrac{Y^{23}Y^5}{Y^3 + Y^4 + Y^5} \\ \hline \mathbf{p}' & \dfrac{(Y^5 - G^{pg})Y^{23}}{Y^3 + Y^4 + Y^5} & \begin{array}{c} Y^5 + Y^6 + Y^7 + G^{pp} + \\ + \dfrac{(G^{pg} - Y^5)Y^5}{Y^3 + Y^4 + Y^5} \end{array} \end{array} = \begin{array}{c|c|c} & \mathbf{a}' & \mathbf{p}' \\ \hline \mathbf{a}' & Y^{aa} & Y^{ap} \\ \hline \mathbf{p}' & Y^{pa} & Y^{pp} \end{array} \qquad 15.39$$

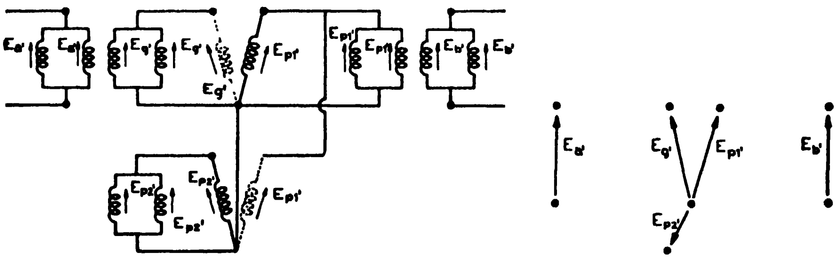
The difference of potential $\Delta E p'$ is found by equation 15.30

XI. FEEDBACK AMPLIFIER

Let the circuit of Fig. 15.8*a* be analyzed. The network has fourteen coils, eight junctions, three sub-networks, hence $8 - 3 = 5$ junction-pairs and $14 - 5 = 9$ meshes.



(*a*) Given network



(*b*) Voltages across individual coils (*c*) Assumed junction-pairs

FIG. 15.8.—Feed-back Amplifier

The admittance tensor of the network is before interconnection

$\mathbf{Y} =$

	1	2	3	4	5	6	7	8	9	10	g_1	p_1	g_2	p_2
1	Y^1	Y^{12}	Y^{13}											
2	Y^{12}	Y^2	Y^{23}											
3	Y^{13}	Y^{23}	Y^3											
4				Y^4										
5					Y^5									
6						Y^6								
7							Y^7	Y^{78}						
8							Y^{78}	Y^8						
9									Y^9					
10										Y^{10}				
g_1														
p_1											G^{pg1}	G^{pp1}		
g_2														
p_2													G^{pg2}	G^{pp2}

15.40

To find **Y** of the outside network from its known **Z**, it is necessary to calculate the inverse of a two- and a three-rowed matrix, representing the impedance tensors of the two transformers.

The assumed five junction-pair voltages are shown in Fig. 15.8*c*, and the differences of potential across the individual coils are shown in Fig. 15.8*d*.

Equating the old and the new voltages across each coil, the transformation tensor (using E instead of ΔE) is

$$
\begin{aligned}
E_1 &= E_{a'} \\
E_2 &= E_{g'} \\
E_3 &= E_{p2'} \\
E_4 &= E_{a'} \\
E_5 &= E_{g'} \\
E_6 &= E_{p2'} \\
E_7 &= E_{p1'} \\
E_8 &= E_b \\
E_9 &= E_{p1'} \\
E_{10} &= E_b \\
E_{g1} &= E_{g'} \\
E_{p1} &= E_{p1'} \\
E_{g2} &= E_{p1'} \\
E_{p2} &= E_{p'2}
\end{aligned}
\qquad
\mathbf{A} = \begin{array}{c|c|c|c|c|c|}
 & \mathrm{a}' & \mathrm{g}' & \mathrm{p}_{2'} & \mathrm{p}_{1'} & \mathrm{b} \\
\hline
1 & 1 & & & & \\
2 & & 1 & & & \\
3 & & & 1 & & \\
4 & 1 & & & & \\
5 & & 1 & & & \\
6 & & & 1 & & \\
7 & & & & 1 & \\
8 & & & & & 1 \\
9 & & & & 1 & \\
10 & & & & & 1 \\
\mathrm{g}_1 & & 1 & & & \\
\mathrm{p}_1 & & & & 1 & \\
\mathrm{g}_2 & & & & 1 & \\
\mathrm{p}_2 & & & 1 & & \\
\hline
\end{array}
\qquad 15.41
$$

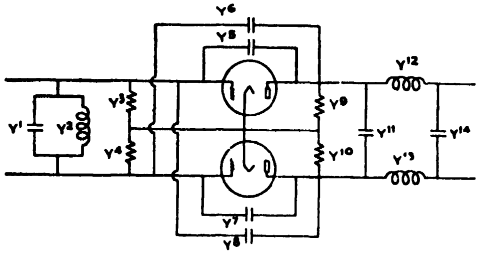
The admittance tensor of the resultant system, by $\mathbf{A}_t \cdot \mathbf{Y} \cdot \mathbf{A}$, is

$$
\mathbf{Y}' = \begin{array}{c|c|c|c|c|c|}
 & \mathrm{a}' & \mathrm{g}' & \mathrm{p}_{2'} & \mathrm{p}_{1'} & \mathrm{b} \\
\hline
\mathrm{a}' & Y^1+Y^4 & Y^{12} & Y^{13} & & \\
\mathrm{g}' & Y^{12} & Y^2+Y^5 & Y^{23} & & \\
\mathrm{p}_{2'} & Y^{13} & Y^{23} & Y^3+Y^6+G^{pp2} & G^{pg2} & \\
\mathrm{p}_{1'} & & G^{pg1} & & Y^7+Y^9+G^{pp1} & Y^{78} \\
\mathrm{b} & & & & Y^{78} & Y^8+Y^{10} \\
\hline
\end{array}
\qquad 15.42
$$

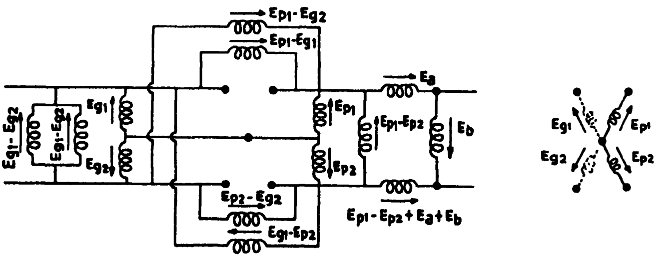
If the current $I^{a'}$ is known and E_b is to be found, then *eliminating the three rows and columns* **g**$'$, $\mathbf{p}_{2'}$, $\mathbf{p}_{1'}$, an equation analogous to equation 15.29 is left whose solution also applies here.

XII. PUSH-PULL AMPLIFIER

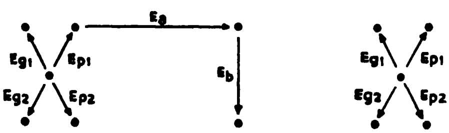
As an example of analyzing separately the network and the tube, consider the high-power radio-frequency amplifier of Fig. 15.9*a* in which the internal intergrid capacity coupling is neutralized by additional capacity coupling externally applied.



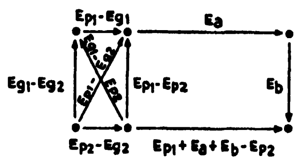
(*a*) Given network



(*b*) Voltages across individual coils of the two component networks



(*c*) Assumed junction-pairs



(*d*) Differences of potential

FIG. 15.9.—Push-pull Amplifier

The component static network of Fig. 15.9*b* has fourteen coils and seven junctions, hence six junction-pairs and eight meshes. Its admittance tensor *before* interconnections has fourteen rows and columns and contains *only diagonal components.*

Its transformation tensor, from Figs. 15.9*b* and *d* (using E instead of ΔE), is

$$
\begin{aligned}
E_1 &= E_{g1} - E_{g2}\\
E_2 &= E_{g1} - E_{g2}\\
E_3 &= E_{g1}\\
E_4 &= E_{g2}\\
E_5 &= -E_{g1} + E_{p1}\\
E_6 &= E_{p1} - E_{g2}\\
E_7 &= -E_{g2} + E_{p2}\\
E_8 &= E_{g1} - E_{p2}\\
E_9 &= E_{p1}\\
E_{10} &= E_{p2}\\
E_{11} &= E_{p1} - E_{p2}\\
E_{12} &= E_a\\
E_{13} &= E_{p1} - E_{p2} + E_a + E_b\\
E_{14} &= E_b
\end{aligned}
$$

A =

	g_1	p_1	g_2	p_2	a	b
1	1		−1			
2	1		−1			
3	1					
4			1			
5	−1	1				
6		1	−1			
7			−1	1		
8	1			−1		
9		1				
10				1		
11		1		−1		
12					1	
13		1		−1	1	1
14						1

15.43

The resultant admittance tensor is by $\mathbf{A}_t \cdot \mathbf{Y} \cdot \mathbf{A}$ =

$\mathbf{Y}_1$ =

	g_1	p_1	g_2	p_2	a	b
g_1	$Y^1+Y^2+Y^3+Y^5+Y^8$	$-Y^5$	$-Y^1-Y^2$	$-Y^8$	0	0
p_1	$-Y^5$	$Y^5+Y^6+Y^9$ $Y^{11}+Y^{13}$	$-Y^6$	$-Y^{11}-Y^{13}$	Y^{13}	Y^{13}
g_2	$-Y^1-Y^2$	$-Y^6$	$Y^1+Y^2+Y^4+Y^6+Y^7$	$-Y^7$	0	0
p_2	$-Y^8$	$-Y^{11}-Y^{13}$	$-Y^7$	$Y^7+Y^8+Y^{10}+Y^{11}+Y^{13}$	$-Y^{13}$	$-Y^{13}$
a	0	Y^{13}	0	$-Y^{13}$	$Y^{12}+Y^{13}$	Y^{13}
b		Y^{13}		$-Y^{13}$	Y^{13}	$Y^{13}+Y^{14}$

15.44

The admittance tensor of the two tubes of Fig. 15.9*b* are

$\mathbf{Y}_2$ =

	g_1	p_1	g_2	p_2
p_1	G^{pg1}	G^{pp1}		
p_2			G^{pg2}	G^{pp2}

15.45

Hence *the new components of the admittance tensor of the resultant network* are by $\mathbf{Y}_1 + \mathbf{Y}_2 = \mathbf{Y} =$

$\mathbf{Y}' =$

	g_1	p_1	g_2	p_2	a	b
g_1	$Y^1+Y^2+Y^3+Y^5+Y^8$	$-Y^5$	$-Y^1-Y^2$	$-Y^8$		
p_1	$-Y^5+G^{pg1}$	$Y^5+Y^6+Y^9+Y^{11}+Y^{13}+G^{pp1}$	$-Y^6$	$-Y^{11}-Y^{13}$	Y^{13}	Y^{13}
g_2	$-Y^1-Y^2$	$-Y^6$	$Y^1+Y^2+Y^4+Y^6+Y^7$	$-Y^7$		
p_2	$-Y^8$	$-Y^{11}-Y^{13}$	$-Y^7+G^{pg2}$	$Y^7+Y^8+Y^{10}+Y^{11}+Y^{13}+G^{pp2}$	$-Y^{13}$	$-Y^{13}$
a		Y^{13}		$-Y^{13}$	$Y^{11}+Y^{13}$	Y^{13}
b		Y^{13}		$-Y^{13}$	Y^{13}	$Y^{13}+Y^{14}$

15.46

(The $\mathbf{C}_t^{-1}$ connecting the tube and the network has a unit matrix.)

The current vector $\Delta\mathbf{I}$ *is impressed,* not across a junction-pair, but *across two junctions belonging to two different junction-pairs* E_{g1} and E_{g2}. That is, $\Delta\mathbf{I}$ may be assumed to be impressed across coil Y^1 so that

$$\Delta\mathbf{I} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & & 14 \end{array} \\ \boxed{\begin{array}{c|c|c|c|c} \Delta I^1 & 0 & 0 & .. & 0 \end{array}} \end{array} \qquad 15.47$$

$$\Delta\mathbf{I}' = \mathbf{A}_t\cdot\Delta\mathbf{I} = \begin{array}{c} \begin{array}{cccccc} g_1 & p_1 & g_2 & p_2 & a & b \end{array} \\ \boxed{\begin{array}{c|c|c|c|c|c} \Delta I^1 & & -\Delta I^1 & & & \end{array}} \end{array} \qquad 15.48$$

Hence there are two impressed currents.

The six equations $\Delta\mathbf{I}' = \mathbf{Y}'\cdot\Delta\mathbf{E}'$ may be reduced to three by the reduction formulas. The three remaining equations may be solved for ΔE_b.

XIII. THE CRITERION OF OSCILLATION

When a junction network oscillates, although no impressed current $\Delta\mathbf{I}$ exists, still a difference of potential $\Delta\mathbf{E}$ appears across some junction-pairs. The reason for the appearance of $\Delta\mathbf{E}$ is that *mesh-currents* $\Delta\mathbf{i}$ *do exist,* which, however, are not included in the equation.

(*a*) The question now arises: If in the equation $\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}$ (or $\Delta\mathbf{I} = \mathbf{Y}\cdot\Delta\mathbf{E}$) the impressed current $\mathbf{I}$ is zero, can $\mathbf{E}$ still remain different from zero?

Solving the equation gives $\mathbf{E} = \mathbf{Y}^{-1}\cdot\mathbf{I}$. If $\mathbf{I} = 0$, then $\mathbf{E} = \mathbf{Y}^{-1}\cdot 0$. Since $\mathbf{Y}^{-1}$ has the form of a fraction whose numerator is a matrix $\mathbf{Y}_c$ containing all the cofactors of $\mathbf{Y}$ and the denominator is the determi-

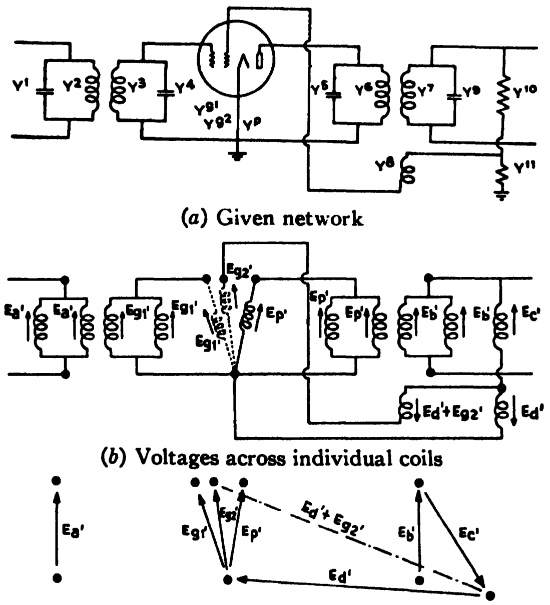
nant D of $\mathbf{Y}$, the last equation may be written as

$$\mathbf{E} = \mathbf{Y}^{-1}\cdot 0 = \frac{\mathbf{Y}_c}{D}\cdot 0 \qquad 15.49$$

In order that the components of $\mathbf{E}$ *should not all become zero it is sufficient that the determinant* D *of* $\mathbf{Y}$ *should be zero.* In that case $\mathbf{E} = 0/0$, which may be an actual number, and consequently the network may oscillate.

Hence, if the determinant of the admittance tensor $\mathbf{Y}$ *is zero, the network is oscillatory.* Equating the determinant of $\mathbf{Y}$ to zero, the necessary relations between the design constants of the network to maintain oscillations is accordingly obtained.

(*b*) *An alternative method consists of dividing the system into an amplifier network and the remaining feedback network* having common input and output junction pairs, and then finding their respective admittance tensors $\mathbf{Y}_1$ and $\mathbf{Y}_2$. If the ratios of the input and output voltages of each component system are found by equation 15.31 then (calling the voltage-ratio of the amplifier network by μ and that of the feedback network by β), *the criterion of oscillation is*



(*a*) Given network

(*b*) Voltages across individual coils

(*c*) Assumed junction-pairs

FIG. 15.10.—Tetrode Circuit

$$\boxed{\mu\beta = 1} \qquad 15.50$$

XIV. TETRODE CIRCUIT

As an example in which *two grids have* ΔE *appearing on them* let the automatic selectivity control circuit of Fig. 15.10*a* be considered.

The network has fourteen coils, also nine junctions, two subnetworks, hence $9 - 2 = 7$ junction-pairs and $14 - 7 = 7$ meshes. The assumed seven junction-pairs and the voltage across the individual coils are shown in Fig. 15.10*c*.

The admittance tensor of the primitive network is

$$
\mathbf{Y} = \begin{array}{c|cccccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & g_1 & g_2 & p \\
\hline
1 & Y^1 & & & & & & & & & & & & & \\
2 & & Y^2 & Y^{23} & & & & & & & & & & & \\
3 & & Y^{23} & Y^3 & & & & & & & & & & & \\
4 & & & & Y^4 & & & & & & & & & & \\
5 & & & & & Y^5 & & & & & & & & & \\
6 & & & & & & Y^6 & Y^{67} & Y^{68} & & & & & & \\
7 & & & & & & Y^{67} & Y^7 & Y^{78} & & & & & & \\
8 & & & & & & Y^{68} & Y^{78} & Y^8 & & & & & & \\
9 & & & & & & & & & Y^9 & & & & & \\
10 & & & & & & & & & & Y^{10} & & & & \\
11 & & & & & & & & & & & Y^{11} & & & \\
g_1 & & & & & & & & & & & & & & \\
g_2 & & & & & & & & & & & & & & \\
p & & & & & & & & & & & & G^{pg1} & G^{pg2} & G^{pp} \\
\hline
\end{array} \qquad 15.51
$$

Equating the old and the new voltages across each coil, the transformation tensor is

$$
\begin{aligned}
E_1 &= E_a \\
E_2 &= E_a \\
E_3 &= E_{g1} \\
E_4 &= E_{g1} \\
E_5 &= E_p \\
E_6 &= E_p \\
E_7 &= E_b \\
E_8 &= E_{g2} + E_d \\
E_9 &= E_b \\
E_{10} &= E_c \\
E_{11} &= E_d \\
E_{g1} &= E_{g1} \\
E_{g2} &= E_{g2} \\
E_p &= E_p
\end{aligned}
\qquad
\mathbf{A} = \begin{array}{c|ccccccc|}
 & a & g_1 & g_2 & p & b & c & d \\
\hline
1 & 1 & & & & & & \\
2 & 1 & & & & & & \\
3 & & 1 & & & & & \\
4 & & 1 & & & & & \\
5 & & & & 1 & & & \\
6 & & & & 1 & & & \\
7 & & & & & 1 & & \\
8 & & & 1 & & & & 1 \\
9 & & & & & 1 & & \\
10 & & & & & & 1 & \\
11 & & & & & & & 1 \\
g_1 & & 1 & & & & & \\
g_2 & & & 1 & & & & \\
p & & & & 1 & & & \\
\hline
\end{array} \qquad 15.52
$$

The resultant admittance tensor is by $\mathbf{A}_t \cdot \mathbf{Y} \cdot \mathbf{A}$

		a	g₁	g₂	p	b	c	d
	a	Y_1+Y_2	Y_{23}					
	g₁	Y^{23}	Y^3+Y^4					
	g₂			Y^8	Y^{68}	Y^{78}		Y^8
$\mathbf{Y}' =$	p		G^{pg1}	$Y^{68}+G^{pg2}$	$Y^5+Y^6+G^{pp}$	Y^{67}		Y^{68}
	b			Y^{78}	Y^{67}	Y^9+Y^7		Y^{78}
	c						Y^{10}	
	d			Y^8	Y^{68}	Y^{78}		Y^8+Y^{11}

15.53

The presence of heavy lines speeds up the multiplications.

Since the input terminal is **a** and the output terminal is **b**, the remaining five axes may be eliminated by the reduction formula. The two equations **a** and **b** are solved for E_b by equation 15.30.

XV. THE IMPEDANCE TENSOR OF TUBES

It may happen that a tube circuit has fewer meshes than junction-pairs. It is then easier to analyze the circuit as a mesh network.

The impedance tensor **z** *of a tube is found by taking the inverse of its admittance tensor* **Y** *given* in equations 15.22 to 15.26. Hence the impedance tensor of a tetrode is

$$
\mathbf{z}_{mn} = \begin{array}{c|ccc} m \diagdown n & a & b & p \\ \hline a & \dfrac{1-\mu_b\nu_b}{r_b r_p D} & \dfrac{\eta_b\nu_a-\eta_a}{r_a r_p D} & \dfrac{\eta_a\nu_b-\nu_a}{r_a r_b D} \\ b & \dfrac{\mu_a\nu_b-\eta_b}{r_b r_p D} & \dfrac{1-\mu_a\nu_a}{r_a r_p D} & \dfrac{\eta_b\nu_a-\nu_b}{r_a r_p D} \\ p & \dfrac{\eta_b\mu_b-\mu_a}{r_b r_p D} & \dfrac{\mu_a\eta_a-\mu_b}{r_a r_p D} & \dfrac{1-\eta_a\eta_b}{r_a r_p D} \end{array} \qquad 15.54
$$

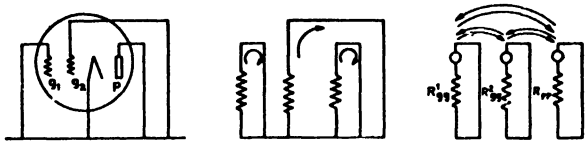
where

$$
D = \frac{1 + \mu_a(\eta_a\nu_b - \nu_a) + \mu_b(\eta_b\nu_a - \nu_b) - \eta_a\eta_b}{r_a r_b r_p}
$$

The impedance tensor of a triode is

$$
\mathbf{z}_{mn} = \begin{array}{c|cc} m \diagdown n & g & p \\ \hline & \dfrac{r_g}{1-\mu_g\mu_p} & \dfrac{-\mu_g r_p}{1-\mu_g\mu_p} \\ & \dfrac{-\mu_p r_g}{1-\mu_g\mu_p} & \dfrac{r_p}{1-\mu_g\mu_p} \end{array} = \begin{array}{c|cc} m \diagdown n & g & p \\ \hline g & Z_{gg} & Z_{gp} \\ p & Z_{pg} & Z_{pp} \end{array} \qquad 15.55
$$

That is, in a mesh network an n-electrode tube may be represented by $n - 1$ coils with unilateral self- and mutual impedances, just as an $n - 1$ winding transformer. That is, a tetrode is represented by three coils, a triode by two coils, their interconnection representing the filament, Fig. 15.11.



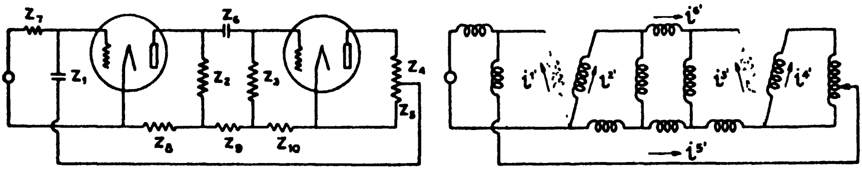
(a) Tetrode (b) Equivalent mesh network (c) Its primitive mesh network

FIG. 15.11

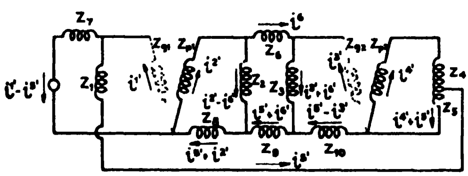
The analysis of a mesh network with tubes follows the analysis of any other mesh network. Since the grid resistances are infinite, *at the end of the analysis, in the admittance tensor* $\mathbf{y} = \mathbf{z}^{-1}$ all r_g may be put equal to infinity and the equations reduced.

XVI. DEGENERATIVE FEEDBACK AMPLIFIER

Let the circuit of Fig. 15.12*a* be considered representing a degenerative feedback amplifier.



(a) Given network (b) Assumed branch-currents



(c) Currents in individual coils

FIG. 15.12.—Degenerative Feed-back Amplifier

It has fourteen coils, nine junctions, eight junction-pairs, and $14 - 8 = 6$ meshes. Assuming six new currents shown in Fig. 15.12*b* (four of them in the four tube-coils), the resultant currents flowing in each coil are shown in Fig. 15.12*c*.

Equating the old and the new currents flowing in each coil (using i^a for Δi^a)

$$
\begin{aligned}
i^1 &= i^{5'} \\
i^2 &= i^{2'} - i^{6'} \\
i^3 &= i^{3'} + i^{6'} \\
i^4 &= i^{4'} \\
i^5 &= i^{4'} + i^{5'} \\
i^6 &= i^{6'} \\
i^7 &= i^{1'} - i^{5'} \\
i^8 &= i^{2'} + i^{5'} \\
i^9 &= i^{5'} + i^{6'} \\
i^{10} &= -i^{3'} + i^{5'} \\
i^{g1} &= i^{1'} \\
i^{p1} &= i^{2'} \\
i^{g2} &= i^{3'} \\
i^{p2} &= i^{4'}
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ g1 \\ p1 \\ g2 \\ p2
\end{array}
\begin{array}{cccccc}
1' & 2' & 3' & 4' & 5' & 6' \\
\hline
 & & & & 1 & \\
 & 1 & & & & -1 \\
 & & 1 & & & 1 \\
 & & & 1 & & \\
 & & & 1 & 1 & \\
 & & & & & 1 \\
1 & & & & -1 & \\
 & 1 & & & 1 & \\
 & & & & 1 & 1 \\
 & & -1 & & 1 & \\
1 & & & & & \\
 & 1 & & & & \\
 & & 1 & & & \\
 & & & 1 & &
\end{array}
\tag{15.56}
$$

The impedance tensor of the primitive network is

$$
\mathbf{z} = \begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ g_1 \\ p_1 \\ g_2 \\ p_2
\end{array}
\begin{array}{cccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & g_1 & p_1 & g_2 & p_2 \\
\hline
Z_1 & & & & & & & & & & & & & \\
 & Z_2 & & & & & & & & & & & & \\
 & & Z_3 & & & & & & & & & & & \\
 & & & Z_4 & & & & & & & & & & \\
 & & & & Z_5 & & & & & & & & & \\
 & & & & & Z_6 & & & & & & & & \\
 & & & & & & Z_7 & & & & & & & \\
 & & & & & & & Z_8 & & & & & & \\
 & & & & & & & & Z_9 & & & & & \\
 & & & & & & & & & Z_{10} & & & & \\
 & & & & & & & & & & Z_{gg1} & Z_{gp1} & & \\
 & & & & & & & & & & Z_{pg1} & Z_{pp1} & & \\
 & & & & & & & & & & & & Z_{gg2} & Z_{gp2} \\
 & & & & & & & & & & & & Z_{pg2} & Z_{pp2}
\end{array}
\tag{15.57}
$$

The impedance tensor of the network is by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ =

	1′	2′	3′	4′	5′	6′
1′	Z_7+Z_{gg1}	Z_{gp1}			$-Z_7$	
2′	Z_{pg1}	$Z_2+Z_8+Z_{pp1}$			Z_8	$-Z_2$
3′			$Z_3+Z_{10}+Z_{gg2}$	Z_{gp2}	$-Z_{10}$	Z_3
$\mathbf{z}'$ = 4′			Z_{pg2}	$Z_4+Z_5+Z_{pp2}$	Z_5	
5′	$-Z_7$	Z_8	$-Z_{10}$	Z_5	$Z_1+Z_5+Z_7$ $+Z_8$ $+Z_9+Z_{10}$	Z_9
6′		$-Z_2$	Z_3		Z_9	Z_2+Z_3 $+Z_6+Z_9$

15.58

To find any particular current, the remaining rows may be eliminated by the reduction formulas. In the admittance tensor $\mathbf{y}'$, r_g is put equal to infinity and the matrix simplified.

XVII. SIMPLIFIED REPRESENTATION OF A TRIODE

Since knowledge of the grid current Δi^g of a triode is not needed (its value is practically always zero), *the axis of* $\mathbf{g}$ *in its impedance tensor may be eliminated by the reduction formula* $\mathbf{z}' = \mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3$ so that the equivalent impedance r'_p of the plate is

$$r'_p = \frac{r_p}{D} - \left(\frac{-\mu_p r_g}{D}\right)\left(\frac{D}{r_g}\right)\left(\frac{-\mu_g r_p}{D}\right) = \frac{r_p(1-\mu_p\mu_g)}{1-\mu_p\mu_g} = r_p \qquad 15.59$$

Hence *the grid coil of the triode may be left out by changing the self-impedance of the plate coil from r_p/D to r_p.*

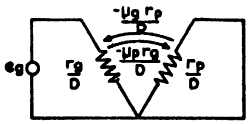
If a voltage e_g is *impressed* in series with the grid coil, or a difference of potential e_g appears across it, then the equivalent impressed voltage on the plate coil is, by the reduction formula,

$$\mathbf{e}' = \mathbf{e}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{e}_2$$

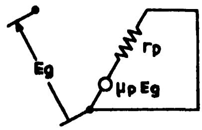
$$e'_p = e_p - \left(\frac{-\mu_p r_g}{D}\right)\left(\frac{D}{r_g}\right) e_g = e_p + \mu_p E_g \qquad 15.60$$

Hence a voltage E_g appearing across the grid coil appears in series

with the plate coil as $\mu_p E_g$ as shown in Fig. 15.13*b* and the grid coil and its mutual impedances may be left out.



(*a*) Triode as a two-mesh network



(*b*) Triode as an orthogonal network (1 mesh, 1 junction-pair)

Fig. 15.13

However, *a network containing the tube of Fig.* 15.13*b* *is not a mesh network but an orthogonal network* as shown in Fig. 15.14 since the

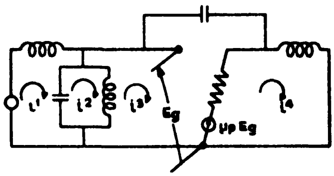


Fig. 15.14.—Triode Circuit as an Orthogonal Network

junction-pair voltage E_g must be known first in order to find the impressed mesh voltage $\mu_p E_g$. The study of these types of orthogonal networks is taken up in Chapter XXI.

CHAPTER XVI

ORTHOGONAL NETWORKS

I. VOLTAGES AND CURRENTS IN MESH AND JUNCTION NETWORKS

(*a*) In a *mesh* network n coils form k meshes. *There are in general n known impressed voltages* $\mathbf{e}$, *each of them being assumed in series with a particular coil.* Since there are only k known response currents $\mathbf{i}'$, the n known coil voltages $\mathbf{e}$ are replaced by k impressed mesh voltages $\mathbf{e}'$ by $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$ in setting up the k equations $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$.

The k new variables $\mathbf{i}'$ are usually assumed to flow in k of the branches. It was also shown in Section VII, Chapter VI, that the k new components of the impressed voltages $\mathbf{e}'$ may be assumed to be concentrated in the same k branches in which the new components $\mathbf{i}'$ are assumed. That is, all the impressed voltages around a mesh can be concentrated into one voltage e' impressed in that branch in which i' flows.

When an impressed voltage exists in an impedanceless branch, it is assumed that this branch has an impedance with zero value so that this branch is counted as a coil and it appears as such in the primitive mesh network. Such an impedanceless branch with a known impressed voltage will be called an "*apparent coil.*" Owing to the presence of apparent coils more reference axes have to be assumed in the network than are absolutely necessary. These additional reference axes have to be introduced in order to treat the network as a mesh and not as an orthogonal network having a more complicated equation of voltage.

(*b*) In a *junction* network n coils form $n - k$ junction-pairs. *There are in general n known impressed currents* $\mathbf{I}$ (or currents withdrawn by outside loads), *each of them assumed to be impressed into the junctions of a particular coil.* Since there are only $n - k$ response voltages $\mathbf{E}'$, the n known impressed coil currents $\mathbf{I}$ are replaced by $n - k$ impressed junction currents $\mathbf{I}'$ by $\mathbf{I}' = \mathbf{C}^{-1} \cdot \mathbf{I}$ in setting up the $n - k$ equations $\mathbf{I}' = \mathbf{Y}' \cdot \mathbf{E}'$.

The $n - k$ new variables $\mathbf{E}'$ and the $n - k$ new impressed currents $\mathbf{I}'$ are both assumed along the same $n - k$ junction-pairs.

When an impressed current (outside load) exists across two junctions with no coils between them, it is assumed that the two junctions

are connected by an admittanceless coil, so that this coil appears in the primitive junction network introducing an additional reference axis. Such an admittanceless branch with a known impressed current will also be called an "apparent coil."

(*c*) One of the purposes of the following analysis is to remove from the primitive networks those coils that have zero impedance or admittance, so that a network should be completely characterized by those coils only that have an actual impedance or admittance. That is, *all apparent coils and hence all superfluous reference axes will disappear from the analysis of orthogonal networks.*

(*d*) *A "mesh" and a "junction-pair" (or their equivalent "branch" and "open-mesh") will be called in general a "reference axis."*

II. VOLTAGES AND CURRENTS IN ORTHOGONAL NETWORKS

(*a*) Every k-mesh network of n coils contains also $n - k$ junction-pairs, across which measurable differences of potentials $\mathbf{E}'$ appear, whose presence, however, has up till now been disregarded in setting up the equation of voltage. (The differences of potential appearing across all individual coils calculated by the auxiliary equation $\mathbf{e}_c = \mathbf{z}\cdot\mathbf{C}\cdot\mathbf{i}'$ do not represent junction-pair voltages.) It may also be assumed, for the sake of completeness, that across the $n - k$ junction-pairs, currents $\mathbf{I}'$ with zero value are impressed.

Accordingly, any mesh network may be assumed to be actually an orthogonal network in which $\mathbf{e}$ *and* $\mathbf{I}$ *are impressed and in which* $\mathbf{i}$ *and* $\mathbf{E}$ *appear in response.* Owing to the absence of $\mathbf{I}$ and to the lack of interest in $\mathbf{E}$, the junction-pair axes have up till now been ignored.

(*b*) Similarly, every junction network with $n - k$ junction-pairs contains also k meshes in which mesh-currents $\mathbf{i}'$ flow whose presence has up till now been disregarded in setting up the equation of current. (The currents flowing in all *individual* coils calculated by the auxiliary equation $\mathbf{I}_c = \mathbf{Y}\cdot\mathbf{C}_t^{-1}\cdot\mathbf{E}'$ represent total currents and not response currents.) It may also be assumed for sake of completeness that there are mesh voltages $\mathbf{e}$ with zero value also impressed.

Accordingly, *any junction network may be assumed actually to be an orthogonal network in which* $\mathbf{I}$ *and* $\mathbf{e}$ *are impressed and in which* $\mathbf{E}$ *and* $\mathbf{i}$ *appear in response.* Because of the absence of $\mathbf{e}$ and the lack of interest in $\mathbf{i}$, the mesh axes have up till now been ignored.

(*c*) In a general *orthogonal* network n coils form k meshes and $n - k$ junction-pairs. *There are in general n voltages* $\mathbf{e}$ *impressed, each in series with a coil, and also n currents* $\mathbf{I}$ *impressed (or withdrawn), each in shunt with a coil.* In replacing the $2n$ impressed quantities $\mathbf{e}$ and $\mathbf{I}$

from the individual coils to their value $\mathbf{e}'$ and $\mathbf{I}'$ around the meshes and junction-pairs by $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$ and $\mathbf{I}' = \mathbf{C}^{-1} \cdot \mathbf{I}$, it will be found that (if $\mathbf{C}$ is non-singular) there are still n (instead of k) known impressed voltages $\mathbf{e}'$ and n (instead of $n - k$) known impressed currents $\mathbf{I}'$. That is, *there exist now known impressed voltages* $\mathbf{e}'$ *also around the junction-pairs (open-mesh) and known impressed currents* $\mathbf{I}'$ *also around the meshes (closed mesh).*

As a result of the n impressed coil voltages $\mathbf{e}$ and coil currents $\mathbf{I}$ there appear k response currents $\mathbf{i}$ around the meshes and $n - k$ response voltages $\mathbf{E}$ across the junction-pairs. Altogether there are n response quantities acting as the n variables.

(*d*) The method of analysis of orthogonal networks to be presented presently is governed by two considerations:

1. *The number of variables and the number of equations must be the same as the number of coils, irrespective of their manner of interconnection or excitation, or the number of apparent coils.*

2. *The power input* $\mathbf{e}^* \cdot \mathbf{i}$ *and the power output* $\mathbf{E}^* \cdot \mathbf{I}$ *must remain invariant under all manner of interconnection and excitation of the n coils.*

These postulates make possible the setting up of a non-singular transformation tensor $\mathbf{C}$, having n rows and columns, for all n-coil networks. Thereby *it will be possible in their analysis and synthesis to pass from any n-coil network to any other n-coil network* without examining the method of excitation of each of them for the existence of apparent coils.

Since many of the coils may be combined into a single equivalent coil, it will be understood throughout that the network under consideration represents the *reduced* original network.

III. COVARIANT AND CONTRAVARIANT VARIABLES

(*a*) In an orthogonal network there are two types of variable quan ities:

1 The k *contravariant* variables i^m flowing around the closed meshes.

2. The $n - k$ *covariant* variables E_u appearing across the open meshes or junction-pairs.

An interesting interdependence exists between the contravariant and the covariant variables that offers a simple method of analysis of orthogonal networks.

(*b*) When a branch containing a mesh current i^a is open-circuited, this particular contravariant variable becomes zero. *In its place appears another variable, which, however, is covariant,* namely, the difference of potential E_a across the open branch.

Similarly, when a junction-pair with the difference of potential E_a is short-circuited, this particular variable becomes zero. However, *in its place there appears another new, but contravariant variable*, namely, the mesh current i^a flowing around the new mesh.

In general, in any dynamical system, whenever a contravariant variable (say a *velocity* v^a in a certain direction) is reduced to zero by the introduction of some constraint, its place is immediately taken by a covariant variable (say the reaction force f_a of the constraint). Similarly, when a covariant variable (say the reaction force f_a) is reduced to zero, in its place appears immediately a new contravariant variable (the velocity v^a along the direction of the vanished force).

During the addition and removal of the constraints, the structure of the dynamical system is not changed and *the effect of constraints is only to change the relative number of covariant and contravariant variables, leaving their sum constant.*

IV. KNOWN RESPONSE QUANTITIES

(*a*) It is not necessary that the variable along a certain axis should become zero before its dual appears. *If the variable* $\mathbf{i}'$ (*or* $\mathbf{E}'$) *assumes a known value along a certain direction, then its dual variable* $\mathbf{E}'$ (*or* $\mathbf{i}'$) *still appears along the same direction.* Of course the known $\mathbf{i}'$ (or $\mathbf{E}'$) is no longer a variable quantity and there is still one variable (either a covariant or a contravariant) along each axis. (In dynamics such cases occur with "moving constraints.") Hence *along each axis two response quantities* $\mathbf{i}'$ *and* $\mathbf{E}'$ *may exist, one having a known value, the other being a variable.*

(*b*) *Apparent coils represent such reference axes along which two types of response quantities appear.* In particular:

1. If a known impressed voltage exists in an impedanceless branch, then the known voltage is denoted by E and the unknown current by i.

2. If a known current is impressed across an admittanceless branch, then the known current is denoted by i and the unknown difference of potential by E.

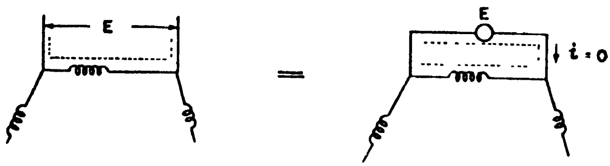
That is, *in the presence of apparent coils the number of response quantities* $\mathbf{i}$ *and* $\mathbf{E}$ *may be* $2n$ *just as the number of impressed quantities* $\mathbf{e}$ *and* $\mathbf{I}$ *is* $2n$. However, the number of variables in all cases is still n.

(*c*) Hence, along each of the n reference axes four quantities may exist, two voltages e and E and two currents I and i. The two voltages e and E cannot be combined into one expression, since one of them is known and the other is unknown. Similarly the two currents flowing in the same coil I and i cannot be combined.

V. ORTHOGONAL NETWORKS AS ALL-MESH OR ALL-JUNCTION NETWORKS

(*a*) Since it is possible to introduce apparent coils into a network without disturbing its performance, *a "closed mesh" may be changed into an "open mesh," and vice versa, by an apparent coil.* In particular:

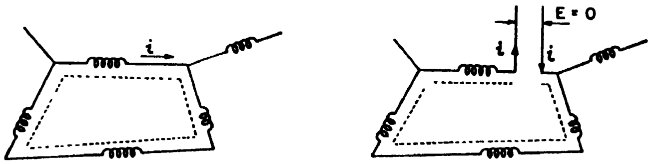
1. Any open mesh may be considered as being closed by an apparent coil having an unknown impressed E and zero i (Fig. 16.1 *a* and *b*).



(*a*) Open mesh (*b*) Closing an open mesh

FIG. 16.1 —Changing an Open Mesh into a Closed Mesh

2. Any closed mesh may be considered as being open, having across the opening an apparent coil with an unknown i at its junctions and with zero E across them (Fig. 16.2*a* and *b*). (The two junctions created by opening a coil are not connected together by a coil, and, since identical currents flow into these junctions, the two junctions are considered to be connected by an apparent coil.)



(*a*) Closed mesh (*b*) Opening a closed mesh

FIG. 16.2.—Changing a Closed Mesh into an Open Mesh

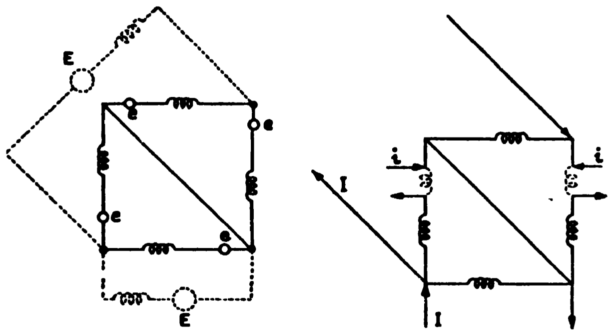
(*b*) Hence whether an axis assumed represents a mesh or a junction-pair depends on the value of the variables i and E existing along the axis:

1. *If E is unknown, the axis is a "junction-pair."*
2. *If i is unknown, the axis is a "mesh."*

(*c*) Since every junction-pair of an orthogonal network may be considered as being closed through an unknown voltage E, forming thereby an apparent coil, *every orthogonal network is equivalent to an all-mesh network in which voltages are impressed not only in series with the actual coils* (as shown in Fig. 16.3*a*), *but also in series with some of the apparent coils.*

Similarly since every mesh of an orthogonal network may be con-

sidered as being opened up by impressing an unknown current i at each opening, *every orthogonal network is equivalent to an all-junction network in which currents are impressed not only in shunt with each actual*



(*a*) Closing the junction-pairs (*b*) Opening the meshes

FIG. 16.3.—Equivalence of Meshes and Junction-pairs

coil (as shown in Fig. 16.3*b*) *but also in shunt with some of the apparent coils.*

(*d*) Each closed junction-pair contributes an apparent coil to the all-mesh network, and each opened mesh contributes an apparent coil to an all-junction network. The voltages appearing in apparent coils are now denoted by E, and the currents appearing across apparent coils are denoted by i. *These apparent coils do not appear in the primitive network.*

Summarizing, it may be stated that:

1. The impressed vectors **e** and **i** are associated with *actual* coils.
2. The response vectors **i** and **E** are associated with *apparent* coils.

VI. INVARIANCE OF THE POWER INPUT AND OUTPUT

(*a*) In an all-mesh network containing only **e** and **i** (Fig. 4.1) the power input $\mathbf{e}^* \cdot \mathbf{i}$ remains invariant no matter how the coils are connected together, since each coil is short-circuited upon itself in any interconnection and the current through it is unchanged. When, in an all-mesh network with **e** and **i**, also currents **I** are impressed in shunt with each coil, the power input $\mathbf{e}^* \cdot \mathbf{i}$ still remains invariant in any interconnection. However, when the n coils are interconnected into a k-mesh network, the currents **i** in the coils become different and $\mathbf{e}^* \cdot \mathbf{i}$ is no longer invariant.

When, in place of the two concepts **e** *and* **i**, *four electromagnetic concepts, namely* **e**, **i**, **E**, *and* **I**, *are associated with a network, then with any impressed voltages and currents all networks (mesh or orthogonal) may be considered as all-mesh networks, hence with any interconnection*

of the coils into networks the power input $\mathbf{e}^* \cdot \mathbf{i}$ *and output* $\mathbf{E}^* \cdot \mathbf{I}$ *still remain invariant.*

This is true since across each coil still the same difference of potential E may be assumed to exist in all interconnections, as shown in Fig. 16.4 where three coils are arranged in four different all-mesh

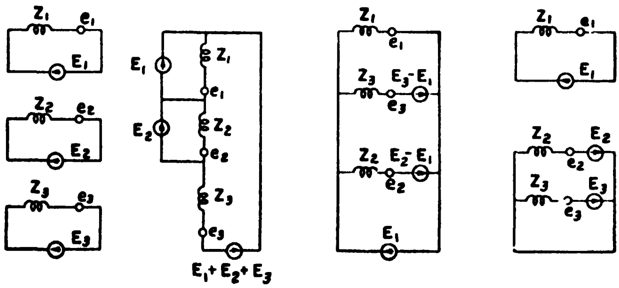


FIG. 16.4.—Various Types of Interconnections of Three Coils into Three Meshes (the same E Existing Across the Same Coils)

networks. (The manner of rearranging the voltages E will follow automatically from the equations to be developed presently.) It should be noted that a voltage E impressed in series with an impedance-less coil is in series also with several other coils, hence the invariance of $\mathbf{E}^* \cdot \mathbf{I}$ is not so apparent.

(*b*) Similar reasoning applies to an all-junction network. When four electrical concepts **e**, **i**, **E**, and **I** are associated with a network instead of two (**E** and **I**) then *with any impressed voltages and currents all networks may be considered as all-junction networks and with any*

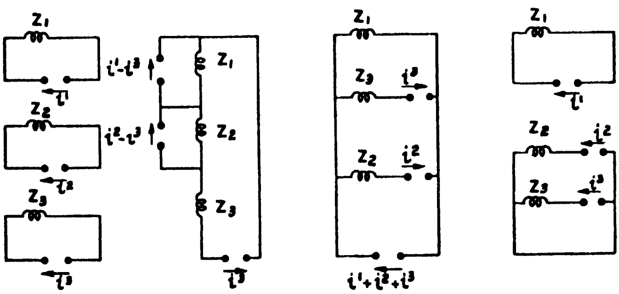


FIG. 16.5.—Various Types of Interconnections of Three Coils into Three Junction-pairs (the same i Flowing Through the Same Coils)

interconnection of the coils the power input $\mathbf{e}^* \cdot \mathbf{i}$ *and output* $\mathbf{E}^* \cdot \mathbf{I}$ *still remain invariant.*

This is true since across each coil the same current i may be assumed to flow in all interconnections as shown in Fig. 16.5. (It should be

noted that current i impressed across an admittanceless coil flows through several coils, hence the invariance of $\mathbf{e}^*\cdot\mathbf{i}$ is not readily apparent.)

(c) Summarizing, in interconnecting n coils into various types of networks, the power input $\mathbf{e}^*\cdot\mathbf{i}$ and output $\mathbf{E}^*\cdot\mathbf{I}$ can be made invariant as

$\mathbf{e}^*\cdot\mathbf{i} = \mathbf{e}^{*\prime}\cdot\mathbf{i}'$	$e_\alpha i^\alpha = e_{\alpha'} i^{\alpha'}$	16.1
$\mathbf{E}^*\cdot\mathbf{I} = \mathbf{E}^{*\prime}\cdot\mathbf{I}'$	$E_\alpha I^\alpha = E_{\alpha'} I^{\alpha'}$	16.2
$(\mathbf{e}+\mathbf{E})^*\cdot(\mathbf{I}+\mathbf{i}) = (\mathbf{e}'+\mathbf{E}')^*\cdot(\mathbf{I}'+\mathbf{i}')$	$(e_\alpha + E_\alpha)(I^\alpha + i^\alpha) = (e_{\alpha'} + E_{\alpha'})(I^{\alpha'} + i^{\alpha'})$	16.3

under all manner of interconnection of the coils by introducing four electrical quantities **e**, **i**, **E**, and **I** in place of two (**e**, **i** or **E**, **I**) and keeping any three of the four quantities unchanged across each coil while the interconnections are changed. *In interconnecting n coils, the value of n (out of $4n$) electrical quantities (or any n linear combination of them) is variable.*

In general, every network may be considered as an all-mesh network or as an all-junction network, or as an orthogonal network, and so on, depending on what components of **i** and **E** are assumed as variables. In particular:

1. All **i** are variables: all-mesh.
2. All **E** are variables: all-junction.
3. Some **E** and some **i**: orthogonal.
4. Some **i** are variables: mesh.
5. Some **E** are variables: junction.

VII. THE EQUATIONS OF VOLTAGE AND CURRENTS

(a) Although along each reference axis two types of voltages e and E, and two types of currents I and i occur—some of them being known, some of them unknown—*Ohm's law is satisfied by the resultant currents and voltages at each instant.* Consequently:

1. *The equation of voltage of orthogonal networks is, by considering the latter as all-mesh networks,*

$$\boxed{\mathbf{E} + \mathbf{e} = \mathbf{z}\cdot(\mathbf{i} + \mathbf{I})} \qquad \boxed{E_{\bar\alpha} + e_{\bar\alpha} = z_{\bar\alpha\beta}(i^\beta + I^\beta)} \qquad 16.4$$

2. *The equation of currents of orthogonal networks is, by considering the latter as all-junction networks,*

$$\boxed{\mathbf{i} + \mathbf{I} = \mathbf{Y}\cdot(\mathbf{E} + \mathbf{e})} \qquad \boxed{i^\alpha + I^\alpha = Y^{\alpha\bar\beta}(E_{\bar\beta} + e_{\bar\beta})} \qquad 16.5$$

In setting up the equations the components of **e** and **I** are all assumed to be known, while those of **i** and **E** are partly known and partly unknown. Along each axis only one variable quantity (an i or an E) may exist. However, in manipulating the equations any n of the $4n$ quantities may be unknown.

The effect of considering a known impressed voltage arbitrarily a component of **E** or of **e** (depending whether or not in establishing the network the terminals of the generator are assumed as junction-pairs) is simply to redistribute the known voltage components among **E** and **e**, leaving **z** unchanged. Similarly the effect of considering a known impressed current arbitrarily a component of **i** or of **I** (depending whether or not in establishing the network the coil connecting the two junctions is assumed as two coils in series) is again simply to redistribute the known current components among **i** and **I**, leaving **Y** unchanged.

(*b*) In general, each vector has as many components as there are reference axes. *The two vectors* e_α *and* E_α (or i^α and I^α) *cannot be added* since then each component of the resultant vector divides into two parts, one part being known, the other being unknown, and to solve for the unknowns, the resultant vector must be decomposed into its known and unknown component vectors.

VIII. STEPS IN SETTING UP THE EQUATION OF VOLTAGE

(*a*) *The equation of voltage of an orthogonal network is established by considering the latter as an all-mesh network in which all junction-pairs have been closed with apparent coils.*

Just as in all networks previously considered, the *response* quantities **i**′ and **E**′ are assumed along n arbitrary axes, while the *impressed* quantities **e**′ and **I**′ along the same axes are calculated by a transformation from **e** and **I** of the primitive network.

Since the apparent coils do not appear in the primitive network, all known impressed quantities associated with apparent coils have to be considered as known response quantities **i**′ and **E**′. (That is, they cannot appear in **I**′ or **e**′ that are derived from **I** and **e** of the primitive network.)

The n reference axes are arbitrarily assumed along the various coils with the precaution that all apparent coils must be assumed as reference axes.

(*b*) When the network diagram corresponding to an actual physical set-up has been established, say that of Fig. 16.7, the following designation is used for the *known* quantities (assuming primed quantities

to refer to the actual network and unprimed quantities to the primitive network):

1. All known voltages in series with coils are denoted by e.

2. All known voltages in series with an impedanceless branch are denoted by E'. (Previously they have also been denoted by e.)

3. All known currents impressed across a coil are denoted by I.

4. All known currents impressed across an admittanceless branch are denoted by i'. (Previously they have also been denoted by I.)

(*c*) Its primitive mesh network shown in Fig. 16.6 now also contains four quantities instead of two, namely, two voltages in series,

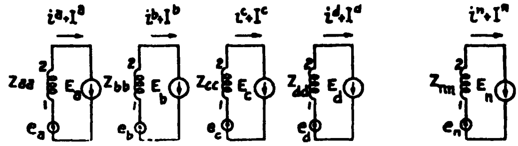


FIG. 16.6.—General Form of the Primitive Mesh Network

e and **E**, and two currents **i** and **I**. The components of **e** are the known voltages impressed *in series with each actual coil* in the network, and the components of **I** are the known currents impressed (or withdrawn by the loads) *across each actual coil* in the network.

(*d*) *The non-singular transformation tensor* **C** *of an orthogonal network is set up in exactly the same manner as in case of any other all-mesh network shown in Section XIIB, Chapter IV.* In setting up the relation $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ no attention is paid to whether or not the components of $\mathbf{i}'$ are known or are variables.

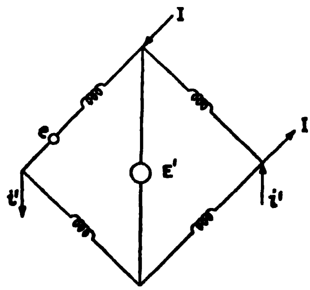


FIG. 16.7.—Designation of *Known* Currents and Voltages

(*e*) Once **C** has been established, then the new components of the impedance tensor $\mathbf{z}'$ of the network are found by $\mathbf{C}_t\cdot\mathbf{z}\cdot\mathbf{C}$.

The new components of the *response* quantities are partly known and partly unknown. In particular:

(*A*) The components of $\mathbf{i}'$ are *assumed* in the following manner:

1. Along the axes across actual coils, they are zero (known).

2. Along the axes across admittanceless coils, they are usually known.

3. Along all the other axes (meshes) they are usually unknown.

(That is, along the junction-pair axes they are known; along the mesh axes they are unknown.)

(*B*) The components of $\mathbf{E}'$ are *assumed* in the following manner:

1. Along the actual coils they are zero (known).
2. Along the impedanceless coils they are usually known.
3. Along all the other axes (junction-pairs) they are usually unknown.

(That is, along the mesh axes they are known; along the junction-pair axes they are unknown.)

The new components of the *impressed* quantities are calculated from those of the primitive network. In particular:

(*C*) The components of $\mathbf{e}'$ are *calculated* by $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$, where the components of $\mathbf{e}$ are usually known.

(*D*) The components of $\mathbf{I}'$ are *calculated* by $\mathbf{I}' = \mathbf{C}^{-1}\, \mathbf{I}$, where the components of $\mathbf{I}$ are usually known.

When the junction-pair axes are assumed across the same coils along which currents $\mathbf{I}$ are impressed, then the components of $\mathbf{I}'$ are already known without any calculation.

(In the networks of Fig. 16.4 the components of $\mathbf{E}'$ are not assumed but are calculated from $\mathbf{E}$ of the primitive mesh network by $\mathbf{E}' = \mathbf{C}_t \cdot \mathbf{E}$. This procedure is followed in establishing Fig. 16.4 to show how $\mathbf{E}'$ of the various networks is established, keeping $\mathbf{E}$ across each coil unchanged and thereby keeping the power invariant. In all other networks of this chapter $\mathbf{E}$ of the primitive network may be calculated from the *assumed* $\mathbf{E}'$ of the actual network by the reversed step $\mathbf{E} = \mathbf{C}_t^{-1} \cdot \mathbf{E}'$. This calculation is not made, though, since it is not needed in the analysis of the actual network.)

(*f*) *In the final set of equations* $\mathbf{E}' + \mathbf{e}' = \mathbf{z}' \cdot (\mathbf{i}' + \mathbf{I}')$, *there are as many current variables* $\mathbf{i}'$ *as there are meshes and as many voltage variables* $\mathbf{E}'$ *as there are junction-pairs. The total number of variables is the same as the number of actual coils.*

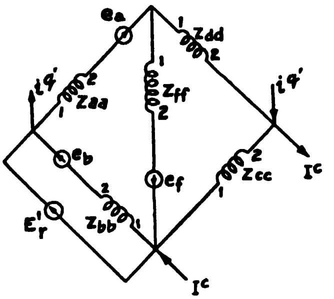
The apparent coils of the actual network do not appear any more as coils, hence they do not increase the number of reference axes. Their effect is to introduce known components of $\mathbf{E}'$ *and* $\mathbf{i}'$.

IX. EXAMPLE OF AN ORTHOGONAL NETWORK

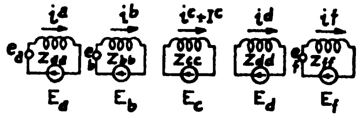
(*a*) Let the equation of voltage of the orthogonal network of Fig. 16.8*a* be established. The currents and voltages shown on it are all known. Some of the voltages and currents are impressed across apparent coils, hence they are denoted by E' and i' instead of e and i.

There are five coils and three meshes, hence there are two junction-pairs. One of the junction-pairs has to be assumed at the junctions of $i^{q'}$; the other is arbitrarily assumed across coil Z_{aa}. The equivalent

all-mesh network is shown in Fig. 16.9, in which the two junction-pairs appear as additional apparent coils.



(a) Given network



(b) Its primitive mesh network

FIG. 16.8.—Orthogonal Network with Impressed **e**, **I**, **E**′ and **i**′

(b) Its primitive mesh network is shown in Fig. 16.8*b*. The components of its geometric objects are

$$\mathbf{e} = \begin{array}{c} \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \end{array} \\ \boxed{\begin{array}{c|c|c|c|c} e_a & e_b & 0 & 0 & e_f \end{array}} \end{array}$$

$$\mathbf{I} = \begin{array}{c} \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \end{array} \\ \boxed{\begin{array}{c|c|c|c|c} 0 & 0 & I^c & 0 & 0 \end{array}} \end{array}$$

$$\mathbf{z} = \begin{array}{c|c|c|c|c|c} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \\ \hline \mathbf{a} & Z_{aa} & & & & \\ \mathbf{b} & & Z_{bb} & & & \\ \mathbf{c} & & & Z_{cc} & & Z_{cf} \\ \mathbf{d} & & & & Z_{dd} & \\ \mathbf{f} & & & Z_{fc} & & Z_{ff} \end{array}$$

The network is assumed to be asymmetrical so that Z_{cf} is not equal to Z_{fc}. The components of **e** and **I** are all known. The components of **i** and **E** are of no interest in the calculation of the new network, since

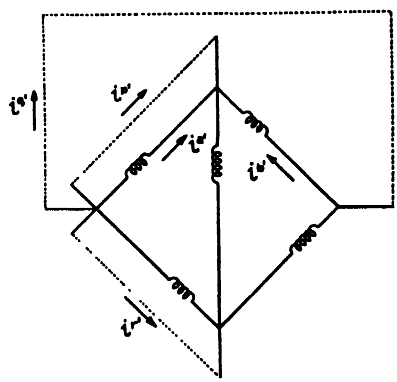


FIG. 16.9.—Equivalent All-mesh Network

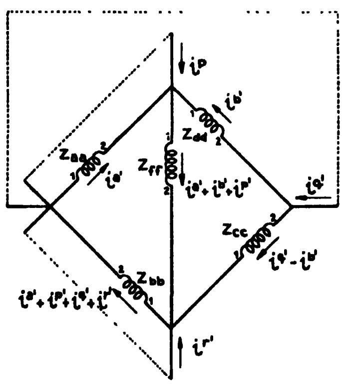


FIG. 16.10.—Currents in the Coils of the All-mesh Network

$\mathbf{i}'$ and $\mathbf{E}'$ are not derived from them but are assumed arbitrarily (just as in mesh and in junction networks).

(*c*) The transformation matrix of the all-mesh network of Fig. 16.9 is established by assuming first five new current-variables $\mathbf{i}'$ as shown on the figure. *Three of the new currents are assumed to flow in the three apparent coils.*

With the aid of Kirchhoff's first law, the currents flowing in the individual coils are established as shown in Fig. 16.10.

Equating the old and the new currents flowing in each coil

$$\begin{aligned} i^a &= i^{a'} \\ i^b &= -i^{a'} + i^{p'} + i^{q'} + i^{r'} \\ i^c &= i^{b'} - i^{q'} \\ i^d &= -i^{b'} \\ i^f &= i^{a'} + i^{b'} + i^{p'} \end{aligned} \qquad \mathbf{C} = \begin{array}{c|ccccc} & \mathbf{a}' & \mathbf{b}' & \mathbf{p}' & \mathbf{q}' & \mathbf{r}' \\ \hline \mathbf{a} & 1 & & & & \\ \mathbf{b} & 1 & & 1 & 1 & 1 \\ \mathbf{c} & & 1 & & -1 & \\ \mathbf{d} & & -1 & & & \\ \mathbf{f} & 1 & 1 & 1 & & \end{array} \qquad (16.6)$$

The coefficients of the new currents give the non-singular transformation matrix $\mathbf{C}$.

(*d*) The contravariant response quantities of the network are

$$\mathbf{i}' = \begin{array}{ccccc} \mathbf{a}' & \mathbf{b}' & \mathbf{p}' & \mathbf{q}' & \mathbf{r}' \\ \hline i^{a'} & i^{b'} & 0 & i^{q'} & i^{r'} \end{array}$$

Along the junction-pair axis $\mathbf{p}'$ the component $i^{p'}$ is known to be zero and along the original apparent coil $\mathbf{q}'$ the impressed current $i^{q'}$ is known. Hence, *the contravariant variables are those existing along the three original meshes*, namely, $i^{a'}$, $i^{b'}$, and $i^{r'}$.

The covariant response quantities are

$$\mathbf{E}' = \begin{array}{ccccc} \mathbf{a}' & \mathbf{b}' & \mathbf{p}' & \mathbf{q}' & \mathbf{r}' \\ \hline 0 & 0 & E_{p'} & E_{q'} & E_{r'} \end{array}$$

Along the mesh axes $\mathbf{a}'$ and $\mathbf{b}'$ the components $E_{a'}$ and $E_{b'}$ are known to be zero, and along the original apparent coil $\mathbf{r}'$ the impressed voltage $E_{r'}$ is known. Hence, *the covariant variables are those existing along the two original junction-pairs, namely $E_{p'}$ and $E_{q'}$.*

The total number of variables (unknowns) is five, one existing along each reference axis.

(*e*) Once the non-singular transformation tensor has been set up, the geometric objects $\mathbf{z}'$, $\mathbf{I}'$, and $\mathbf{e}'$ of the new network can be established automatically.

The new components of the impedance tensor are found by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ as

$$\mathbf{z}' = \begin{array}{c|c|c|c|c|c|} & \mathbf{a}' & \mathbf{b}' & \mathbf{p}' & \mathbf{q}' & \mathbf{r}' \\ \hline \mathbf{a}' & Z_{aa}+Z_{bb}+Z_{ff} & Z_{fc}+Z_{ff} & Z_{bb}+Z_{ff} & Z_{bb}-Z_{fc} & Z_{bb} \\ \hline \mathbf{b}' & Z_{cf}+Z_{ff} & Z_{cc}+Z_{cf}+Z_{dd}+Z_{fc}+Z_{ff} & Z_{cf}+Z_{ff} & -Z_{cc}-Z_{fc} & 0 \\ \hline \mathbf{p}' & Z_{bb}+Z_{ff} & Z_{fc}+Z_{ff} & Z_{bb}+Z_{ff} & Z_{bb}-Z_{fc} & Z_{bb} \\ \hline \mathbf{q}' & Z_{bb}-Z_{cf} & -Z_{cc}-Z_{cf} & Z_{bb}-Z_{cf} & Z_{bb}+Z_{cc} & Z_{bb} \\ \hline \mathbf{r}' & Z_{bb} & 0 & Z_{bb} & Z_{bb} & Z_{bb} \\ \hline \end{array} \qquad 16.7$$

(*f*) The new components of the impressed voltage vector $\mathbf{e}'$ are by $\mathbf{C}_t \cdot \mathbf{e}$

$$\mathbf{e}' = \begin{array}{|c|c|c|c|c|} \mathbf{a}' & \mathbf{b}' & \mathbf{p}' & \mathbf{q}' & \mathbf{r}' \\ \hline e_a + e_b + e_f & e_f & e_b + e_f & e_b & e_b \\ \hline \end{array} \qquad 16.8$$

All these components are known.

To find the impressed current vector $\mathbf{I}'$, *the inverse of* $\mathbf{C}$ *has to be calculated* by solving the set of equations 16.6 for the primed quantities as

$$\begin{aligned} i^{a'} &= i^a \\ i^{b'} &= -i^d \\ i^{p'} &= -i^a + i^d + i^f \\ i^{q'} &= -i^{c'} - i^d \\ i^{r'} &= i^b + i^c - i^f \end{aligned} \qquad \mathbf{C}^{-1} = \begin{array}{c|c|c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \\ \hline \mathbf{a}' & 1 & & & & \\ \hline \mathbf{b}' & & & & -1 & \\ \hline \mathbf{p}' & -1 & & & 1 & 1 \\ \hline \mathbf{q}' & & & -1 & -1 & \\ \hline \mathbf{r}' & & 1 & 1 & & -1 \\ \hline \end{array} \qquad 16.9$$

The new impressed mesh currents are by $\mathbf{C}^{-1} \cdot \mathbf{I} =$

$$\mathbf{I}' = \begin{array}{|c|c|c|c|c|} \mathbf{a}' & \mathbf{b}' & \mathbf{p}' & \mathbf{q}' & \mathbf{r}' \\ \hline 0 & 0 & 0 & -i^c & i^c \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \mathbf{a}' & \mathbf{b}' & \mathbf{p}' & \mathbf{q}' & \mathbf{r}' \\ \hline I^{a'} & I^{b'} & I^{p'} & I^{q'} & I^{r'} \\ \hline \end{array} \qquad 16.10$$

All these components are known.

(*g*) The equation of voltage of the network is

$$\mathbf{E}' + \mathbf{e}' = \mathbf{z}' \cdot (\mathbf{i}' + \mathbf{I}') \quad \Big| \quad E_{\alpha'} + e_{\alpha'} = z_{\alpha'\beta'}\,(i^{\beta'} + I^{\beta'}) \qquad 16.11$$

All the components of $\mathbf{e}'$ and $\mathbf{I}'$ are known while the components of $\mathbf{E}'$ and $\mathbf{i}'$ are partly known and partly unknown. *The equation cannot be solved for the unknowns in its present form.*

X. THE "ORTHOGONAL" EQUATIONS OF VOLTAGE

(*a*) In order to solve the equation of voltage of orthogonal networks, *it is necessary to subdivide it into two invariant equations along the mesh and junction-pair axes.* Hence dividing $\mathbf{e}$ into $\mathbf{e}_1 + \mathbf{e}_2$, etc. (see Section II, Chapter IX), *the "orthogonal" equations of voltage are* (leaving out the primes):

$$\left.\begin{aligned} \mathbf{E}_1 + \mathbf{e}_1 &= \mathbf{z}_1\cdot(\mathbf{i}' + \mathbf{I}') + \mathbf{z}_2\cdot(\mathbf{i}^2 + \mathbf{I}^2) \\ \mathbf{E}_2 + \mathbf{e}_2 &= \mathbf{z}_3\cdot(\mathbf{i}' + \mathbf{I}') + \mathbf{z}_4\cdot(\mathbf{i}^2 + \mathbf{I}^2) \end{aligned}\right\}$$

$$\left.\begin{aligned} E_m + e_m &= z_{mn}\,(i^n + I^n) + z_{mv}\,(i^v + I^v) \\ E_u + e_u &= z_{un}\,(i^n + I^n) + z_{uv}\,(i^v + I^v) \end{aligned}\right\} \qquad 16.12$$

These equations may be subjected to various types of manipulations.

(*b*) The following special cases may be distinguished:

1. If no known voltages are impressed in impedanceless branches, then $\mathbf{E}_1 = 0$.

2. If no known currents are impressed (or load connected) across two junctions that are not connected by a coil, then $\mathbf{i}^2 = 0$.

That is, *in the absence of apparent coils* the above equations become

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_1\cdot(\mathbf{i}^1+\mathbf{I}^1)+\mathbf{z}_2\cdot\mathbf{I}^2 \\ \mathbf{E}_2+\mathbf{e}_2 &= \mathbf{z}_3\cdot(\mathbf{i}^1+\mathbf{I}^1)+\mathbf{z}_4\cdot\mathbf{I}^2 \end{aligned} \quad\Bigg|\quad \begin{aligned} e_m &= z_{mn}(i^n+I^n)+z_{mv}I^v \\ E_u+e_u &= z_{un}(i^n+I^n)+z_{uv}I^v \end{aligned} \qquad 16.13$$

The variables are $\mathbf{i}^1$ and $\mathbf{E}_2$.

3. If all load terminals are selected as new junction-pairs, then no mesh impressed currents exist and $\mathbf{I}^1 = 0$.

With all three cases present, the equations become

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_1\cdot\mathbf{i} + \mathbf{z}_2\cdot\mathbf{I} \\ \mathbf{E} + \mathbf{e}_2 &= \mathbf{z}_3\cdot\mathbf{i} + \mathbf{z}_4\cdot\mathbf{I} \end{aligned} \quad\Bigg|\quad \begin{aligned} e_m &= z_{mn}i^n + z_{mv}I^v \\ E_u + e_u &= z_{un}i^n + z_{uv}I^v \end{aligned} \qquad 16.14$$

The variables are $\mathbf{i}$ and $\mathbf{E}$.

4. If known impressed voltages occur only around closed meshes and not around open meshes, then $\mathbf{e}_2 = 0$.

Hence, *if no apparent coils exist and if all known impressed voltages occur around the closed meshes and all known impressed currents across the junction-pairs, then the orthogonal equations of voltage reduce to*

$$\begin{aligned} \mathbf{e} &= \mathbf{z}_1\cdot\mathbf{i} + \mathbf{z}_2\cdot\mathbf{I} \\ \mathbf{E} &= \mathbf{z}_3\cdot\mathbf{i} + \mathbf{z}_4\cdot\mathbf{I} \end{aligned} \quad\Bigg|\quad \begin{aligned} e_m &= z_{mn}i^n + z_{mv}I^v \\ E_u &= z_{un}i^n + z_{uv}I^v \end{aligned} \qquad 16.15$$

The variables are $\mathbf{i}$ (the mesh-currents) and $\mathbf{E}$ (the junction-pair voltages).

(*c*) If the unknowns are the mesh currents $\mathbf{i}^1$ and the junction-pair differences of potential $\mathbf{E}_2$, the unknown $\mathbf{i}^1$ may be found from the first orthogonal equation alone as

$$\mathbf{z}_1 \cdot \mathbf{i}^1 = \mathbf{E}_1 + \mathbf{e}_1 - \mathbf{z}_2 \cdot (\mathbf{i}^2 + \mathbf{I}^2) - \mathbf{z}_1 \cdot \mathbf{I}^1$$

$$\boxed{\mathbf{i}^1 = \mathbf{z}_1^{-1} \cdot [\mathbf{E}_1 + \mathbf{e}_1 - \mathbf{z}_2 \cdot (\mathbf{i}^2 + \mathbf{I}^2)] - \mathbf{I}^1} \qquad 16.16$$

Substituting the value of $\mathbf{i}^1$ into the second orthogonal equation, the unknown $\mathbf{E}_2$ is found without any calculation of inverses as

$$\boxed{\mathbf{E}_2 = \mathbf{z}_3 \cdot (\mathbf{i}^1 + \mathbf{I}^1) + \mathbf{z}_4 \cdot (\mathbf{i}^2 + \mathbf{I}^2) - \mathbf{e}_2} \qquad 16.17$$

It may also be expressed in terms of known impressed quantities $\mathbf{E}_1$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{i}^2$ and $\mathbf{I}^2$ alone as

$$\mathbf{E}_2 = \mathbf{z}_3 \cdot \mathbf{z}_1^{-1} \cdot (\mathbf{E}_1 + \mathbf{e}_1) - \mathbf{e}_2 + (\mathbf{z}_4 - \mathbf{z}_3 \cdot \mathbf{z}_1^{-1} \cdot \mathbf{z}_2) \cdot (\mathbf{i}^2 + \mathbf{I}^2) \qquad 16.18$$

(*d*) It should be noted that *although there are as many unknowns as there are coils (five), still the matrix of* $\mathbf{z}_1$ *whose inverse has to be calculated has only as many rows and columns as there are closed meshes. No other inverse has to be calculated in finding the unknowns.*

It should also be noted that, even though for the solution of the mesh currents $\mathbf{i}^1$ only as many equations are needed as there are meshes, still it is necessary to consider the presence of the junction-pairs in order to find $\mathbf{z}_2$ needed in finding $\mathbf{i}_1$. That is, knowledge of $\mathbf{z}_1$ (that can be found by a singular transformation tensor $\mathbf{C}$ by ignoring the junction-pairs) is not sufficient to find $\mathbf{i}^1$ when impressed currents $\mathbf{I}$ also exist.

For the special case of equation 16.15, the unknowns are found by

$$\begin{aligned} \mathbf{i} &= \mathbf{z}_1^{-1} \cdot (\mathbf{e} - \mathbf{z}_2 \cdot \mathbf{I}) \\ \mathbf{E} &= \mathbf{z}_3 \cdot \mathbf{i} + \mathbf{z}_4 \cdot \mathbf{I} \\ \mathbf{E} &= \mathbf{z}_3 \cdot \mathbf{z}_1^{-1} \cdot \mathbf{e} + (\mathbf{z}_4 - \mathbf{z}_3 \cdot \mathbf{z}_1^{-1} \cdot \mathbf{z}_2) \cdot \mathbf{I} \end{aligned} \qquad 16.19$$

XI. "ACTIVE" JUNCTION-PAIRS

(*a*) When some of the junction pairs have no currents $\mathbf{I}'$ impressed upon them and if knowledge of their difference of potential $\mathbf{E}'$ is not needed, the presence of those junction-pairs may be ignored in setting up $\mathbf{z}$ and only as many junction-pairs are considered as have known $\mathbf{I}'$ or needed $\mathbf{E}'$. Or if the impedance tensor $\mathbf{z}'$ has already been estab-

lished containing all junction-pair axes, *each time a junction-pair is ignored the corresponding row and column of* $\mathbf{z}'$ *may be cancelled*, until finally $\mathbf{z}'$ has as many rows and columns as there are meshes. However, *no rows and columns of* $\mathbf{z}'$ *can be eliminated that refer to a mesh axis* since that eliminates a variable.

That is, in case of orthogonal networks the number of equations to be set up may be less than the number of coils. *In general, as many equations of voltage are set up as there are meshes and "active" junction-pairs.* An "active" junction-pair will be defined as one in which the impressed current I' or voltage E' is known, or in their absence the difference of potential E' has to be calculated for some reason.

(*b*) An orthogonal network in which all reference axes are considered (hence it has a non-singular $\mathbf{C}$) will be referred to as a "*completely orthogonal network*" or a "*complete network.*" The expression "orthogonal network" will be used whenever two types of variables exist (or two types of impressed quantities) without assuming that $\mathbf{C}$ is non-singular.

Hence, in general, *in an orthogonal network as many reference axes are assumed as there are meshes and "active" junction-pairs*, and the network is analyzed as a mesh network with a singular $\mathbf{C}$.

In many problems, especially in problems of synthesis, all networks have to be considered as completely orthogonal networks, irrespective of the type of impressed quantities, in order to establish for them a non-singular $\mathbf{C}$. *When each network is defined by a non-singular transformation tensor* $\mathbf{C}$, *it is an easy routine procedure to pass from one network to any other network in the search for a desired performance, without using cut-and-try methods for finding the connection diagrams.*

XII. NUMERICAL EXAMPLE

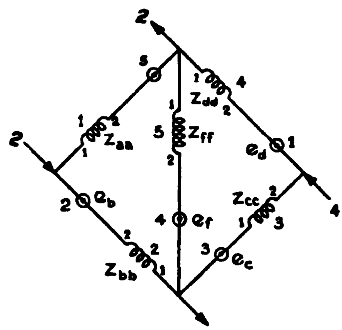


FIG. 16.11.—Given Orthogonal Network

(*a*) Let it be assumed that the impedances of Fig. 16.8 have the numerical values shown in Fig. 16.11. Also the impressed coil voltages $\mathbf{e}$ and impressed currents $\mathbf{I}$ have the values shown. The unknown quantities are the mesh currents $\mathbf{i}'$ and the junction-pair voltages $\mathbf{E}'$, that is, the unknowns are the two mesh currents $i^{a'}$ and $i^{b'}$ and the previous three junction-pair voltages $E_{p'}$, $E_{q'}$, and $E_{r'}$. No known $\mathbf{i}'$ or $\mathbf{E}'$ is impressed.

It should be noted that, although the network supplies currents to

two outside loads connected across $\mathbf{Z}_{aa}$ and $\mathbf{Z}_{ce}$, still *only one of the load terminals is assumed as a junction-pair.* The other two junction-pairs do not involve the loads themselves. Such a selection of junction-pairs may be required in many types of problems.

(*b*) If the network is considered as an all-mesh network (Fig. 16.9) the geometric objects **z**, **e**, and **I** of its primitive mesh network are

e =

a	b	c	d	f
5	2	3	1	4

I =

a	b	c	d	f
2	0	−4	0	0

z =

	a	b	c	d	f
a	1				
b		2			
c			3		
d				4	
f					5

16.20

(*c*) The new components of the geometric objects **z**′, **e**′, and **I**′ for the all-mesh network are found by substituting the components of **z** into equation 16.7 and finding $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$ and $\mathbf{I}' = \mathbf{C}^{-1} \cdot \mathbf{I}$ by equations 16.6 and 16.9, giving

e′ =

a′	b′	p′	q′	r′
11	6	6	−1	2

I′ =

a′	b′	p′	q′	r′
2	0	−2	4	−4

i′ =

a′	b′	p′	q′	r′
$i^{a'}$	$i^{b'}$	0	0	0

E′ =

a'	b'	p'	q'	r'
0	0	$E_{p'}$	$E_{q'}$	$E_{r'}$

z′ =

	a′	b′	p′	q′	r′
a′	8	5	7	2	2
b′	5	12	5	−3	0
p′	7	5	7	3	2
q′	2	−3	2	5	2
r′	2	0	2	2	2

16.21

(*d*) To solve for the unknown components of **i**′ and **E**′, each vector is divided into two components and **z**′ into four components along the mesh (**a**′, **b**′) and junction-pair (**p**′, **q**′, **r**′) axes as

$$\mathbf{i}^{1'} = \boxed{i^{a'} \mid i^{b'}} \quad \mathbf{i}^{2'} = \boxed{0 \mid 0 \mid 0} \qquad \mathbf{e}'_1 = \boxed{11 \mid 6} \quad \mathbf{e}'_2 = \boxed{6 \mid -1 \mid 2}$$

$$\mathbf{E}'_1 = \boxed{0 \mid 0} \quad \mathbf{E}'_2 = \boxed{E_{p'} \mid E_{q'} \mid E_{r'}} \qquad \mathbf{I}^{1'} = \boxed{2 \mid 0} \quad \mathbf{I}^{2'} = \boxed{-2 \mid 4 \mid -4}$$

Because of the absence of apparent coils, the known components of **i**′ and **E**′ (namely $\mathbf{i}^{2'}$ and $\mathbf{E}'_2$) are all zero.

$$\mathbf{z}_1' = \begin{array}{|c|c|}\hline 8 & 5 \\ \hline 5 & 12 \\ \hline\end{array} \qquad \mathbf{z}_2' = \begin{array}{|c|c|c|}\hline 7 & 2 & 2 \\ \hline 5 & -3 & 0 \\ \hline\end{array} \qquad 16.22$$

$$\mathbf{z}_3' = \begin{array}{|c|c|}\hline 7 & 5 \\ \hline 2 & -3 \\ \hline 2 & 0 \\ \hline\end{array} \qquad \mathbf{z}_4' = \begin{array}{|c|c|c|}\hline 7 & 2 & 2 \\ \hline 2 & 5 & 2 \\ \hline 2 & 2 & 2 \\ \hline\end{array}$$

By equation 16.16 the unknown $\mathbf{i}^{1'}$ (since $\mathbf{E}_1'$ and $\mathbf{i}^{2'}$ are zero) is

$$\mathbf{i}^{1'} = \mathbf{z}_1'^{-1} \cdot (\mathbf{e}_1' - \mathbf{z}_2' \cdot \mathbf{I}^{2'}) - \mathbf{I}^{1'}$$

Hence

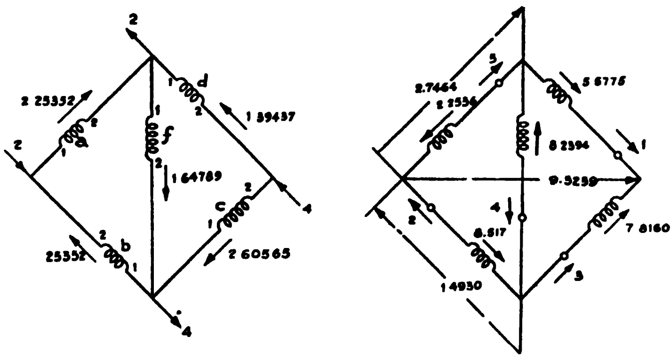
$$\mathbf{z}_2' \cdot \mathbf{I}^{2'} = \begin{array}{|c|}\hline -7\times2+2\times4-2\times4 \\ \hline -5\times2-3\times4-0\times4 \\ \hline\end{array} = \begin{array}{|c|}\hline -14 \\ \hline -22 \\ \hline\end{array} \qquad \mathbf{z}_1'^{-1} = \begin{array}{|c|c|}\hline .169014 & -.070423 \\ \hline -.070423 & .112676 \\ \hline\end{array}$$

$$\mathbf{e}_1' - \mathbf{z}_2' \cdot \mathbf{I}^{2'} = \begin{array}{|c|}\hline 11+14 \\ \hline 6+22 \\ \hline\end{array} = \begin{array}{|c|}\hline 25 \\ \hline 28 \\ \hline\end{array} \qquad \mathbf{i}^{1'} = \begin{array}{|c|}\hline 2.253521-2 \\ \hline 1.39437-0 \\ \hline\end{array} = \begin{array}{c|c|}\hline \mathbf{a}' & .253521 \\ \hline \mathbf{b}' & 1.39437 \\ \hline\end{array} \qquad 16.23$$

The unknown differences of potential $\mathbf{E}_2'$ are by equation 16.17

$$\mathbf{E}_2' = \mathbf{z}_3' \cdot (\mathbf{i}^{1'} + \mathbf{I}^{1'}) + \mathbf{z}_4' \cdot \mathbf{I}^{2'} - \mathbf{e}_2'$$

$$\mathbf{E}_2' = \begin{array}{|c|}\hline (7\times160+99\times5)/71 \\ \hline (2\times160-99\times3)/71 \\ \hline 2\times160/71 \\ \hline\end{array} + \begin{array}{|c|}\hline -7\times2+2\times4-2\times4 \\ \hline -2\times2+5\times4-2\times4 \\ \hline -2\times2+2\times4-2\times4 \\ \hline\end{array} - \begin{array}{|c|}\hline 6 \\ \hline -1 \\ \hline 2 \\ \hline\end{array} = \begin{array}{c|c|}\hline \mathbf{p}' & 2.7464 \\ \hline \mathbf{q}' & 9.3239 \\ \hline \mathbf{r}' & -1.4930 \\ \hline\end{array} \qquad 16.24$$



(*a*) Current distribution (*b*) Voltage distribution

FIG. 16.12

(*e*) The correctness of the results may be checked by putting the numerical values of the currents and voltages on the network as shown in Fig. 16.12. They must satisfy the two laws of Kirchhoff that all

currents around any junction are zero and all voltages around any closed meshes are zero. (It should be noted that in coil Z_{aa} flows $i^a + I^a = 0.25 + 2 = 2.25$; similarly for all other coils.)

XIII. STEPS IN SETTING UP THE EQUATION OF CURRENT

(*a*) The equation of current of orthogonal networks is established by steps similar to those used in setting up the equation of voltage, with the difference that the dual quantities are interchanged.

The equation of current $\mathbf{i} + \mathbf{I} = \mathbf{Y} \cdot (\mathbf{E} + \mathbf{e})$ *of orthogonal networks is established by considering them as all-junction networks by opening each mesh and producing thereby apparent coils.* For the known quantities, the same notation is used as if it were an all-mesh network as shown in Fig. 16.7.

When some of the meshes have no voltages $\mathbf{e}'$ impressed around them, and if knowledge of some of the mesh currents $\mathbf{i}'$ is not needed, their presence may be ignored. That is, *in general, as many equations of currents are set up as there are junction-pairs and "active" meshes* by considering it as a junction network with singular $\mathbf{C}_t^{-1}$.

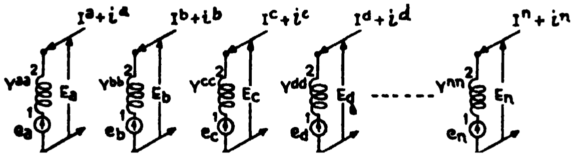


FIG. 16.13.—General Form of the Primitive Junction Network

(*b*) Their primitive junction network shown in Fig. 16.13 also contains four quantities, the known $\mathbf{e}$ and $\mathbf{I}$ and the unknown $\mathbf{i}$ and $\mathbf{E}$.

Its transformation tensor $\mathbf{C}_t^{-1}$ is set up in exactly the same manner as in any other all-junction network, shown in Section VI*b*, Chapter XIV, with the precaution that all apparent coils (that is, the opened mesh axes and the admittanceless branches with known impressed currents) are also assumed as new reference axes. In setting up the relation $\mathbf{E} = \mathbf{C}_t^{-1} \cdot \mathbf{E}'$ no attention is paid to whether or not the components of $\mathbf{E}'$ are known or are variables.

(*c*) Once $\mathbf{C}_t^{-1} = \mathbf{A}$ has been established then $\mathbf{Y}'$ is found by $\mathbf{A}_t \cdot \mathbf{Y} \cdot \mathbf{A}$. Also the known and unknown components of the variables $\mathbf{i}'$ and $\mathbf{E}'$ and of the known quantities $\mathbf{e}'$ and $\mathbf{I}'$ are established in exactly the same manner as in all-mesh networks shown in Section VIII.

(*d*) The final equation of current is

$$\mathbf{i}' + \mathbf{I}' = \mathbf{Y}' \cdot (\mathbf{E}' + \mathbf{e}') \quad \Big| \quad i^{\alpha'} + I^{\alpha'} = Y^{\alpha'\beta'}(E_{\beta'} + e_{\beta'}) \qquad 16.25$$

The components of $\mathbf{e}'$ and $\mathbf{I}'$ are known; the components of $\mathbf{E}'$ and $\mathbf{i}'$ are partly known and partly unknown.

In the set of equations there are as many variables $\mathbf{i}'$ as there are *active* meshes, and as many variables $\mathbf{E}'$ as there are junction-pairs.

The "orthogonal" equations of currents are

$$\left.\begin{aligned} \mathbf{i}^1 + \mathbf{I}^1 &= \mathbf{Y}^1 \cdot (\mathbf{e}_1 + \mathbf{E}_1) + \mathbf{Y}^2 \cdot (\mathbf{e}_2 + \mathbf{E}_2) \\ \mathbf{i}^2 + \mathbf{I}^2 &= \mathbf{Y}^3 \cdot (\mathbf{e}_1 + \mathbf{E}_1) + \mathbf{Y}^4 \cdot (\mathbf{e}_2 + \mathbf{E}_2) \end{aligned}\right\} \qquad \left.\begin{aligned} i^m + I^m &= Y^{mn}(e_n + E_n) + Y^{mv}(e_v + E_v) \\ i^u + I^u &= Y^{un}(e_n + E_n) + Y^{uv}(e_v + E_v) \end{aligned}\right\} \tag{16.26}$$

These equations may be subjected to various types of manipulations.

When no apparent coils exist, $\mathbf{E}_1$ and $\mathbf{i}^2$ are zero. When impressed voltages exist only around the meshes, then $\mathbf{e}_2 = 0$, and when impressed currents exist only across the new junction-pairs selected, then $\mathbf{I}^1 = 0$. In the presence of all these special cases the orthogonal equation of current reduces to

$$\begin{aligned} \mathbf{i} &= \mathbf{Y}^1 \cdot \mathbf{e} + \mathbf{Y}^2 \cdot \mathbf{E} \\ \mathbf{I} &= \mathbf{Y}^3 \cdot \mathbf{e} + \mathbf{Y}^4 \cdot \mathbf{E} \end{aligned} \qquad \Bigg| \qquad \begin{aligned} i^m &= Y^{mn} e_n + Y^{mv} E_v \\ I^u &= Y^{un} e_n + Y^{uv} E_v \end{aligned} \tag{16.27}$$

The variables are $\mathbf{i}$ and $\mathbf{E}$.

(*e*) If the unknowns are the junction-pair differences of potential $\mathbf{E}_2$ and the active mesh currents $\mathbf{i}^1$, the unknown $\mathbf{E}_2$ may be found from the second equation as

$$\mathbf{Y}^4 \cdot \mathbf{E}_2 = \mathbf{i}^2 + \mathbf{I}^2 - \mathbf{Y}^3 \cdot (\mathbf{e}_1 + \mathbf{E}_1) - \mathbf{Y}^4 \cdot \mathbf{e}_2$$

$$\boxed{\mathbf{E}_2 = (\mathbf{Y}^4)^{-1} \cdot [\mathbf{i}^2 + \mathbf{I}^2 - \mathbf{Y}^3 \cdot (\mathbf{e}_1 + \mathbf{E}_1)] - \mathbf{e}_2} \tag{16.28}$$

Substituting the value of $\mathbf{E}_2$ into the first orthogonal equation, the unknown $\mathbf{i}^1$ is found without any calculation of inverses as

$$\boxed{\mathbf{i}^1 = \mathbf{Y}^1 \cdot (\mathbf{e}_1 + \mathbf{E}_1) + \mathbf{Y}^2 \cdot (\mathbf{e}_2 + \mathbf{E}_2) - \mathbf{I}^1} \tag{16.29}$$

It may also be expressed in terms of known impressed quantities as

$$\mathbf{i}^1 = \mathbf{Y}^2 \cdot \mathbf{Y}^{4-1} \cdot (\mathbf{i}^2 + \mathbf{I}^2) - \mathbf{I}^1 + (\mathbf{Y}^1 - \mathbf{Y}^2 \cdot \mathbf{Y}^{4-1} \cdot \mathbf{Y}^3) \cdot (\mathbf{e}_1 + \mathbf{E}_1) \tag{16.30}$$

Here again *the tensor* $\mathbf{Y}^4$ *whose inverse has to be calculated has only as many rows and columns as there are junction-pairs.*

In the special case of equation 16.27, these equations simplify to

$$\begin{aligned} \mathbf{E} &= (\mathbf{Y}^4)^{-1} \cdot (\mathbf{I} - \mathbf{Y}^3 \cdot \mathbf{e}) \\ \mathbf{i} &= \mathbf{Y}^1 \cdot \mathbf{e} + \mathbf{Y}^2 \cdot \mathbf{E} \\ \mathbf{i} &= \mathbf{Y}^2 \cdot \mathbf{Y}^{4-1} \cdot \mathbf{I} + (\mathbf{Y}^1 - \mathbf{Y}^2 \cdot \mathbf{Y}^{4-1} \cdot \mathbf{Y}^3) \cdot \mathbf{e} \end{aligned} \tag{16.31}$$

XIV. SUMMARY OF STEPS

(*a*) Summarizing, if an orthogonal network contains fewer meshes than junction-pairs, then (if possible) its equation of voltage should be established by assuming it as a mesh network; and if it contains fewer junction-pairs than meshes, then its equation of current should be established by assuming it as a junction network. With these assumptions the amount of inverse calculations is reduced. Of course, other considerations may prevent such assumptions; sometimes the network impedances alone may be known, or the admittances, and so on.

(*b*) In finding **z** of an orthogonal network it should be remembered that:

1. The presence of these junction-pairs that are not active may simply be ignored and the network analyzed as a mesh network having a singular **C**.

2. The presence of meshes can be ignored only with the aid of the reduction of formulas of Chapter X *after* **z′** has been established.

(*c*) In finding **Y** of an orthogonal network it should be remembered that:

1. The presence of inactive meshes may be ignored and the network analyzed as a junction network having a singular $\mathbf{C}_t^{-1}$.

2. The presence of junction-pairs may be ignored only by the use of reduction formulas, *after* **Y′** has already been established.

XV. THE NON-SINGULAR C OF ANY NETWORK

(*a*) When a non-singular transformation matrix **C** is set up for a completely orthogonal network, whenever a junction-pair axis is not needed the corresponding column of **C** may be dropped. All junction-pair axes may be left out so that **C** finally has as many columns as there are meshes, representing the familiar singular transformation matrix of a ***mesh network***. For instance, if the junction-pair axes **p′**, **q′**, **r′** of Fig. 16.3 are not needed, the transformation matrix of equation 16.6 reduces to

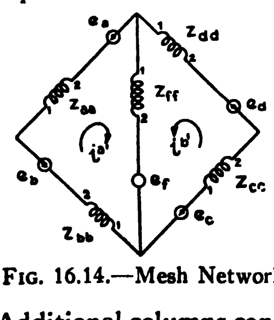


FIG. 16.14.—Mesh Network

$$\mathbf{C} = \begin{array}{c|cc|ccc|} & \mathbf{a'} & \mathbf{b'} & \mathbf{p'} & \mathbf{q'} & \mathbf{r'} \\ \hline \mathbf{a} & 1 & & & & \\ \mathbf{b} & 1 & & 1 & 1 & 1 \\ \mathbf{c} & & 1 & & -1 & \\ \mathbf{d} & & -1 & & & \\ \mathbf{f} & 1 & 1 & 1 & & \\ \hline \end{array} \qquad 16.32$$

Additional columns cannot be dropped without destroying the network.

When the non-singular $\mathbf{C}_t^{-1}$ is set up, the columns corresponding to the mesh axes may be dropped until the *junction network* is left with a transformation matrix $\mathbf{C}_t^{-1}$ (see equation 16.9)

$$\mathbf{C}_t^{-1} = \begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \\ \mathbf{f} \end{array} \begin{array}{ccccc} \mathbf{a}' & \mathbf{b}' & \mathbf{p}' & \mathbf{q}' & \mathbf{r}' \\ \hline 1 & & -1 & & \\ & & & & 1 \\ & & & -1 & 1 \\ & -1 & 1 & -1 & \\ & & 1 & & -1 \end{array} \qquad 16.33$$

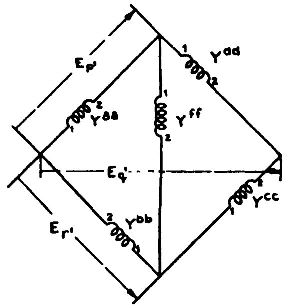


FIG. 16.15.—Junction Network

Here also no additional columns may be dropped.

That is, *it should be remembered that:*

1. *In setting up* $\mathbf{z}$, *the presence of inactive junction-pairs may be ignored. However, none of the meshes may be ignored.*

2. *In setting up* $\mathbf{Y}$, *the presence of inactive meshes may be ignored. However, no junction-pairs may be ignored.*

As extreme cases, the ignoring of all junction-pairs establishes the $\mathbf{z}$ of a mesh network, and the ignoring of all meshes establishes the $\mathbf{Y}$ of a junction network.

(*b*) To keep the physical picture clear, in ignoring the presence of inactive junction-pairs or meshes it must be remembered that *any singular (rectangular) transformation matrix of a mesh or junction network given in the earlier chapters represents only a part of a non-singular (square) transformation matrix that can be established for any mesh or junction network by simply supplying the missing orthogonal axes, either by considering the junction-pairs closed or by considering the meshes open.*

It is shown in equation 16.16 that, in order to find the mesh currents $\mathbf{i}^1$, it is sufficient to find the inverse of $\mathbf{z}_1$ only. Also, if no impressed currents $\mathbf{I}$ (and $\mathbf{i}^2$) exist, the value of $\mathbf{z}_2$ is not needed. Now the value of $\mathbf{z}_1$ needed can be found from $\mathbf{z}$ of the primitive network with the aid of the singular $\mathbf{C}$ in which the junction-pair axes are missing. That is, *the justification of using in the analysis of mesh networks a singular* $\mathbf{C}$ *is that, for finding the mesh currents* $\mathbf{i}^1$, *the values of* $\mathbf{z}_2$, $\mathbf{z}_3$, *and* $\mathbf{z}_4$ *in equation* 16.16 *are not needed, only those of* $\mathbf{z}_1$. *And the* $\mathbf{Y}$ *may be found even in the absence of the junction-pair axes in* $\mathbf{C}$.

Similar justification applies for the use of a singular $\mathbf{C}_t^{-1}$ in the analysis of junction networks, in which only the value of $\mathbf{Y}^4$ needs to be calculated with its aid.

(*c*) Since any network may be looked upon as a completely orthogonal network if it is assumed to contain four electrical quantities, *it is possible to find the components of the impedance or admittance tensors* $\mathbf{z}'$ *or* $\mathbf{Y}'$ *of any network of n coils by starting with the known components of* $\mathbf{z}$ *or* $\mathbf{Y}$ *of any other network of n coils, instead of those of the primitive networks. The reference network may have any number of meshes and junction-pairs. Both networks must be analyzed, however, as complete networks.*

The primitive network is used as a reference network because:

1. It is easy to set up the components of its geometric objects.
2. It is easy to set up the transformation tensor relating the new network to the primitive network.

(*d*) In order to find the transformation matrix changing a particular network 1 to network 2, even then it is often simpler to introduce the primitive network as a reference network. Let:

1. $\mathbf{C}^0_1$ change the primitive network to network 1.
2. $\mathbf{C}^0_2$ change the primitive network to network 2.
3. $\mathbf{C}^1_2$ change network 1 to network 2.

Then *by virtue of their group property*

$$\mathbf{C}^0_1 \cdot \mathbf{C}^1_2 = \mathbf{C}^0_2 \qquad 16.34$$

From this *the transformation matrix changing network* 1 *to network* 2 *is*

$$\boxed{\mathbf{C}^1_2 \cdot (\mathbf{C}^0_1)^{-1} \cdot \mathbf{C}^0_2} \qquad 16.35$$

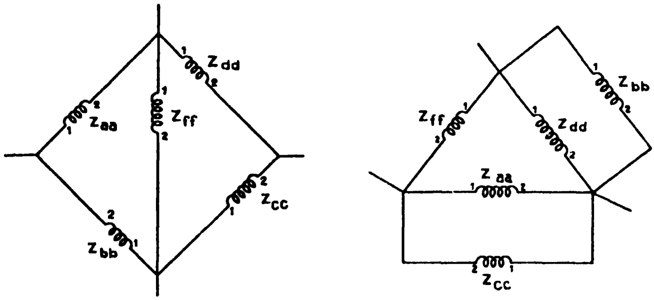
where $\mathbf{C}^0_1$ *and* $\mathbf{C}^0_2$ *are the transformation matrices of the two networks, using the primitive network as their reference network.*

(*e*) It should also be remembered that *the singular transformation tensor* $\mathbf{C}_2$ *used in neglecting magnetizing current is also part of a non-singular transformation tensor that divides the actual currents into hypothetical "load" currents and "magnetizing" currents* as shown in Section XII, Chapter VI.

XVI. TRANSFORMATION OF ANY TWO NETWORKS

(*a*) As an example to show that any *n*-coil network may be used as a reference network for the analysis of any other *n*-coil network, consider the five-coil network of Fig. 16.16*a* (reproduced from Fig. 16.8) containing two meshes and three junction-pairs whose tensors have already been set up in Section IX. Assume that its five coils are inter-

connected to form *another* network with three meshes and two junction-pairs shown in Fig. 16.16*b*. The problem is:



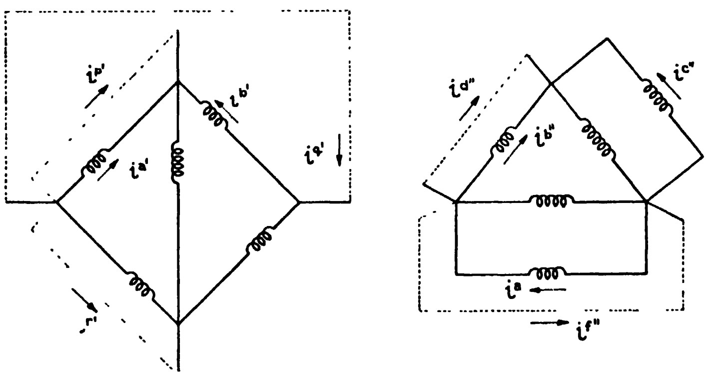
(*a*) Reference network (*b*) Derived network

FIG. 16.16.—Transformation of Networks

1. To set up a $\mathbf{C}'$ changing the first network to the second network.
2. To find $\mathbf{e}''$, $\mathbf{I}''$, and $\mathbf{z}''$ of the second network from $\mathbf{e}'$, $\mathbf{I}'$, and $\mathbf{z}'$ of the first network given in Section VI with the aid of $\mathbf{C}'$.

The method of analysis is the same as when the primitive network is used as the reference network.

(*b*) Assume both networks as five-mesh networks by considering the junction-pairs as apparent coils. Then



(*a*) Reference network (*b*) Derived network

FIG. 16.17.—Changing to All-mesh Networks

1. Assume the five new variables $\mathbf{i}''$ of the new network, two of them along the junction-pairs, the remaining ones along the meshes as shown in Fig. 16.17*b*. In the old network three junction-pairs and two mesh variables are assumed as shown in Fig. 16.17*a*.

2. Write along each coil of both networks the currents flowing through them with the aid of Kirchhoff's first law, as shown in Fig. 16.18*a* and *b*.

3. Equate the old and the new currents flowing in each coil of Figs. 16.18*a* and *b* as

$$\begin{array}{rcl} i^{a'} & = & i^{a''} - i^{b''} - i^{d''} - i^{f''} \\ i^{a'} + i^{p'} - i^{q'} + i^{r'} & = & i^{c''} \\ i^{b'} - i^{q'} & = & i^{a''} \\ -i^{b'} & = & i^{b''} + i^{c''} + i^{d''} \\ i^{a'} + i^{b'} + i^{p'} & = & -i^{b''} \end{array} \qquad 16.36$$

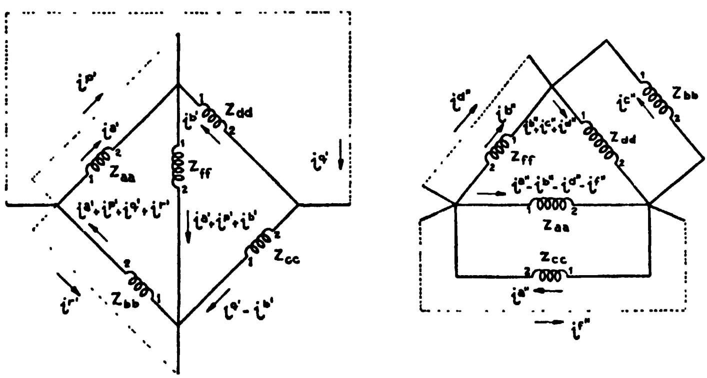
This set of equations can always be rearranged so that only *one* old current occurs on the left-hand side of each equation as

$$\begin{array}{l} i^{a'} = i^{a''} - i^{b''} - i^{d''} - i^{f''} \\ i^{b'} = -i^{b''} - i^{c''} - i^{d''} \\ i^{p'} = -i^{a''} + i^{b''} + i^{c''} + 2i^{d''} + i^{f''} \\ i^{q'} = -i^{a''} - i^{b''} - i^{c''} - i^{d''} \\ i^{r'} = i^{a''} + i^{b''} - i^{c''} \end{array}$$

$$\mathbf{C}' = \begin{array}{c|ccccc} & a'' & b'' & c'' & d'' & f'' \\ \hline a' & 1 & -1 & 0 & -1 & -1 \\ b' & 0 & -1 & -1 & -1 & 0 \\ p' & -1 & 1 & 1 & 2 & 1 \\ q' & -1 & -1 & -1 & -1 & 0 \\ r' & 1 & 1 & -1 & 0 & 0 \end{array} \qquad 16.37$$

The presence of the integer 2 should be noted.

4. The coefficients of the new currents give the transformation matrix **C**′ changing the network of Fig. 16.18*a* to that of Fig. 16.18*b*.



(*a*) Reference networks (*b*) Derived network

FIG. 16.18.—Currents in Individual Coils

(*c*) The new components of the impedance tensor of the new network are by $\mathbf{C}'_t \cdot \mathbf{z}' \cdot \mathbf{C}' =$

$$
z'' = \begin{array}{c|c|c|c|c|c|}
 & a'' & b'' & c'' & d'' & f'' \\
\hline
a'' & Z_{aa} + Z_{cc} & -Z_{aa} - Z_{cf} & 0 & -Z_{aa} & -Z_{aa} \\
\hline
b'' & -Z_{aa} - Z_{fc} & Z_{aa} + Z_{dd} + Z_{ff} & Z_{dd} & Z_{aa} + Z_{dd} & Z_{aa} \\
\hline
c'' & 0 & Z_{dd} & Z_{bb} + Z_{dd} & Z_{dd} & 0 \\
\hline
d'' & -Z_{aa} & Z_{aa} + Z_{dd} & Z_{dd} & Z_{aa} + Z_{dd} & Z_{aa} \\
\hline
f'' & -Z_{aa} & Z_{aa} & 0 & Z_{aa} & Z_{aa} \\
\hline
\end{array} \qquad 16.38
$$

The new components of the impressed coil voltages are by $C'_t \cdot e' = e'' =$

$$
e'' = \begin{array}{|c|c|c|c|c|}
\multicolumn{1}{c}{a''} & \multicolumn{1}{c}{b''} & \multicolumn{1}{c}{c''} & \multicolumn{1}{c}{d''} & \multicolumn{1}{c}{f''} \\
\hline
e_a + e_c & -e_a + e_d - e_f & -e_b + e_d & -e_a + e_d & -e_a \\
\hline
\end{array} \qquad 16.39
$$

The vector I″ may be found by calculating first the inverse of C′.

XVII. A CHECK ON THE TRANSFORMATION

The correctness of C′, z″, e″, and I″ of Fig. 16.16*b* may be checked by deriving them from the primitive network instead of from Fig. 16.16*a*.

The transformation matrix C″ of Fig. 16.8*b* (by using its primitive mesh network) is:

$$
\begin{aligned}
i^a &= i^{a''} - i^{b''} - i^{d''} - i^{f''} \\
i^b &= -i^{c''} \\
i^c &= i^{a''} \\
i^d &= i^{b''} + i^{c''} + i^{d''} \\
i^f &= -i^{b''}
\end{aligned}
\qquad
C'' = \begin{array}{c|c|c|c|c|c|}
 & a'' & b'' & c'' & d'' & f'' \\
\hline
a & 1 & -1 & & -1 & -1 \\
\hline
b & & & -1 & & \\
\hline
c & 1 & & & & \\
\hline
d & & 1 & 1 & 1 & \\
\hline
f & & -1 & & & \\
\hline
\end{array} \qquad 16.40
$$

This matrix C″ must be the product of C of equation 16.6 and C′ of equation 16.37 or $C \cdot C' = C''$. That is

$$
\begin{array}{|c|c|c|c|c|}
\hline
1 & & & & \\
\hline
1 & & 1 & 1 & 1 \\
\hline
 & 1 & & -1 & \\
\hline
 & -1 & & & \\
\hline
1 & 1 & 1 & & \\
\hline
\end{array}
\cdot
\begin{array}{|c|c|c|c|c|}
\hline
1 & -1 & & -1 & -1 \\
\hline
 & -1 & -1 & -1 & \\
\hline
-1 & 1 & 1 & 2 & 1 \\
\hline
-1 & -1 & -1 & -1 & \\
\hline
1 & 1 & -1 & & \\
\hline
\end{array}
=
\begin{array}{|c|c|c|c|c|}
\hline
1 & -1 & & -1 & -1 \\
\hline
 & & -1 & & \\
\hline
1 & & & & \\
\hline
 & 1 & 1 & 1 & \\
\hline
 & -1 & & & \\
\hline
\end{array} \qquad 16.41
$$

The two matrices 16.40 and 16.41 are the same.

The impedance matrix is

$$
\mathbf{C}_t''\cdot\mathbf{z}\cdot\mathbf{C}'' = \begin{array}{c|c|c|c|c|c|}
 & \mathbf{a}'' & \mathbf{b}'' & \mathbf{c}'' & \mathbf{d}'' & \mathbf{f}'' \\
\hline
\mathbf{a}'' & Z_{aa} + Z_{cc} & -Z_{aa} - Z_{cf} & 0 & -Z_{aa} & -Z_{aa} \\
\hline
\mathbf{b}'' & -Z_{aa} - Z_{fc} & Z_{aa} + Z_{dd} + Z_{ff} & Z_{dd} & Z_{aa} + Z_{dd} & Z_{aa} \\
\hline
\mathbf{c}'' & 0 & Z_{dd} & Z_{bb} + Z_{dd} & Z_{dd} & 0 \\
\hline
\mathbf{d}'' & -Z_{aa} & Z_{aa} + Z_{dd} & Z_{dd} & Z_{aa} + Z_{dd} & Z_{aa} \\
\hline
\mathbf{f}'' & -Z_{aa} & Z_{aa} & 0 & Z_{aa} & Z_{aa} \\
\hline
\end{array}
\qquad 16.42
$$

This matrix checks with equation 16.38. Also $\mathbf{C}_t''\cdot\mathbf{e} = \mathbf{e}'' =$

$$
\mathbf{e}'' = \begin{array}{|c|c|c|c|c|}
\mathbf{a}'' & \mathbf{b}'' & \mathbf{c}'' & \mathbf{d}'' & \mathbf{f}'' \\
\hline
e_a + e_c & -e_a + e_d - e_f & -e_b + e_d & -e_a + e_d & -e_a \\
\hline
\end{array}
\qquad 16.43
$$

This 1-matrix also checks equation 16.39.

Hence *it does not make any difference what n-coil network is used as the reference network for the calculation of any n-coil network. The final n-matrices and equations are the same, irrespective of the starting point.*

XVIII. THE EQUIVALENCE OF ALL n-COIL NETWORKS

(a) Now that it has been shown that it is possible to establish a non-singular transformation matrix $C_{\alpha'}^{\alpha}$ between any two n-coil networks no matter how many meshes, junction-pairs, and sub-networks each one has, it follows that the six "n-matrices" associated with each network, namely, $e_{(a)}$, $E_{(a)}$, $i_{(a)}$, $I_{(a)}$, $z_{(a)(\beta)}$, and $Y_{(a)(\beta)}$, are not independent of one another, but are different sets of components along different reference frames of six entities, or "geometric objects" namely, of e_α, E_α, i^α, I^α, $z_{\alpha\beta}$, and $Y^{\alpha\beta}$.

These great varieties of n-matrices are bound together into the six entities by the group of non-singular transformation matrices $C_{\alpha'}^{\alpha}$ because of the fact that each of them may be found from any of the others solely by using some particular $C_{\alpha'}^{\alpha}$ and no other n-matrix. *The components of all transformation matrices of this group contain only integers* This group may be called the "*connection group.*"

Similarly *the various equations of performance of all n-coil networks are not independent, but can be transformed into one another with the aid of $C_{\alpha'}^{\alpha}$.* Hence, if the various types of equations have once been established for the primitive network (or for any other network), similar equations can be established for all asymmetrical, active networks by a routine transformation.

It is however emphasized that no groups of transformation matrices have as yet been established between networks having *different* number of coils. Such groups will be introduced in connection with the synthesis of networks.

(*b*) To summarize, in investigating the characteristics, the equations, or the behavior of networks, it is not necessary to deal explicitly with one particular network; it is often possible to deal with all analogous networks at the same time by setting up *invariant* equations. This is possible only because of the existence of a group of non-singular transformation matrices $C_{\alpha'}^{\alpha}$ that can change the equations of performance of any n-coil network to that of any other n-coil network. *The transformation matrices that may be established between all possible n-coil networks from a group (the connection group), since they satisfy the four group conditions* (Section III, Chapter XI), namely:

(1) The product of any two C-s is also an element of the group; that is if $\mathbf{C}_2^1$ changes network 1 to 2 and $\mathbf{C}_3^2$ changes network 2 to 3 then their product

$$\mathbf{C}_2^1 \cdot \mathbf{C}_3^2 = \mathbf{C}_3^1$$

changes network 1 to 3.

(2) The associative law holds:

$$(\mathbf{C}_2^1 \cdot \mathbf{C}_3^2) \cdot \mathbf{C}_4^3 = \mathbf{C}_2^1 \cdot (\mathbf{C}_3^2 \cdot \mathbf{C}_4^3)$$

That is, several successive transformations may be performed in any grouping.

(3) A network remains unchanged by transforming it with the unit matrix **I**.

(4) Each transformation matrix **C** has an inverse $\mathbf{C}^{-1}$ so that it is possible to reestablish the equations of the original network from those of the new network with the aid of $\mathbf{C}^{-1}$.

It should be remembered that the existence of this group of **C** implies that *four* electromagnetic quantities (**e**, **E**, **i** and **I**) are associated with every network and not just two (**e**, **i** or **E**, **I**).

CHAPTER XVII

INTERLINKED ELECTRIC AND MAGNETIC NETWORKS

I. THE MAGNETIC AND DIELECTRIC NETWORKS

(*a*) In setting up the equations of performance of networks, say $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$, no attention was paid to the components of the various geometric objects. It was understood that they can be constants, functions, linear operators, etc.

Now let a certain special form of $\mathbf{z}$ *and* $\mathbf{Y}$ *be investigated in greater detail.* Let their matrix be *symmetrical*, that is, let the network be *stationary* and symmetrical, and let it contain only lumped resistances, inductances, and elastances. The problem of this chapter is to express the $\mathbf{z}$ and $\mathbf{Y}$ tensors of these special networks in terms of more fundamental tensors.

(*b*) *In the study of such networks three different types of continuous paths will be distinguished*, each contributing its share of design constants in establishing $\mathbf{z}$ and $\mathbf{Y}$. These are:

1. The paths described by *electric charges*. These paths will be called the "electrical network."

2. The paths described by *magnetic flux lines*, to be called "magnetic network."

3. The paths described by *dielectric flux lines*, to be called "dielectric network."

Each of these networks is built up of "coils" and "junctions" (0-cells and 1-cells). (In rotating electrical machinery these three types of networks are generalized to include electric, magnetic, and dielectric "layers" (2-cells) also. In addition there are also the *mechanical networks* to be considered in the analysis of rotating machinery, or of other electromechanical systems.)

(*c*) Now, *an electrical network consisting of interconnected continuous copper conductors will be considered to have no other design constants besides resistances* $r_{\alpha\beta}$ representing the opposition offered by the network to the flow of the electric charges. The inductances of the coils $l_{\alpha\beta}$ are due to *magnetic flux lines that do not travel along the conductors as the electric charges do, but follow entirely different paths* lying outside the continuous copper conductors. Similarly, the elastances $s_{\alpha\beta}$ are due to dielectric flux lines located partly in condensers (in *discontinuities introduced in the conductors*). *The condensers will not be considered as part of the electrical network*, since the electric charges do not travel along the discontinuities.

(*d*) Hence, when it is assumed that the impedance tensor $z_{\alpha\beta}$ of a network of coils contains other design constants besides resistance $r_{\alpha\beta}$, then it is tacitly assumed that ***underlying the electrical network there are other networks interlinked with it, namely, magnetic and dielectric networks, whose presence, however, has been masked by the artifice of endowing the individual coils with self- and mutual inductances and elastances.***

However, the components of $z_{\alpha\beta}$, namely, the self- and mutual inductances and elastances of the individual coils, may be changed by ***interconnecting*** the component members of the underlying magnetic and dielectric networks in a different manner, without changing the interconnection of the electrical network itself; hence in establishing the design constants $z_{\alpha\beta}$ of the electrical network the study of the interlinking magnetic and dielectric networks is necessary.

On their own right, magnetic and dielectric networks also occur in electrical engineering studies, without interlinking with electrical networks. As independent networks their engineering importance is far less, however, than that of electrical networks. Their analysis as independent networks has been covered in the previous chapters.

II. EQUATIONS OF PERFORMANCE OF ISOLATED MESH NETWORKS

(*a*) If isolated electric, magnetic, or electrostatic *fields* are considered, the relations between the field intensities (impressed quantity) and the resultant flux densities (response quantity) along a direction assumed at each point of a *stationary homogeneous medium* are respectively

1. In the electric field: $e = ri$ or $i = ge$ 17.1
2. In the magnetic field: $h = \rho b$ $b = \mu h$ 17.2
3. In the electrostatic field: $e = sd$ $d = ce$ 17.3

The various symbols are defined in Table 17.1.

TABLE 17.1

	Electrical		Material
e	Electric field intensity	r	Resistivity
h	Magnetic field intensity	ρ	Reluctivity
e	Electric field intensity	s	Elastivity
i	Current density	g	Conductivity
b	Magnetic flux density	μ	Permeability
d	Electrostatic flux density	c	Dielectric constant

(*b*) In a stationary *non-homogeneous medium* a field intensity acting in one direction produces flux lines that have components in other directions also, hence the proportionality factor between the two quantities cannot be a scalar. Considering the impressed and response quantities at each point as vectors, *the constants of the medium become tensors of valence two in place of scalars, and the above relations in a non-homogeneous medium become*

1. $e_\alpha = r_{\alpha\beta} i^\beta$ or $i^\alpha = g^{\alpha\beta} e_\beta$ 17.4

2. $h^\alpha = \rho^{\alpha\beta} b_\beta$ $\quad b_\alpha = \mu_{\alpha\beta} h^\beta$ 17.5

3. $e_\alpha = s_{\alpha\beta} d^\beta$ $\quad d^\alpha = c^{\alpha\beta} e_\beta$ 17.6

The position of the indices follows from considerations to be introduced later.

(*c*) In network studies, instead of considering the field intensity at a point, its line integral between two points (a junction-pair) or around a closed circuit (a mesh) is considered. Also instead of considering the flux density at a point, *its surface integral* over the cross-section of a coil is considered.

A network with several meshes or junction-pairs may be considered as a non-homogeneous field in which only a finite number of directions may be assumed (in all other directions the field is zero).

With such interpretations, equations 17.4 *to* 17.6 *remain unchanged, representing the equations of performance of the three types of mesh networks*, each type being an isolated network.

(*d*) First the study of *isolated* magnetic and dielectric networks is undertaken; afterward the effect of their *interrelation* with electrical networks is studied.

III. MAGNETIC NETWORKS

(*a*) A magnetic network consists of magnetic conductors (usually iron and air) and their junctions as shown in Fig. 17.1*a*. The analogous electrical network is shown in Fig. 17.1*b*. Such magnetic networks play an important part in multiwinding transformers, rotating machines, measuring instruments, etc. The design constants are the *reluctances* ρ^{aa}, ρ^{bb}, etc., of the various magnetic members or their permeances μ_{aa}, μ_{bb}, etc. There are no mutual reluctances ρ^{ab} or permeances μ_{ab} between the individual members, just as there are no mutual resistances between the individual coils of an electrical network.

The magnetic network can also be considered as a mesh, or as a junction, or as an orthogonal network, depending on the point of view.

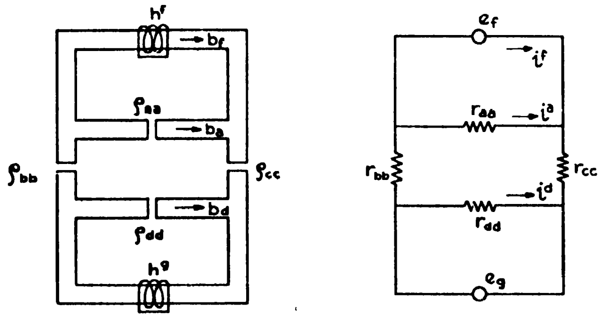
(*b*) The superimposed electromagnetic quantities are of two types:

1. *Magnetizing forces* h^α, usually produced by currents flowing in coils wound around the magnetic paths at isolated points (m.m.f.'s).

Permanent magnets also have a magnetizing force; however, they will not be considered at this point.

2. *Magnetic flux lines* b_α.

Each of them may be an *impressed* or a *response* quantity.



(*a*) Magnetic network (*b*) Its electrical analogue

Fig. 17.1

(*c*) In their *physical* action m.m.f.'s are analogous to voltages and flux lines to currents. In particular:

1. In a *mesh network* m.m.f.'s **h** are impressed at various points and flux lines **b** appear around the closed meshes in response, analogously to impressed **e** and response **i**.

2. In a *junction network* flux lines **B** are impressed across the junction-pairs and m.m.f.'s **H** (or rather differences of magnetic potential) appear in response across the junction-pairs, analogously to impressed **I** and response **E**.

Because of certain interdependence existing between the vectors of the electrical and magnetic networks in an interlinked system (*to be shown in Section VIIIc*) *the flux lines linking the various coils are represented as a covariant vector* b_α, *while the physically analogous current is a contravariant vector* i^α. Similarly, the m.m.f. of a coil is a contravariant vector h^α while the physically analogous voltage is a covariant vector e_α.

Corresponding to this change, the reluctance tensor is $\rho^{\alpha\beta}$ (physically analogous to $r_{\alpha\beta}$), and *the permeance tensor is* $\mu_{\alpha\beta}$ (physically analogous to $Y^{\alpha\beta}$.)

The various equations of performance are analogous to those of electrical networks, except that the position of the indices is interchanged. That is:

1. For *mesh* networks the equation of m.m.f. is

$$\mathbf{h} = \boldsymbol{\rho}\cdot\mathbf{b} \quad \Big| \quad h^m = \rho^{mn}b_n \tag{17.7}$$

2. For *junction* networks the equation of flux is

$$\mathbf{B} = \boldsymbol{\mu}\cdot\mathbf{H} \quad \Big| \quad B_u = \mu_{uv}H^v \tag{17.8}$$

3. For *orthogonal* networks the equations of m.m.f. and flux are respectively

$$\mathbf{H} + \mathbf{h} = \boldsymbol{\rho}\cdot(\mathbf{b} + \mathbf{B}) \quad \Big| \quad H^\alpha + h^\alpha = \rho^{\alpha\beta}(b_\beta + B_\beta) \tag{17.9}$$

$$\mathbf{b} + \mathbf{B} = \boldsymbol{\mu}\cdot(\mathbf{H} + \mathbf{h}) \quad \Big| \quad b_\alpha + B_\alpha = \mu_{\alpha\beta}(H^\beta + h^\beta) \tag{17.10}$$

Since the covariant and contravariant indices of physically analogous quantities are interchanged, *the same dualism exists between electrical and magnetic networks as exists between mesh and junction networks.*

IV. EXAMPLE OF A MAGNETIC MESH NETWORK

(a) Let the network of Fig. 17.2 be given in which an electrical network is interlinked with a magnetic network, and let the performance of the electrical network be found.

In all electrical network problems hitherto considered it has been assumed that the self- and mutual inductances of the individual coils (**z** of the primitive network) are known. In the present example it is assumed, however, that only the individual reluctances $\rho^{\alpha\beta}$ of the magnetic members are known. Hence *in order to find the performance of the electrical network it is necessary first to remove all electrical interconnections and to find the self- and mutual inductances (permeances) of the individual coils* (shown again in Fig. 17.3a) from the individual reluctances of the magnetic members. Similar cases occur in all multiwinding transformers.

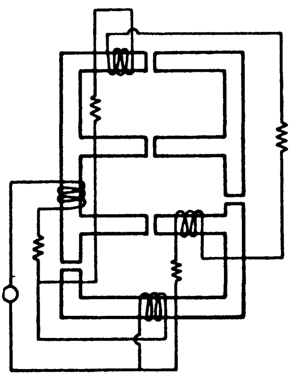


FIG. 17.2—Interlinked Electric and Magnetic Network

The problem of finding the self- and mutual permeances (or inductances) of the individual coils placed on a magnetic network is analogous to finding the self- and mutual admittances of the individual *coils* of an electrical network *after* the coils are interconnected into several meshes, shown in

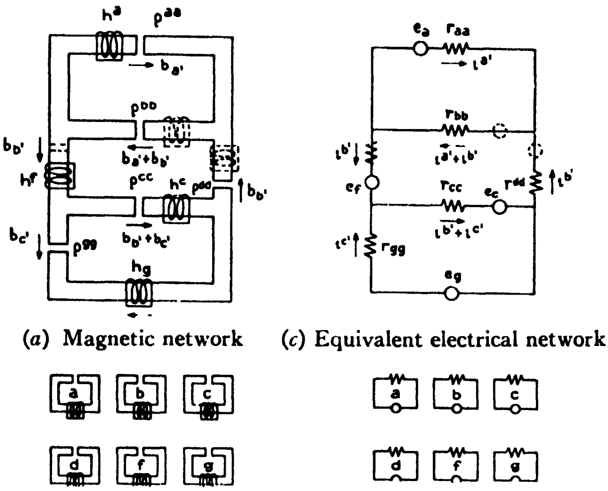
Section VIII, Chapter V. That is, the calculation is made in two steps:

1. First the self- and mutual reluctances of the magnetic *meshes* are found from those of the individual magnetic members.

2. Then the self- and mutual permeances of the *individual magnetic members* (or of the coils wound around them) is found.

In this section the first step is calculated. The second step is calculated in Section XIII.

(*b*) The electrical network analogous to the magnetic network of Fig. 17.3*a* is shown in Fig. 17.3*c*. Their analysis follows parallel



(*a*) Magnetic network (*c*) Equivalent electrical network

(*b*) Its primitive network (*d*) Primitive electrical network

FIG. 17.3

The primitive magnetic mesh network is shown in Fig. 17.3*b*. Its reluctance tensor is

$$\mathbf{\rho} = \rho^{mn} = \begin{array}{c|c|c|c|c|c|c|} {}_{m}\diagdown^{n} & a & b & c & d & f & g \\ \hline a & \rho^{aa} & & & & & \\ \hline b & & \rho^{bb} & & & & \\ \hline c & & & \rho^{cc} & & & \\ \hline d & & & & \rho^{dd} & & \\ \hline f & & & & & 0 & \\ \hline g & & & & & & \rho^{gg} \\ \hline \end{array} \qquad 17.11$$

The components of this tensor, representing the reluctances of the various paths, are usually calculated by a *field* method. Instead of reluctances ρ^{aa} . . . of course *permeances* μ_{aa} . . . may be used where $\rho^{aa} = 1/\mu_{aa}$, etc.

In the magnetic network with every m.m.f. is associated a magnetic reluctance member which may have zero reluctance; and with every *reluctance* is associated an m.m.f. whose value may be zero. As a consequence the primitive magnetic mesh network contains an equal number of electric and magnetic members. The introduction of additional coils and reluctances is analogous to the introduction of additional voltage e_a alongside an impedance Z_{aa} and an additional impedance alongside a voltage, so that in the primitive electrical network an impedance Z_{aa} is always associated with a voltage e_a alongside of it.

The impressed m.m.f. vector is

$$\mathbf{h} = h^m = \begin{array}{c|c|c|c|c|c} a & b & c & d & f & g \\ \hline h^a & 0 & h^c & 0 & h^f & h^g \end{array} \qquad 17.12$$

(*c*) Since there are *three* meshes, any three of the flux lines may be assumed as variables as shown in Fig. 17.3*a*. The flux lines passing through each individual magnetic reluctance are also shown.

The equation of transformation $b_m = C_m^{m'} b_{m'}$ or $\mathbf{b} = \mathbf{C}_t^{-1} \cdot \mathbf{b}'$ is established by ***equating the old and the new fluxes passing through each magnetic reluctance*** ρ^{aa}, ρ^{bb} as

$$\begin{aligned} b_a &= b_{a'} \\ b_b &= b_{a'} + b_{b'} \\ b_c &= b_{b'} + b_{c'} \\ b_d &= b_{b'} \\ b_f &= b_{b'} \\ b_g &= b_{c'} \end{aligned} \qquad \mathbf{C}_t^{-1} = \mathbf{A} = \begin{array}{c|c|c|c|} & \mathbf{a'} & \mathbf{b'} & \mathbf{c'} \\ \hline \mathbf{a} & 1 & & \\ \mathbf{b} & 1 & 1 & \\ \mathbf{c} & & 1 & 1 \\ \mathbf{d} & & 1 & \\ \mathbf{f} & & 1 & \\ \mathbf{g} & & & 1 \\ \hline \end{array} \qquad 17.13$$

The coefficients of the new fluxes give $\mathbf{C}_t^{-1}$.

If $\mathbf{C}_t^{-1}$ is replaced by $\mathbf{A}$, then all the formulas of the electrical mesh network may be used for the magnetic and mesh network by replacing in them (expressed in direct notation) $\mathbf{C}$ by $\mathbf{A}$.

(*d*) The new components of the reluctance tensor $\boldsymbol{\rho}'$ are found by $\mathbf{A}_t \cdot \boldsymbol{\rho} \cdot \mathbf{A}$ and those of $\mathbf{h}'$ by $\mathbf{A}_t \cdot \mathbf{h}$ as

$$\mathbf{p \cdot A} = \begin{array}{c|ccc} & a' & b' & c' \\ \hline a & \rho^{aa} & & \\ b & \rho^{bb} & \rho^{bb} & \\ c & & \rho^{cc} & \rho^{cc} \\ d & & \rho^{dd} & \\ f & & & \\ g & & & \rho^{gg} \end{array} \tag{17.14}$$

$$\rho^{m'n'} = \begin{array}{c|ccc} m' \backslash n' & a' & b' & c' \\ \hline a' & \rho^{aa} + \rho^{bb} & \rho^{bb} & 0 \\ b' & \rho^{bb} & \rho^{bb} + \rho^{cc} + \rho^{dd} & \rho^{cc} \\ c' & 0 & \rho^{cc} & \rho^{cc} + \rho^{gg} \end{array} \tag{17.15}$$

$$\mathbf{h}' = h^{m'} = \begin{array}{ccc} a' & b' & c' \\ \hline h^a & h^c + h^f & h^c + h^g \end{array} \tag{17.16}$$

(*e*) *If the impressed m.m.f.'s are assumed to be known*, then the flux lines existing in the meshes are found by $\mathbf{b}' = \mathbf{p}'^{-1} \cdot \mathbf{h}' = \boldsymbol{\mu}' \cdot \mathbf{h}'$.

If the three components of $\mathbf{b}'$ have been calculated, *the flux lines in the individual reluctances* are found by $\mathbf{b} = \mathbf{A} \cdot \mathbf{b}'$. The *magnetic potential drops* (m.m.f.) across each individual reluctance are found by $\mathbf{h}_c = \mathbf{p} \cdot \mathbf{A} \cdot \mathbf{b}'$, where $\mathbf{p} \cdot \mathbf{A}$ has already been calculated in equation 17.14 so that

$$\mathbf{h}_c = \begin{array}{cccccc} a & b & c & d & f & g \\ \hline \rho^{aa} b_{a'} & \rho^{bb}(b_{a'} + b_{b'}) & \rho^{cc}(b_{b'} + b_{c'}) & \rho^{dd} b_{b'} & 0 & \rho^{gg} b_{c'} \end{array} \tag{17.17}$$

V. DIELECTRIC NETWORKS

(*a*) A dielectric network consists of electric conductors and insulators (say copper and air) forming paths for the electrostatic flux lines. They play important parts in all electrical apparatus subjected to high voltages.

The design constants are the *elastances* s_{aa}, s_{bb}, etc., of the various members, or their inverse, the capacitances C^{aa}, C^{bb}, etc. They are assumed to be concentrated in the air gap between the conductors. They are practically always calculated as field problems. There are again no mutual capacitances C^{ab} between individual members.

The dielectric network may also be considered as a mesh, junction, or orthogonal network.

(*b*) The superimposed electromagnetic quantities are:

1. *Electromotive forces* e_α. They are the same that accelerate the electric charges in an electrical network.

2. *Electrostatic flux lines* d^α (also called "displacement").

Instead of saying that $+q$ electric charges are placed on point A and $-q$ on point B, it will be said that d electric flux lines enter point A and leave point B; that is, d electric flux lines flow from point A to

point B through whatever paths they encounter between A and B. *If so desired, electric charges* $\mathbf{q}$ *and electrostatic fluxes* $\mathbf{d}$ *may be interchanged in the following.*

Each of these may be an impressed or a response quantity.

(*c*) *There is a dualism between a dielectric and magnetic network, but there is no dualism between a dielectric and an electrical network.* That is, the electric flux lines d^α (or charges q^α) and the physically analogous current i^α are both contravariant vectors.

The difference between an electric and a dielectric network lies in the manner in which *time* enters into their equation of performance.

(*d*) The various equations of performance for the three types of dielectric networks are analogous to those of electrical networks. That is,

Mesh:	$\rightarrow$	$\mathbf{e}=\mathbf{s}\cdot\mathbf{d}$	$e_m=s_{mn}d^n$	17.18
Junction:	$\rightarrow$	$\mathbf{D}=\mathbf{C}\cdot\mathbf{E}$	$D^u=C^{uv}E_v$	17.19
Orthogonal:	$\rightarrow$	$\mathbf{E}+\mathbf{e}=\mathbf{s}\cdot(\mathbf{d}+\mathbf{D})$	$E_\alpha+e_\alpha=s_{\alpha\beta}(d^\beta+D^\beta)$	17.20
	$\rightarrow$	$(\mathbf{d}+\mathbf{D})=\mathbf{C}\cdot(\mathbf{E}+\mathbf{e})$	$d^\alpha+D^\alpha=C^{\alpha\beta}(E_\beta+e_\beta)$	17.21

In these equations $\mathbf{d}$ may be replaced by $\mathbf{q}$.

(*e*) *In engineering problems magnetic networks usually appear as mesh networks, while dielectric networks usually appear as junction networks.*

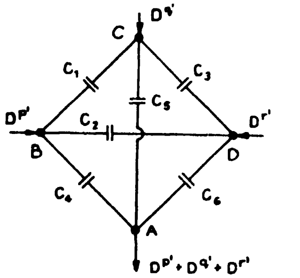
An electric charge appearing at a point is equivalent to electric flux lines entering that point as if it were a junction point of several dielectric members. That is, *the magnetic flux lines appearing in engineering problems (produced by electric currents) are considered as closed lines, while the electrostatic flux lines are considered as open lines.* (They start at a positive charge and end at a negative charge.) Or magnetic flux lines form closed meshes, while dielectric flux lines form open meshes.

Similarly in magnetic networks the *impressed m.m.f.'s* are usually concentrated at a point *in series* with each magnetic member; on the other hand, in dielectric networks the impressed *differences of potential* are usually applied *across* the dielectric members, not in series with them.

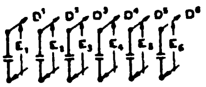
VI. EXAMPLE OF A DIELECTRIC JUNCTION NETWORK

(*a*) Let the dielectric network of Fig. 17.4*a* be given, consisting of four conductors A, B, C, and D (say the four electrodes of a tube,

or the three conductors of a transmission line and the ground, etc.). There are three meshes and three junction-pairs, say A-D, A-C, and A-B. (Junction A may be the ground.) Let flux lines enter the junctions and leave at junction A. (That is, let electric charges be



(*a*) Dielectric junction network



(*b*) Its primitive junction network

FIG. 17.4

placed on conductors B, C, and D, assuming conductor A as the ground.) The differences of potentials appearing across any two conductors are to be investigated.

(*b*) The capacitance tensor of the primitive junction network of Fig. 17.4*b* is

$C^{uv} =$

$u \backslash v$	1	2	3	4	5	6	7
1	C^1						
2		C^2					
3			C^3				
4				C^4			
5					C^5		
6						C^6	
7							C^7

17.22

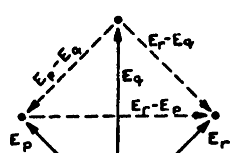


FIG. 17.5.—Differences of Potential

The three differences of potential assumed are E_p, E_q, and E_r shown in Fig. 17.5, where are also shown the differences of potentials appearing across each member.

Equating the old and the new differences of potentials appearing across each member

$$
\begin{aligned}
E_1 &= E_{p'} - E_{q'} \\
E_2 &= -E_{p'} + E_{r'} \\
E_3 &= -E_{q'} + E_{r'} \\
E_4 &= E_{q'} \\
E_5 &= E_{q'} \\
E_6 &= E_{r'}
\end{aligned}
\qquad
C_u^{u'} = \mathbf{A} = \begin{array}{c|ccc}
u \backslash u' & p' & q' & r' \\
\hline
1 & 1 & -1 & \\
2 & -1 & & 1 \\
3 & & -1 & 1 \\
4 & 1 & & \\
5 & & 1 & \\
6 & & & 1
\end{array}
\qquad 17.23
$$

(*c*) The capacitance tensor of the network is by $\mathbf{A}_t \cdot \mathbf{C} \cdot \mathbf{A} = \mathbf{C}'$

$$
C^{u'v'} = \begin{array}{c|ccc}
u' \backslash v' & p' & q' & r' \\
\hline
p' & C_1 + C_2 + C_4 & -C_1 & -C_2 \\
q' & -C_1 & C_1 + C_3 + C_5 & -C_3 \\
r' & -C_2 & -C_3 & C_2 + C_3 + C_6
\end{array}
\qquad 17.24
$$

In the equation of flux $D^{u'} = C^{u'v'}E_{v'}$ the flux lines $D^{u'}$ across the junction-pairs are given as

$$
D^{u'} = \begin{array}{ccc}
p' & q' & r' \\
\hline
D^{p'} & D^{q'} & D^{r'}
\end{array}
\qquad 17.25
$$

(It is not necessary to find them by $D^{u'} = D^u C_u^{u'}$.)

The unknown differences of potential across the junction-pairs are found by $E_{v'} = S_{v'u'}D^{u'}$, that is, by finding the inverse of $C^{u'v'}$ as

$$
S_{u'v'} = \begin{array}{c|ccc}
u' \backslash v' & p' & q' & r' \\
\hline
p' & S_{pp} & S_{pq} & S_{pr} \\
q' & S_{pq} & S_{qq} & S_{qr} \\
r' & S_{pr} & S_{qr} & S_{rr}
\end{array}
\qquad 17.26
$$

(*d*) Multiplying $\mathbf{S}$ by $\mathbf{A}$ as $\mathbf{A} \cdot \mathbf{S} \cdot \mathbf{A}_t$ gives the self- and mutual elastances of the various condensers S'_{uv} while interconnected as

$$S'_{uv} = \begin{array}{c|c|c|c|c|c|c|}
u \backslash v & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & S_{pp}-2S_{pq}+S_{qq} & S_{pr}-S_{qr}-S_{pp}+S_{pq} & S_{pr}-S_{qr}-S_{pq}+S_{qq} & S_{pp}-S_{pq} & S_{pq}-S_{qq} & S_{pr}-S_{qr} \\
\hline
2 & S_{pr}-S_{pp}-S_{qr}+S_{pq} & S_{rr}-2S_{pr}+S_{pp} & S_{rr}-S_{pr}-S_{qr}+S_{pq} & S_{pr}-S_{pp} & S_{qr}-S_{pq} & S_{rr}-S_{pr} \\
\hline
3 & S_{pr}-S_{pq}-S_{qr}+S_{qq} & S_{rr}-S_{qr}-S_{pr}-S_{pq} & S_{rr}-2S_{qr}+S_{qq} & S_{pr}-S_{pq} & S_{qr}-S_{qq} & S_{rr}-S_{qr} \\
\hline
4 & S_{pp}-S_{pq} & S_{pr}-S_{pp} & S_{pr}-S_{pq} & S_{pp} & S_{pq} & S_{pr} \\
\hline
5 & S_{pq}-S_{qq} & S_{qr}-S_{pq} & S_{qr}-S_{qq} & S_{pq} & S_{qq} & S_{qr} \\
\hline
6 & S_{pr}-S_{qr} & S_{rr}-S_{pr} & S_{rr}-S_{qr} & S_{pr} & S_{qr} & S_{rr} \\
\hline
\end{array} \qquad 17.27$$

so that $E'_u = S'_{uv}D^v$, where D^v is the impressed flux lines (charges) *across* each condenser and E'_u is the difference of potential appearing *across* each condenser.

VII. INTERRELATED NETWORKS

(*a*) The three types of networks, namely, the electric, magnetic, and dielectric networks, in most problems are not isolated from one another, but are interrelated in some way. In network studies the following two types of interrelation may be considered:

1. The electric and dielectric networks are *interconnected* with each other, so that a physical contact exists between them as shown in Fig. 17.6.

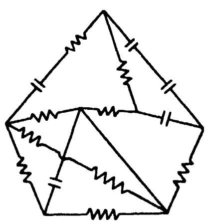


FIG. 17.6—Interconnected Electric and Dielectric Networks

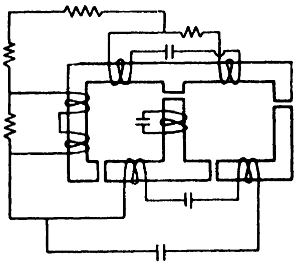


FIG. 17.7—Magnetic Network Interlinked with Electric and Dielectric Networks

2. The magnetic network is *interlinked* with both the electric and the dielectric networks, so that no physical contact exists between them (Fig. 17.7.)

(*b*) It is emphasized that these are sharply defined, *limiting* cases. There are many other possibilities. For instance, the magnetic iron path itself may form part of the electrical network. But then a new

problem arises; that is, in that case a *distinction has to be made between:* (1) the material particles forming the underlying network (the *material* network); (2) the superimposed electromagnetic quantities that also form a network (the *geometrical* network).

That is, *the network formed by the superimposed electromagnetic quantities (the network of "chains") is different from the network formed by the underlying material network (the network of "cells"). In stationary symmetrical network problems these two types of networks coincide in their entirety, but not so in rotating machinery.*

It should be noted that in a dielectric network the path of the electric charges is *discontinuous*, since the charges are not assumed to pass between the two plates of the condenser. That is, *in a dielectric network both the material paths and the geometrical paths of the electric charges are discontinuous.* Hence for the present purpose it can be assumed that a dielectric network is the same as an electrical network having discontinuities.

However, a magnetic network is fundamentally different from an electrical network since no electric charges flow along it or exist upon it. It is the "dual" of the dielectric network. *Theoretically* there ought to be a magnetic network in which magnetic charges flow and produce dissipation of heat serving as the "dual" of the electrical network. There appears to be no such magnetic phenomenon, however.

VIII. INTERRELATED ELECTROMAGNETIC PHENOMENA

(*a*) Just as the three types of physical networks are interrelated, similarly the electromagnetic phenomena superimposed upon them (e_α and i^α, h^α and b_α, e_α and d^α) are interrelated. These interrelations are given by the field equations of Maxwell. The interlinkage between the magnetic and dielectric networks is ignored in this volume.

(*b*) Considering a single mesh of each type of network only, when the electric circuit is wound N times around the magnetic core, then

$$Nb = \phi = \text{flux linkage (vector-potential)} \qquad 17.28$$

$$Ni = m = \text{magneto-motive force} \qquad 17.29$$

In terms of these new quantities the interrelations between the three types of meshes are

1.	magnetic → electric	$\dfrac{d\phi}{dt} = e$	17.30
2.	electric → magnetic	$m = h$	17.31
3.	dielectric → electric	$\dfrac{dq}{dt} = i$	17.32

(*c*) When *several* electric and dielectric meshes are interlinked with magnetic meshes, the number of interlinkages being represented by the tensor **N** containing the number of turns of each coil, then the above relations become for mesh networks

1. magnetic → electric:	$\dfrac{d\boldsymbol{\phi}}{dt} = \mathbf{e}$	$\dfrac{d\phi_m}{dt} = e_m$	17.33
2. electric → magnetic:	$\mathbf{m} = \mathbf{h}$	$m^m = h^m$	17.34
3. dielectric → electric:	$\dfrac{d\mathbf{q}}{dt} = \mathbf{i}$	$\dfrac{dq^m}{dt} = i^m$	17.35

where

$\boldsymbol{\phi} = \mathbf{N}_t \cdot \mathbf{b}$	$\phi_m = N_m^n b_n$	17.36
$\mathbf{m} = \mathbf{N} \cdot \mathbf{i}$	$m^m = N_n^m i^n$	17.37

(*d*) The flux-linkage vector $\boldsymbol{\phi}$ may be expressed in terms of the current vector **i** as

$$\boldsymbol{\phi} = \mathbf{N}_t \cdot \mathbf{b} = \mathbf{N}_t \cdot \boldsymbol{\mu} \cdot \mathbf{h} = \mathbf{N}_t \cdot \boldsymbol{\mu} \cdot \mathbf{m} = \mathbf{N}_t \cdot \boldsymbol{\mu} \cdot \mathbf{N} \cdot \mathbf{i} \qquad 17.38$$

If the inductance tensor is defined in terms of the permeance tensor as

$$\boxed{\mathbf{l} = \mathbf{N}_t \cdot \boldsymbol{\mu} \cdot \mathbf{N}} \quad \Big| \quad \boxed{l_{mn} = \mu_{pq} N_m^p N_n^q} \qquad 17.39$$

then the flux-linkage vector is in terms of the inductance tensor

$$\boxed{\boldsymbol{\phi} = \mathbf{l} \cdot \mathbf{i}} \quad \Big| \quad \boxed{\phi_m = l_{mn} i^n} \qquad 17.40$$

The inverse relation may be written as

$$\mathbf{i} = \mathbf{k} \cdot \boldsymbol{\phi} \quad \Big| \quad i^m = k^{mn} \phi_n \qquad 17.41$$

IX. THE SELF- AND MUTUAL-INDUCTANCES

(*a*) Knowledge of the self- and mutual permeances $\boldsymbol{\mu}'$ of the *magnetic meshes* as calculated in Section IV is only the first step in calculating **l** of the *individual coils*. It is necessary to go a step further and to calculate the self- and mutual permeances of the individual magnetic members, that gives by equation 17.44 *the self- and mutual inductances of the individual coils* that interlink the individual magnetic members.

The step from the *mesh* permeances to the *coil* permeances is exactly the same as the step from *mesh* admittances to *coil* admittances

shown in Section VIII, Chapter V. That is, if the mesh permeance $\boldsymbol{\mu}'$ has been calculated by taking the inverse of $\boldsymbol{\rho}'$, then the *coil* permeance, by equation 5.30, is

$$\boxed{\boldsymbol{\mu}_c = \mathbf{C}_t^{-1} \cdot \boldsymbol{\mu}' \cdot \mathbf{C}^{-1}} \qquad \boxed{\mu_{mn}' = \mu_{m'n'} C_m^{m'} C_n^{n'}} \tag{17.42}$$

$$\boxed{\boldsymbol{\mu}_c = \mathbf{A} \cdot \boldsymbol{\mu}' \cdot \mathbf{A}_t} \tag{17.43}$$

(*b*) If only the self- and mutual reluctances $\boldsymbol{\rho}$ of the primitive magnetic network are known, then *after interconnecting the magnetic mesh network by* $\mathbf{C}_t^{-1} = \mathbf{A}$, the self- and mutual permeances of the individual coils on the magnetic network, by equation 5.29 (if the electric circuits are not yet interconnected), are

$$\begin{aligned} \boldsymbol{\mu}_c &= \mathbf{A} \cdot (\mathbf{A}_t \cdot \boldsymbol{\rho} \cdot \mathbf{A})^{-1} \cdot \mathbf{A}_t \\ \boldsymbol{\mu}_c &= \mathbf{C}_t^{-1} \cdot (\mathbf{C}^{-1} \cdot \boldsymbol{\rho} \cdot \mathbf{C}_t^{-1})^{-1} \cdot \mathbf{C}^{-1} \end{aligned} \tag{17.44}$$

If the individual coils have different number of turns, represented by the diagonal turn-tensor $\mathbf{N} = N_m^p$, then the self- and mutual inductances $\mathbf{l}$ may be found from the permeances $\boldsymbol{\mu}_c$ by equation 17.39.

(*c*) Hence, *the self- and mutual inductances* $\mathbf{l}$ *of the primitive electric mesh network are found from the reluctances* $\boldsymbol{\rho}$ *of the primitive magnetic mesh network by*

$$\boxed{\mathbf{l} = \mathbf{N}_t \cdot [\mathbf{A} \cdot (\mathbf{A}_t \cdot \boldsymbol{\rho} \cdot \mathbf{A})^{-1} \cdot \mathbf{A}_t] \cdot \mathbf{N}} \tag{17.45}$$

where $\mathbf{A} = \mathbf{C}_t^{-1}$ shows the manner of interconnection of the *magnetic* network and $\mathbf{N}$ represents the number of turns of the coils of the primitive electrical mesh network. There are as many rows and columns in $\mathbf{l}$ as there are coils.

When several of the reluctance members have no coils on them, the corresponding rows and columns in $\mathbf{l}$ automatically drop out by the use of $\mathbf{N}$.

(*d*) Considering the example of Fig. 17.3*a*, its reluctance tensor has already been calculated in equation 17.15. Its inverse, the permeance tensor, is

$$\boldsymbol{\mu}' = \begin{array}{c|ccc|} & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' \\ \hline \mathbf{a}' & \mu_{aa} & \mu_{ab} & \mu_{ac} \\ \mathbf{b}' & \mu_{ab} & \mu_{bb} & \mu_{bc} \\ \mathbf{c}' & \mu_{ac} & \mu_{bc} & \mu_{cc} \\ \hline \end{array} \tag{17.46}$$

Multiplying it by the transformation tensor $\mathbf{C}_t^{-1} = \mathbf{A}$ (given in

equation 17.13) twice in succession, the permeance tensor of the individual coils, by $\mathbf{A}\cdot\boldsymbol{\mu}'\cdot\mathbf{A}_t$, is

$\boldsymbol{\mu}_c =$

	a	b	c	d	f	g
a	μ_{aa}	$\mu_{aa}+\mu_{ab}$	$\mu_{ab}+\mu_{ac}$	μ_{ab}	μ_{ab}	μ_{ac}
b	$\mu_{aa}+\mu_{ab}$	$\mu_{aa}+2\mu_{ab}+\mu_{bb}$	$\mu_{ab}+\mu_{bb}+\mu_{ac}+\mu_{bc}$	$\mu_{ab}+\mu_{bb}$	$\mu_{ab}+\mu_{bb}$	$\mu_{ac}+\mu_{bc}$
c	$\mu_{ab}+\mu_{ac}$	$\mu_{ab}+\mu_{ac}+\mu_{bb}+\mu_{bc}$	$\mu_{bb}+2\mu_{bc}+\mu_{cc}$	$\mu_{bb}+\mu_{bc}$	$\mu_{bb}+\mu_{bc}$	$\mu_{bc}+\mu_{cc}$
d	μ_{ab}	$\mu_{ab}+\mu_{bb}$	$\mu_{bb}+\mu_{bc}$	μ_{bb}	μ_{bb}	μ_{bc}
f	μ_{ab}	$\mu_{ab}+\mu_{bb}$	$\mu_{bb}+\mu_{bc}$	μ_{bb}	μ_{bb}	μ_{bc}
g	μ_{ac}	$\mu_{ac}+\mu_{bc}$	$\mu_{bc}+\mu_{cc}$	μ_{bc}	μ_{bc}	μ_{cc}

17.47

The turn ratio tensor is

$\mathbf{N} =$

	a	b	c	d	f	g
a	n_a					
b						
c			n_c			
d						
f					n_f	
g						n_g

17.48

Hence *the self- and mutual inductances of the four windings* are by $\mathbf{N}_t\cdot\boldsymbol{\mu}_c\cdot\mathbf{N} =$

1 =

	a	c	f	g
a	$(n_a)^2\mu_{aa}$	$n_an_c(\mu_{ab}+\mu_{ac})$	$n_an_f\mu_{ab}$	$n_an_g\mu_{ac}$
c	$n_an_c(\mu_{ab}+\mu_{ac})$	$n_c^2(\mu_{bb}+2\mu_{bc}+\mu_{cc})$	$n_cn_f(\mu_{bb}+\mu_{bc})$	$n_cn_g(\mu_{bc}+\mu_{cc})$
f	$n_an_f\mu_{ab}$	$n_cn_f(\mu_{bb}+\mu_{bc})$	$(n_f)^2\mu_{bb}$	$n_fn_g\mu_{bc}$
g	$n_an_g\mu_{ac}$	$n_cn_g(\mu_{bc}+\mu_{cc})$	$n_fn_g\mu_{bc}$	$(n_g)^2\mu_{cc}$

17.49

X. BASIC AND DERIVED VARIABLES

(*a*) In the equations of an *electrical* network two types of variables have been introduced:

1. The *contravariant* variable $\mathbf{i} = i^m$ used in mesh networks.
2. The *covariant* variable $\mathbf{E} = E_u$ used in junction networks.

When magnetic and dielectric networks link the electrical network *two additional sets of variables are introduced:*

1. The *contravariant* variables $\mathbf{q} = \mathbf{q}^m$, representing the electrical charges in mesh networks.

2. The *covariant* variables $\mathbf{\Phi} = \Phi_u$, representing the flux linkages in junction networks.

(*b*) The previous variables **i** and **E** may be derived from the additional variables **q** and $\mathbf{\Phi}$ by differentiation, as $i^m = dq^m/dt$ and $E_u = d\Phi_u/dt$, hence

1. *q^m and Φ_u may be called "basic variables."*
2. *i^m and E_u may be called "derived variables."*

It should be noted that the basic variable is *not* ϕ_u but Φ_u, a concept different from ϕ_u, since one refers to a junction, the other to a mesh network. Similarly the other basic variable is q^m and not Q^m.

The equations of performance of networks may be expressed in terms of these four types of variables in different combinations.

XI. DUAL QUANTITIES

(*a*) It was shown in Section I, Chapter XIV, that the concepts of "*mesh*" and "*junction-pair*" are dual concepts and all quantities associated with them are also dual to each other. In the previous chapters the following dual tensors were introduced:

1. Impressed quantities: $\mathbf{e} = e_{\bar{m}} \rightarrow \mathbf{I} = I^u$
2. Response quantities: $\mathbf{i} = i^m \rightarrow \mathbf{E} = E_{\bar{u}}$
3. Network constants: $\mathbf{z} = z_{\bar{m}n} \rightarrow \mathbf{Y} = Y^{u\bar{v}}$
4. Transformation tensors: $\mathbf{C} = C^{\alpha}_{\alpha'} \rightarrow \mathbf{C}_t^{*-1} = C^{\bar{\alpha}'}_{\bar{\alpha}}$

(*b*) In the presence of interrelated electric, magnetic, and dielectric networks the following additional dual tensors occur:

5. Response quantities: $\begin{cases} \boldsymbol{\phi} = \phi_{\bar{m}} \rightarrow \mathbf{Q} = Q^u. \\ \mathbf{q} = q^m \rightarrow \mathbf{\Phi} = \Phi_{\bar{u}}. \end{cases}$
6. Electrical constants: $\mathbf{r} = r_{\bar{m}n} \rightarrow \mathbf{G} = G^{u\bar{v}}$.
7. Magnetic constants: $\mathbf{l} = l_{\bar{m}n} \rightarrow \mathbf{K} = K^{u\bar{v}}$.
8. Dielectric constants: $\mathbf{s} = s_{\bar{m}n} \rightarrow \mathbf{C} = C^{u\bar{v}}$.

Also the following dual scalars occur:

1. Kinetic energy $= T \rightarrow V' =$ potential energy.
2. Rate of heat loss $= D \rightarrow D' =$ rate of heat loss.
3. Power input $= P \rightarrow P' =$ power input.

If the "impressed" current I is interpreted as "withdrawn" current (current supplied to outside loads) then P' is a power output.

XII. THE EQUATION OF VOLTAGE

(*a*) The equation of voltage $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ of an interconnected electrical and dielectric mesh network linked by a magnetic network becomes by equations 17.4, 17.18, and 17.33

$$\mathbf{e} = \mathbf{r}\cdot\mathbf{i} + \frac{d\boldsymbol{\phi}}{dt} + \mathbf{s}\cdot\mathbf{q} \qquad \Big| \qquad e_m = r_{mn}i^n + \frac{d\phi_m}{dt} + s_{mn}q^n \qquad 17.50$$

The electrical network contributes $\mathbf{r}\cdot\mathbf{i}$, the interlinked magnetic network $d\boldsymbol{\phi}/dt$, and the interconnected dielectric network $\mathbf{s}\cdot\mathbf{q}$. By Kirchhoff's law the voltages add up around a mesh.

In terms of the *derived* variable $\mathbf{i}$ (since $\boldsymbol{\phi} = \mathbf{l}\cdot\mathbf{i}$ and $\mathbf{q} = \int \mathbf{i}dt$), *the equation of voltage of mesh networks is*

$$\mathbf{e}=\mathbf{r}\cdot\mathbf{i}+\mathbf{l}\cdot\frac{d\mathbf{i}}{dt}+\mathbf{s}\cdot\int\mathbf{i}dt \qquad \Big| \qquad e_m=r_{mn}i^n+l_{mn}\frac{di^n}{dt}+s_{mn}\int i^n dt \qquad 17.51$$

$$\boxed{\mathbf{e} = \mathbf{r}\cdot\mathbf{i} + \mathbf{l}p\cdot\mathbf{i} + (\mathbf{s}/p)\cdot\mathbf{i}} \qquad \Big| \qquad \boxed{e_m = r_{mn}i^n + l_{mn}pi^n + (s_{mn}/p)i^n} \qquad 17.52$$

(*b*) Hence, *the impedance tensor* $\mathbf{z}$ *assumes the following special form for stationary symmetrical networks*

$$\mathbf{z} = \mathbf{r} + \mathbf{l}\frac{d}{dt} + \mathbf{s}\int dt \qquad \Big| \qquad z_{mn} = r_{mn} + l_{mn}\frac{d}{dt} + s_{mn}\int dt \qquad 17.53$$

$$\boxed{\mathbf{z} = \mathbf{r} + \mathbf{l}p + \mathbf{s}/p} \qquad \Big| \qquad \boxed{z_{mn} = r_{mn} + l_{mn}p + s_{mn}/p} \qquad 17.54$$

The electrical network itself contributes $\mathbf{r}$, the magnetic network $\mathbf{l}p$ and the dielectric network $\mathbf{s}/p$.

(*c*) In terms of the *basic variable* $\mathbf{q}$ the equation becomes

$$\boxed{\mathbf{e} = \mathbf{l}\cdot\frac{d^2\mathbf{q}}{dt^2} + \mathbf{r}\cdot\frac{d\mathbf{q}}{dt} + \mathbf{s}\cdot\mathbf{q}} \qquad \Big| \qquad \boxed{e_m = l_{mn}\frac{d^2q^n}{dt^2} + r_{mn}\frac{dq^n}{dt} + s_{mn}q^n} \qquad 17.55$$

representing as many second-order differential equations as there are meshes.

This equation is analogous to the equation of voltage of a single coil having resistance r, inductance l, and elastance s, namely to

$$e = l\frac{d^2q}{dt^2} + r\frac{dq}{dt} + sq \qquad 17.56$$

except that, in accordance with the First Generalization Postulate, each scalar is replaced by an appropriate n-matrix, and in accordance with the Second Generalization Postulate each n-matrix is replaced by a geometric object of valence n.

(*d*) In an orthogonal network **E** and **I** appear also, so that *the equation of voltage of an orthogonal network* (using *spin* indices) is

$$\boxed{(\mathbf{E}+\mathbf{e})=(\mathbf{r}+\mathbf{l}p+\mathbf{s}/p)\cdot(\mathbf{i}+\mathbf{I})} \quad \Bigg| \quad \boxed{E_{\bar{\alpha}}+e_{\bar{\alpha}}=(r_{\bar{\alpha}\beta}+l_{\bar{\alpha}\beta}p+s_{\bar{\alpha}\beta}/p)(i^\beta+I^\beta)} \qquad 17.57$$

The equations of this Section, used in the routine analysis of networks, represent also the *explicit* form of the dynamical equations of Lagrange with *contravariant* variables, as will be shown in Section XVI.

XIII. THE EQUATION OF CURRENT

(*a*) The equation of current $\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}$ of an interconnected electrical and dielectric junction network linked by a magnetic network becomes by equations 17.4, 17.35, and 17.41

$$\mathbf{I} = \mathbf{G}\cdot\mathbf{E} + \frac{d\mathbf{Q}}{dt} + \mathbf{K}\cdot\mathbf{\Phi} \quad \Bigg| \quad I^u = G^{uv}E_v + \frac{dQ^u}{dt} + K^{uv}\Phi_v \qquad 17.58$$

The electrical network contributes $\mathbf{G}\cdot\mathbf{E}$, the interlinked magnetic network $\mathbf{K}\cdot\mathbf{\Phi}$, and the interconnected dielectric network $d\mathbf{Q}/dt$. By Kirchhoff's law the currents add up at a junction.

In terms of the derived variable **E** (since $\mathbf{Q} = \mathbf{C}\cdot\mathbf{E}$ and $\mathbf{\Phi} = \int \mathbf{E}dt$) *the equation of current of junction networks is*

$$\mathbf{I}=\mathbf{G}\cdot\mathbf{E} + \mathbf{C}\cdot\frac{d\mathbf{E}}{dt} + \mathbf{K}\cdot\int \mathbf{E}dt \quad \Bigg| \quad I^u=G^{uv}E_v + C^{uv}\frac{dE_v}{dt} + K^{uv}\int E_v dt \qquad 17.59$$

$$\boxed{\mathbf{I} = \mathbf{G}\cdot\mathbf{E} + \mathbf{C}p\cdot\mathbf{E} + (\mathbf{K}/p)\cdot\mathbf{E}} \quad \Bigg| \quad \boxed{I^u = G^{uv}E_v + C^{uv}pE_v + (K^{uv}/p)E_v} \qquad 17.60$$

(*b*) Hence, *the admittance tensor* **Y** *assumes the following special form for stationary symmetrical networks*

$$\mathbf{Y} = \mathbf{G} + \mathbf{C}\frac{d}{dt} + \mathbf{K}\cdot\int dt \quad \Bigg| \quad Y^{uv} = G^{uv} + C^{uv}\frac{d}{dt} + K^{uv}\int dt \qquad 17.61$$

$$\boxed{\mathbf{Y} = \mathbf{G} + \mathbf{C}p + \mathbf{K}/p} \quad \Bigg| \quad \boxed{Y^{uv} = G^{uv} + C^{uv}p + K^{uv}/p} \qquad 17.62$$

The electrical network itself contributes **G**, the dielectric network $\mathbf{C}p$, and the magnetic network $\mathbf{K}/p$.

(*c*) In terms of the *basic variable* Φ the equation becomes:

$$\boxed{\mathbf{I} = \mathbf{C}\cdot\frac{d^2\mathbf{\Phi}}{dt^2} + \mathbf{G}\cdot\frac{d\mathbf{\Phi}}{dt} + \mathbf{K}\cdot\mathbf{\Phi}} \qquad \boxed{I^u = C^{uv}\frac{d^2\Phi_v}{dt^2} + G^{uv}\frac{d\Phi_v}{dt} + K^{uv}\Phi_v} \qquad 17.63$$

representing as many second-order differential equations as there are junction-pairs.

This equation is analogous to that of a single coil

$$I = C\frac{d^2\Phi}{dt^2} + G\frac{d\Phi}{dt} + K\Phi \qquad 17.64$$

with each scalar replaced by an appropriate geometric object.

(*d*) In an orthogonal network **e** and **i** appear also, so that the equation of current of an orthogonal network (using *spin* indices) is

$$\boxed{\mathbf{i} + \mathbf{I} = (\mathbf{G} + \mathbf{C}p + \mathbf{K}/p)\cdot(\mathbf{E} + \mathbf{e})} \qquad \boxed{i^\alpha + I^\alpha = (G^{\alpha\beta} + C^{\alpha\bar{\beta}}p + K^{\alpha\bar{\beta}}/p)(E_{\bar{\beta}} + e_\beta)}$$

17.65

The equations of this Section, used in the routine analysis of networks, represent also the *explicit* form of the dynamical equations of Lagrange with *covariant* variables, as will be shown in Section XVI.

XIV. THE EQUATION OF POWER

(*a*) If the equation of voltage 17.52 is multiplied by **i**, each term represents a power

$$\mathbf{i}\cdot\mathbf{e} = \mathbf{i}\cdot\mathbf{r}\cdot\mathbf{i} + \mathbf{i}\cdot\mathbf{l}p\cdot\mathbf{i} + \mathbf{i}\cdot\mathbf{s}/p\cdot\mathbf{i}$$

$$e_m i^m = r_{mn} i^m i^n + l_{mn}(p i^n) i^m + s_{mn}\left(\frac{1}{p} i^n\right) i^m$$

Since $\mathbf{i} = d\mathbf{q}/dt = p\mathbf{q}$, therefore the equation of power is

$$\mathbf{i}\cdot\mathbf{e} = \mathbf{i}\cdot\mathbf{r}\cdot\mathbf{i} + p(\tfrac{1}{2}\mathbf{i}\cdot\mathbf{l}\cdot\mathbf{i}) + p(\tfrac{1}{2}\mathbf{q}\cdot\mathbf{s}\cdot\mathbf{q}) \qquad 17.66$$

$$e_m i^m = r_{mn} i^m i^n + p(\tfrac{1}{2} l_{mn} i^m i^n) + p(\tfrac{1}{2} s_{mn} q^m q^n) \qquad 17.67$$

where

1. $\mathbf{i}\cdot\mathbf{e} = P$ = total power input into the electrical mesh network.
2. $\mathbf{i}\cdot\mathbf{r}\cdot\mathbf{i} = D$ = rate of heat dissipation in the electrical network.
3. $\mathbf{i}\cdot\mathbf{l}\cdot\mathbf{i}/2 = T$ = kinetic energy stored in the magnetic network.
4. $\mathbf{q}\cdot\mathbf{s}\cdot\mathbf{q}/2 = V$ = potential energy stored in the dielectric network.

Hence the equation of power of mesh networks is

$$P = D + \frac{dT}{dt} + \frac{dV}{dt} \qquad 17.68$$

(*b*) Multiplying the equation of current 17.60 by **E**

$$\mathbf{E}\cdot\mathbf{I} = \mathbf{E}\cdot\mathbf{G}\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{C}p\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{K}/p\cdot\mathbf{E}$$

$$E_u I^u = G^{uv}E_u E_v + C^{uv}(pE_v)E_u + K^{uv}\left(\frac{1}{p}E_v\right)E_u \qquad 17.69$$

Since $\mathbf{E} = d\mathbf{\Phi}/dt = p\mathbf{\Phi}$, therefore the equation of power is

$$\mathbf{E}\cdot\mathbf{I} = \mathbf{E}\cdot\mathbf{G}\cdot\mathbf{E} + p(\tfrac{1}{2}\mathbf{E}\cdot\mathbf{C}\cdot\mathbf{E}) + p(\tfrac{1}{2}\mathbf{\Phi}\cdot\mathbf{K}\cdot\mathbf{\Phi}) \qquad 17.70$$

$$E_u I^u = G^{uv}E_u E_v + p(\tfrac{1}{2}C^{uv}E_u E_v) + p(\tfrac{1}{2}K^{uv}\Phi_u\Phi_v)$$

where

1. $\mathbf{E}\cdot\mathbf{I} = P'$ = total power input into the electrical junction network.
2. $\mathbf{E}\cdot\mathbf{G}\cdot\mathbf{E} = D'$ = rate of heat dissipation in the electrical network.
3. $\mathbf{E}\cdot\mathbf{C}\cdot\mathbf{E}/2 = V'$ = potential energy stored in the dielectric network.
4. $\mathbf{\Phi}\cdot\mathbf{K}\cdot\mathbf{\Phi}/2 = T'$ = kinetic energy stored in the magnetic network. In terms of P', D', V', and T' the equation of power of junction networks is

$$P' = D' + \frac{dT'}{dt} + \frac{dV'}{dt} \qquad 17.71$$

XV. QUADRATIC AND HERMITIAN FORMS

(*a*) The various expressions for power and energy

$\mathbf{i}\cdot\mathbf{r}\cdot\mathbf{i}=D$	$\mathbf{E}\cdot\mathbf{G}\cdot\mathbf{E}=D'$	$r_{mn}i^m i^n=D$	$G^{uv}E_uE_v=D'$	17.72
$\mathbf{i}\cdot\mathbf{l}\cdot\mathbf{i}=2T$	$\mathbf{\Phi}\cdot\mathbf{K}\cdot\mathbf{\Phi}=2T'$	$l_{mn}i^m i^n=2T$	$K^{uv}\Phi_u\Phi_v=2T'$	17.73
$\mathbf{q}\cdot\mathbf{s}\cdot\mathbf{q}=2V$	$\mathbf{E}\cdot\mathbf{C}\cdot\mathbf{E}=2V'$	$s_{mn}q^m q^n=2V$	$C^{uv}E_uE_v=2V'$	17.74

are all *quadratic* forms. On the other hand, the total power input, $\mathbf{i}\cdot\mathbf{e} = P$ and $\mathbf{E}\cdot\mathbf{I} = P'$ are *linear* forms.

When the components are complex numbers, the corresponding quadratic forms change into bilinear forms as

$\mathbf{i}\cdot\mathbf{r}\cdot\mathbf{i}^*=D$	$\mathbf{E}^*\cdot\mathbf{G}\cdot\mathbf{E}=D'$	$r_{\bar{m}n}i^{\bar{m}}i^n=D$	$G^{u\bar{v}}E_uE_{\bar{v}}=D'$	17.75
$\mathbf{i}\cdot\mathbf{l}\cdot\mathbf{i}^*=2T$	$\mathbf{\Phi}^*\cdot\mathbf{K}\cdot\mathbf{\Phi}=2T'$	$l_{\bar{m}n}i^{\bar{m}}i^n=2T$	$K^{u\bar{v}}\Phi_u\Phi_{\bar{v}}=2T'$	17.76
$\mathbf{q}\cdot\mathbf{s}\cdot\mathbf{q}^*=2V$	$\mathbf{E}^*\cdot\mathbf{C}\cdot\mathbf{E}=2V'$	$s_{\bar{m}n}q^{\bar{m}}q^n=2V$	$C^{u\bar{v}}E_uE_{\bar{v}}=2V'$	17.77

that is, the conjugate of one of the variables is taken.

Since the components of **r**, **l**, etc., are real numbers, $\mathbf{r} = \mathbf{r}^*$ and so on. Because of this property of the coefficients of the six bilinear forms,

the latter are called "*hermitian forms.*" The value of a hermitian form (a scalar) is always a *real* number and not a complex number.

When any of the transformation tensors **C** of this volume are used for stationary, symmetrical networks, *all these six quadratic (or hermitian) forms also remain invariant under the transformation in addition to the two linear forms.*

(*b*) It should be emphasized that *the assumption of the invariance of the power input is far more general than those of the invariance of the various quadratic forms, since in asymmetrical networks in general no quadratic forms exist, their* **z** *or* **Y** *tensor being asymmetrical, but a linear form does exist.*

XVI. THE DUAL EQUATIONS OF MOTION OF LAGRANGE

(*a*) The *explicit* forms of the equations of Lagrange, equations 17.52 and 17.60, that are used in the study of networks, may be brought to their *implicit* forms given by Lagrange by replacing the expressions containing tensors of valence two **r**, **l**, etc. with equivalent expressions containing only scalars and vectors. These implicit equations however are not in a suitable form for the routine calculation of the large variety of systems.

Let in the equation of voltage 17.52 the following substitutions be made (remembering that **r**, **l** and **s** have *symmetrical* matrices in all reference frames and are not functions of the variables or of time).

$$\mathbf{r}\cdot\mathbf{i} = \frac{1}{2}\frac{\partial\mathbf{i}\cdot\mathbf{r}\cdot\mathbf{i}}{\partial\mathbf{i}} = \frac{\partial F}{\partial\mathbf{i}}$$

where

$$F = \text{dissipation function} = \tfrac{1}{2}\mathbf{i}\cdot\mathbf{r}\cdot\mathbf{i} = D/2$$

$$\mathbf{l}p\cdot\mathbf{i} = p\mathbf{l}\cdot\mathbf{i} = p\left(\frac{1}{2}\frac{\partial\mathbf{i}\cdot\mathbf{l}\cdot\mathbf{i}}{\partial\mathbf{i}}\right) = p\,\frac{\partial T}{\partial\mathbf{i}}$$

$$\mathbf{s}/p\cdot\mathbf{i} = \mathbf{s}\cdot\mathbf{q} = \frac{1}{2}\frac{\partial\mathbf{q}\cdot\mathbf{s}\cdot\mathbf{q}}{\partial\mathbf{q}} = \frac{\partial V}{\partial\mathbf{q}}$$

In terms of these scalars and vectors the equation of voltage 17.52 becomes

$$\boxed{\mathbf{e} = \frac{d}{dt}\frac{\partial T}{\partial\mathbf{i}} - \frac{\partial(-V)}{\partial\mathbf{q}} + \frac{\partial F}{\partial\mathbf{i}}} \quad\Bigg|\quad \boxed{e_m = \frac{d}{dt}\frac{\partial T}{\partial i^m} - \frac{\partial(-V)}{\partial q^m} + \frac{\partial F}{\partial i^m}} \qquad 17.78$$

This form of the equation of voltage containing the scalars, T, V, and F is a special case of the equation of motion of Lagrange expressed in terms of the two contravariant variables q^m and i^m.

(*b*) Similarly in the equation of current 17.60 let the following substitutions be made

$$\mathbf{G}\cdot\mathbf{E} = \frac{1}{2}\frac{\partial \mathbf{E}\cdot\mathbf{G}\cdot\mathbf{E}}{\partial \mathbf{E}} = \frac{\partial F'}{\partial \mathbf{E}}$$

where

$$F' = \text{dual dissipation function} = \tfrac{1}{2}\mathbf{E}\cdot\mathbf{G}\cdot\mathbf{E} = D'/2$$

$$\mathbf{C}p\cdot\mathbf{E} = p\mathbf{C}\cdot\mathbf{E} = p\left(\frac{1}{2}\frac{\partial \mathbf{E}\cdot\mathbf{C}\cdot\mathbf{E}}{\partial \mathbf{E}}\right) = p\frac{\partial V'}{\partial \mathbf{E}}$$

$$\mathbf{K}/p\cdot\mathbf{E} = \mathbf{K}\cdot\mathbf{\Phi} = \frac{1}{2}\frac{\partial \mathbf{\Phi}\cdot\mathbf{K}\cdot\mathbf{\Phi}}{\partial \mathbf{\Phi}} = \frac{\partial T'}{\partial \mathbf{\Phi}}$$

In terms of these scalars and vectors the equation of current 17.60 becomes

$$\boxed{\mathbf{I} = \frac{d}{dt}\frac{\partial V'}{\partial \mathbf{E}} - \frac{\partial(-T')}{\partial \mathbf{\Phi}} + \frac{\partial F'}{\partial \mathbf{E}}} \quad\Bigg|\quad \boxed{I^u = \frac{d}{dt}\frac{\partial V'}{\partial E_u} - \frac{\partial(-T')}{\partial \Phi_u} + \frac{\partial F'}{\partial E_u}} \qquad 17.79$$

This form of the equation of current is a special form of the equation of motion of Lagrange expressed in terms of the two *covariant* variables Φ_u and E_u.

(*c*) The dual equations of motion may be given a more symmetrical form by defining:

$$\text{Lagrangean function: } L = T - V \qquad 17.80$$

$$\text{Dual Lagrangean function: } L' = V' - T' \qquad 17.81$$

In particular in terms of *contravariant* variables

$$L = (\mathbf{i}\cdot\mathbf{l}\cdot\mathbf{i} - \mathbf{q}\cdot\mathbf{s}\cdot\mathbf{q})/2 \quad\Big|\quad L = (l_{mn}i^m i^n - s_{mn}q^m q^n)/2 \qquad 17.82$$

and in terms of *covariant* variables

$$L' = (\mathbf{E}\cdot\mathbf{C}\cdot\mathbf{E} - \mathbf{\Phi}\cdot\mathbf{K}\cdot\mathbf{\Phi})/2 \quad\Big|\quad L' = (C^{uv}E_uE_v - K^{uv}\Phi_u\Phi_v)/2 \qquad 17.83$$

In terms of the Lagrangean functions L and L' the dual equations of motion become

$$\boxed{\mathbf{e} = \frac{d}{dt}\frac{\partial L}{\partial \mathbf{i}} - \frac{\partial L}{\partial \mathbf{q}} + \frac{\partial F}{\partial \mathbf{i}}} \quad\Bigg|\quad \boxed{e_m = \frac{d}{dt}\frac{\partial L}{\partial i^m} - \frac{\partial L}{\partial q^m} + \frac{\partial F}{\partial i^m}} \qquad 17.84$$

$$\boxed{\mathbf{I} = \frac{d}{dt}\frac{\partial L'}{\partial \mathbf{E}} - \frac{\partial L'}{\partial \mathbf{\Phi}} + \frac{\partial F'}{\partial \mathbf{E}}} \quad\Bigg|\quad \boxed{I^u = \frac{d}{dt}\frac{\partial L'}{\partial E_u} - \frac{\partial L'}{\partial \Phi_u} + \frac{\partial F'}{\partial E_u}} \qquad 17.85$$

Hence the dual Lagrangean equations of motion give either the equation of voltage of "mesh" networks or the equation of current of "junction" networks, depending whether contravariant variables, q^m *and* i^m *or covariant variables,* Φ_u *and* E_u*, are assumed.*

It should again be emphasized that the contravariant and covariant variables are independent of one another, the first appearing in the meshes, the other in the junction-pairs of a network.

CHAPTER XVIII

THE METRIC TENSOR

I. NETWORKS WITH ZERO DESIGN CONSTANTS

The fact that in stationary networks **z** or **Y** can be subdivided into three component tensors allows the introduction of *new types of design constants* (consisting of ratios only) and new types of equations. Before introducing new types of design constants let networks without design constants be considered in passing. Such networks play important part in relays, switches, control equipment, etc.

The simplest electrical network consists of a number of interconnected branches having zero resistances as shown in Fig. 18.1 for five branches, along which no voltage drops exist. The only electrical concepts associated with it are the currents flowing through the network. By considering Fig. 18.1 as an all-mesh network it is possible to set up for it with the aid of Kirchhoff's First Law a non-singular (square) transformation tensor $C^{\alpha}_{\alpha'}$, in which all abstract properties of

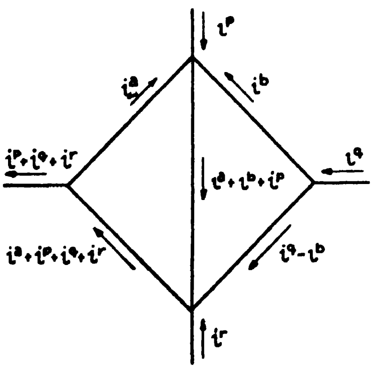


FIG. 18.1.—Network with Zero Impedances

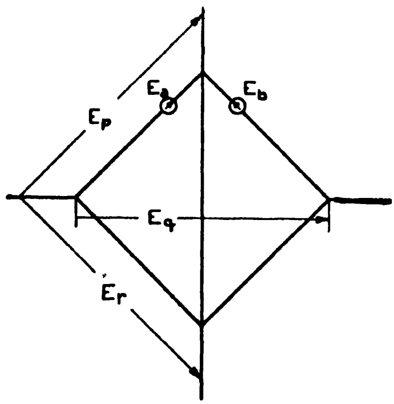


FIG. 18.2.—Network with Zero Admittances

the network are codified. The theory of this simple network is the theory of $C^{\alpha}_{\alpha'}$.

Its dual network consists of a number of interconnected *conductanceless* branches as shown in Fig. 18.2 having no currents in the

branches. The only electrical concepts associated with it are the voltages appearing across the junction-pairs. It is possible to set up for it with the aid of Kirchhoff's Second Law the inverse transformation tensor $C_{\alpha}^{\alpha'}$, codifying all abstract properties of the network.

Elementary topology studies the properties of these types of networks with which no design constants are associated. The network is completely characterized by its transformation tensor.

Geometrically, a network with zero design constants (but with a $C_{\alpha'}^{\alpha}$) is equivalent to a space in which neither the "magnitude" nor the "direction" of vectors introduced into it is defined.

II. THE BASIC INVARIANTS

(*a*) The first step in endowing a network with design constants consists of introducing resistances $\mathbf{r} = r_{mn}$, so that their equation of voltage as a mesh network is $\mathbf{e} = \mathbf{r}\cdot\mathbf{i}$.

When an underlying *magnetic* network *links* the electrical network, the coils are endowed by additional design constants, the inductances $\mathbf{l} = l_{mn}$, so that its equation of voltage is $\mathbf{e} = \mathbf{r}\cdot\mathbf{i} + \mathbf{l}\cdot\dfrac{d\mathbf{i}}{dt}\cdot$

When the electric mesh network is *interconnected* also with a *dielectric* network, it acquires still another set of design constants, the elastances $\mathbf{s} = s_{mn}$, so that its equation of voltage becomes

$$\mathbf{e} = \mathbf{r}\cdot\mathbf{i} + \mathbf{l}\cdot p\mathbf{i} + (\mathbf{s}/p)\cdot\mathbf{i} = \mathbf{l}\cdot\frac{d^2\mathbf{q}}{dt^2} + \mathbf{r}\cdot\frac{d\mathbf{q}}{dt} + \mathbf{s}\cdot\mathbf{q}$$

(*b*) The three independent types of design constants of the network, namely: (1) the resistance tensor $\mathbf{r} = r_{mn}$, (2) the inductance tensor $\mathbf{l} = l_{mn}$, (3) the elastance tensor $\mathbf{s} = s_{mn}$, are called the "*basic invariants*" of the network. They determine the "structure" of the network.

The superimposed electromagnetic quantities are variables, in particular let it be defined arbitrarily:

1. $\mathbf{q} = q^m$ is the basic variable.
2. $\mathbf{i} = i^m$ is a derived variable.
3. $\mathbf{e} = e_m$ is a parameter.

There is also the independent variable t.

All the other invariants of the mesh network are derived from these basic invariants by differentiation, or by combination with each other or with the variables.

Although in the general case the components of the structural basic

invariants **r**, **l**, and **s** are functions of the variable **i**, in this volume it is assumed that their components are all constants. Also in the general case basic invariants are not necessarily tensors.

(*c*) One of the purposes of tensor analysis is to derive other invariants from the given basic invariants and the variables of a geometrical or physical system. There are numerous ways of deriving invariants from the basic invariants. For instance, deriving new invariants from basic invariants by differentiation is shown in Section IV, Chapter XV.

Another way of deriving new invariants is *by forming products* of basic invariants and variables. *Such geometric objects are called "simultaneous invariants."* Simultaneous invariants are, for instance, the four scalar "forms"

$$P = \mathbf{i} \cdot \mathbf{e}, \quad D = \mathbf{i} \cdot \mathbf{r} \cdot \mathbf{i}, \quad T = \mathbf{i} \cdot \mathbf{l} \cdot \mathbf{i}/2 \text{ and } V = \mathbf{q} \cdot \mathbf{s} \cdot \mathbf{q}/2$$

(*d*) A junction network possesses the *dual invariants* of those of a mesh network. From its equation of current

$$\mathbf{I} = \mathbf{G} \cdot \mathbf{E} + \mathbf{C}p \cdot \mathbf{E} + (\mathbf{K}/p) \cdot \mathbf{E} = \mathbf{C} \cdot \frac{d^2\mathbf{\Phi}}{dt^2} + \mathbf{G} \cdot \frac{d\mathbf{\Phi}}{dt} + \mathbf{K} \cdot \mathbf{\Phi}$$

its basic invariants are:

1. The conductance tensor $\mathbf{G} = G^{uv}$.
2. The susceptance tensor $\mathbf{K} = K^{uv}$.
3. The capacitance tensor $\mathbf{C} = C^{uv}$.

Its variables are:

1. $\mathbf{\Phi} = \Phi_u$ is the basic variable.
2. $\mathbf{E} = E_u$ is a derived variable.
3. $\mathbf{I} = I^u$ is a parameter.

Its simultaneous invariants are the same as those above except defined in terms of the junction basic invariants and variables, namely $P' = \mathbf{E} \cdot \mathbf{I}$, $D' = \mathbf{E} \cdot \mathbf{G} \cdot \mathbf{E}$, $T' = \mathbf{\Phi} \cdot \mathbf{K} \cdot \mathbf{\Phi}/2$ and $V' = \mathbf{E} \cdot \mathbf{C} \cdot \mathbf{E}/2$.

III. GENERALIZATION OF THE "PER UNIT" SYSTEM

(*a*) In studying many physical phenomenon or geometrical relation or building any engineering structure, *one of the first steps in simplifying the problem is to eliminate from view the concept of "magnitude" in some way.* If possible small-scale models are built, or the problem is expressed in terms of *ratios* of quantities instead of the actual quantities themselves, etc.

(*b*) The first step in introducing *ratios* into the problem is to select some convenient magnitude of voltage, current, etc., as unity and express all similar quantities as their ratio. For instance the full-load current may be denoted as "1" unit of current and all other currents are expressed as a fraction or multiple of the full-load current. Similar units may be selected to express the voltages, the reactances, etc.

Instead of calling the reference quantity "1," it is also customary to call it "100 per cent" and express all other quantities as its percentage. This is the so-called "per unit" system used in electrical engineering problems.

By the use of such ratios *two apparatus of different magnitude may be compared with each other,* to see which one of them has, say, larger or smaller percentage of short-circuit reactance or of resistance, and so on.

All formulas of this volume are equally valid if all quantities are expressed in this type of a "per unit" system. That is, it does not make any difference as far as the formulas of this volume are concerned in what units the components of the various geometric objects are measured.

(*c*) This first step is not quite satisfactory since it does not correlate the magnitude of quantities of *different types* like that of resistances and reactances. A second step is made in the design of apparatus by expressing its performance in terms of *ratios* of two different types of quantities, such as r/X = resistance/inductance. In comparing two apparatus in terms of such ratios the ratios are a measure of the *quality* of the apparatus in regards to efficiency or cheapness, etc.

(*d*) Tensor analysis supplies a systematic procedure by which the performance of physical systems is expressed in terms of ratios of two different types of design constants, instead of in terms of their actual values. That is, *tensor analysis supplies a routine procedure by which the actual "magnitudes" of physical quantities can be made to disappear from the picture and reappear again at will.* During disappearance their place is taken by those ratios of the physical quantities that play a deciding role in the analysis. *That routine procedure is called the "raising and lowering of indices."*

IV. THE METRIC TENSOR a_{mn}

(*a*) The basic invariant that plays the central role in the disappearance and reappearance of the concept of "magnitude" is the inductance tensor $\mathbf{l} = l_{mn}$ representing the self- and mutual inductances of the various meshes. *Because of its role, the inductance tensor l_{mn} is called in tensor analysis the "metric tensor" and is denoted by a_{mn}* (in geometrical tensor literature by g_{mn}.)

The metric tensor a_{mn} *represents the additional characteristics acquired by the electrical network owing to its linkage with an underlying magnetic network.* It is a symmetrical tensor of valence two. (It may be mentioned that *in the presence of permanent magnets the inductance tensor* l_{mn} *is not symmetrical,* since the linkage is unidirectional, no flux lines linking the magnet, *and it cannot be assumed as the metric tensor* a_{mn}.)

For instance, the metric tensor of a two-winding transformer (Fig. 18.3) is

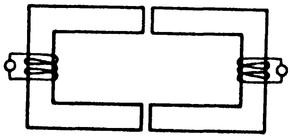


FIG. 18.3.—Two-winding Transformer

$$\mathbf{a} = \begin{array}{c|c|c|} & \mathbf{p} & \mathbf{s} \\ \hline \mathbf{p} & L_p & M \\ \hline \mathbf{s} & M & L_s \\ \hline \end{array} \tag{18.1}$$

In mechanical problems the moments of inertias and the products of inertias of the various masses may form the components of a metric tensor.

(*b*) It is one of the few tensors of valence two whose inverse is denoted by the same base letter as a^{nm}. (Other tensors are, for instance, $C^{a}_{\alpha'}$ whose inverse is $C^{\alpha'}_{a}$, also the unit tensor I^{α}_{β} whose inverse is I^{β}_{α}.) The inverse of that of the two-winding transformer is

$$a^{nm} = \begin{array}{c|c|c|} {}_{n}\diagdown^{m} & p & s \\ \hline p & L_s/D & -M/D \\ \hline s & -M/D & L_p/D \\ \hline \end{array} \tag{18.2}$$

where $D = L_pL_s - M^2$. Since the *short-circuit inductance* of the primary winding is $L'_p = L_p - M^2/L_pL_s$, therefore $D = L_pL'_p = L_sL'_s$. Also, if the leakage coefficients are $\lambda_p = M/L_p$ and $\lambda_s = M/L_s$, the *inverse metric tensor is*

$$a^{nm} = \begin{array}{c|c|c|} {}_{n}\diagdown^{m} & p & s \\ \hline p & 1/L'_p & -\lambda p/L'_p \\ \hline s & -\lambda_s/L'_s & 1/L'_s \\ \hline \end{array} \tag{18.3}$$

(*c*) Since inductances are proportional to permeances (equation 17.39 and the inverse of a permeance is a reluctance), it may be said that:

1. The metric tensor a_{mn} represents the self- and mutual *permeances* of the underlying magnetic network measured with all coils of the electrical network *open-circuited.*

2. The inverse metric tensor a^{nm} represents the self- and mutual *reluctances* of the underlying magnetic network measured with all coils of the electrical network *short-circuited.*

Expressing it in another way:

1. In measuring the components of the metric tensor a_{mn} the magnetic flux lines follow paths with the *maximum possible* permeances (Fig. 18.4*a*).

(*a*) Magnetic paths with maximum permeance

(*b*) Magnetic paths with minimum permeance

FIG. 18.4

2. In measuring the components of the inverse metric tensor the magnetic flux lines follow paths with the *minimum possible* permeances (Fig. 18.4*b*).

V. THE RAISING AND LOWERING OF INDICES

(*a*) The metric tensor a_{mn} has the following important properties:

1. *If an upper (contravariant) index of any tensor is multiplied by the metric tensor a_{mn}, the upper index becomes a lower (covariant) index.* For instance

$$K^{mn}_{\cdot\cdot}a_{nk} = K^{m}_{\cdot k} \quad \text{or} \quad A_{m}^{\cdot nk}a_{kp} = A_{m\cdot p}^{\cdot n} \qquad 18.4$$

All the other indices of the tensor and its base letter remain undisturbed. The indices also keep their proper order as

$$K^{m\cdot k\cdot q}_{\cdot n\cdot p}a_{kr} = K^{m\cdot\cdot\cdot q}_{\cdot nrp} \qquad 18.5$$

where the *third* upper index *k* became the *third* lower index *r*. Similarly:

2. *If a lower (covariant) index of any tensor is multiplied by the inverse metric tensor a^{mn}, the lower index becomes an upper index.* I.e.

$$K^{m\cdot k\cdot q}_{\cdot n\cdot p}a^{nr} = K^{mrk\cdot q}_{\cdot\cdot\cdot p} \quad \text{or} \quad i^{m}a_{mn} = i_{n} \qquad 18.6$$

These rules for changing the indices are *not* valid for geometric objects, only for tensors. (It should be remembered that the indices of *n*-matrices are neither covariant nor contravariant, and these rules have no meaning for them.)

Several indices may be raised and lowered in one step as

$$K^{m\cdot k\cdot q}_{\cdot n\cdot p}a^{nr}a_{qt}a_{ks} = K^{mr\cdot\cdot\cdot}_{\cdot\cdot spt} \qquad 18.7$$

(*b*) Tensors that have the same base letter and the same number of indices, but in different position, are called "*associated tensors*" (Section IV, Chapter VIII.) Hence, ***associated tensors can be changed into each other with the aid of the metric tensor.***

(*c*) If one index of the metric tensor itself is raised the mixed metric tensor is the unit tensor I^m_n. That is

$$a_{mn}a^{nk} = \mathbf{a}\cdot\mathbf{a}^{-1} = \mathbf{I} = a^k_m = I^k_m \tag{18.8}$$

As a consequence, ***two dummy indices, one an upper, the other a lower index, may interchange their position as***

$$R_{mn}i^n = R_m^{\cdot n}{}_{\cdot}i_n \tag{18.9}$$

since $\qquad R_{mn}i^n = R_{mk}I^k_n i^n \quad \text{and} \quad I^k_n = a^{kp}a_{np}$

hence $\qquad R_{mn}i^n = (R_{mk}a^{kp})(a_{np}i^n) = R_m^{\cdot p}i_p = R_m^{\cdot n}i_n$

VI. ASSOCIATED TENSORS OF THE DESIGN CONSTANTS

(*a*) The three design constants or basic invariants of a mesh network are defined as twice covariant 2-tensors r_{mn}, a_{mn}, s_{mn}. If they are multiplied by the inverse metric tensor they become mixed 2-tensors, namely

$\boxed{r_{mn}a^{nk} = r_m^{\cdot k}}$ 18.10 $\quad \boxed{a_{mn}a^{nk} = I^k_m}$ 18.11 $\quad \boxed{s_{mn}a^{nk} = s_m^{\cdot k}}$ 18.12

(*b*) ***The mixed resistance tensor $r_m^{\cdot n}$ contains such ratios as:***

$$\frac{\text{resistance}}{\text{short-circuit inductance}} = \frac{r}{L'} = \delta = \text{decrement factor} \tag{18.13}$$

For a two-winding transformer

$$r_m^{\cdot n} = \begin{array}{|c|c|}\hline r_p & \\ \hline & r_s \\ \hline\end{array} \cdot \begin{array}{|c|c|}\hline 1/L'_p & -\lambda_p/L'_p \\ \hline -\lambda_s/L'_s & 1/L'_s \\ \hline\end{array} = \begin{array}{c|c|c|}m \backslash n & p & s \\ \hline p & r_p/L'_p & -\lambda_p(r_p/L'_p) \\ \hline s & -\lambda_s(r_s/L'_s) & r_s/L'_s \\ \hline\end{array} \tag{18.14}$$

In terms of "*decrement factors*" δ_p, the mixed resistance tensor is

$$r_m^{\cdot n} = \begin{array}{c|c|c|} & p & s \\ \hline p & \delta_p & -\lambda_p\delta_p \\ \hline s & -\lambda_s\delta_s & \delta_s \\ \hline\end{array} \tag{18.15}$$

If a coil with resistance r has no mutual inductance with other coils, its short-circuit inductance is the same as its own inductance L and the corresponding term in $r_{m}^{\cdot n}$ is r/L.

(c) *The mixed elastance tensor $s_{m}^{\cdot n}$ contains such ratios as:*

$$\frac{\text{elastance}}{\text{short-circuit inductance}} = \frac{S}{L'} = \frac{1}{CL'} = \upsilon = \text{frequency factor} \qquad 18.16$$

If a coil with capacitance C has no mutual inductance with other coils, the corresponding term in $C_{m}^{\cdot n}$ is $1/LC$.

VII. ASSOCIATED TENSORS OF THE ELECTROMAGNETIC QUANTITIES

(a) If the current vector i^m is multiplied by the metric tensor a_{mn}, it gives the flux linkage vector ϕ_m

$$\boxed{i^m a_{mn} = i_n = \phi_n} \qquad 18.17$$

Hence, *the flux linkage vector ϕ_n may also be written as a covariant current i_m. Similarly, the current vector i^m may also be written as a contravariant flux linkage*

$$\boxed{\phi_m a^{mn} = \phi^n = i^n} \qquad 18.18$$

In mechanical problems "*velocity*" v^m and "*momentum*" M_m (mass × velocity) are in the same relation as "current" and "flux linkage" are in electrical problems. That is, $v^m = M^m$ and $M_m = v_m$.

(b) If the *impressed* voltage vector e_m is multiplied by the inverse metric tensor, it gives

$$\boxed{e_m a^{mn} = e^n} \qquad 18.19$$

There appears to be no concept in electrical engineering usage corresponding to e^m. Its components contain expressions such as e/L'. *It represents the acceleration of the electric charges.*

In mechanical problems "*acceleration*" a^m and "*applied force*" f_m (mass × acceleration) are in the same relation as e^m and e_m are in electrical problems.

Similarly there is no electrical equivalent to $q_n = q^m a_{mn}$. (However, in *junction* networks both E^u and Q_u have a physical interpretation, on the other hand I_u and Φ^u have none, as will be shown in Section XV.)

VIII. THE MIXED EQUATIONS OF PERFORMANCE

(*a*) The equations of voltage of stationary symmetrical nesh networks

$$e_m = (r_{mn} + l_{mn}p + s_{mn}/p)i^n \qquad 18.20$$

may be expressed in terms of mixed tensors by interchanging the position of the two dummy indices n giving

$$e_m = (r_m^{\cdot n} + l_m^n p + s_m^{\cdot n}/p)\phi_n \qquad 18.21$$

That is, *if in the equation of voltage the current i^m is replaced by the flux linkage ϕ_m as the variable, then only two types of design constants, namely decrement factors δ and frequency factors ν, occur in the equations.* That is

$$\boxed{z_m^{\cdot n} = r_m^{\cdot n} + l_m^n p + s_m^{\cdot n}/p} \qquad 18.22$$

The new $z_m^{\cdot n}$ is found from the usual z_{mn} by multiplying it by the inverse of the inductance or metric tensor.

(*b*) Two special cases are of interest:

1. *If the dielectric network is absent*

$$\boxed{z_m^{\cdot n} = r_m^{\cdot n} + l_m^n p} \qquad 18.23$$

Its matrix contains the differential operator p only in the diagonal components as in

$$z_m^{\cdot n} = \begin{array}{c|ccc} {}_m\backslash^n & a & b & c \\ \hline a & \delta_{aa} + p & \delta_{ab} & \delta_{ac} \\ b & \delta_{ab} & \delta_{bb} + p & \delta_{bc} \\ c & \delta_{ac} & \delta_{bc} & \delta_{cc} + p \end{array} \qquad 18.24$$

This matrix is called the "*characteristic matrix*," and its differential equation $e_m = z_m^{\cdot n}\phi_n$ is called the "*characteristic equation*" or "*secular equation.*" *A very extensive literature is available in the theory of matrices on the properties and methods of solution of these types of differential equations.*

2. *If the resistances can be neglected,* as they can in oscillatory circuits, the mixed impedance tensor becomes

$$z_m^{\cdot n} = l_m^n p + s_m^{\cdot n}/p \qquad 18.25$$

and with no impressed voltages *the differential equation of the oscillatory system is*

$$\boxed{0 = (l^n_m p^2 + s_m^{\cdot n})i_n} \qquad 18.26$$

containing p^2 in each of its diagonal components. It is a "characteristic equation."

(c) Instead of replacing the current i^m by the flux linkage ϕ_m as the variable, the mixed design constants could have been introduced by replacing e_m by e^m *by multiplying every term of the equation by* a^{mn} *as*

$$a^{km}e_m = a^{km}z_{mn}i^n$$

$$\boxed{e^k = z^{k}_{\cdot n}i^n = (r^k_{\ n} + l^k_n p + s^{k}_{\cdot n}/p)i^n} \qquad 18.27$$

(d) It is the best procedure to set up the mixed impedance tensor $z_m^{\cdot n}$first for the primitive mesh network, then transform it for the actual network. *The transformation formula of* $z_m^{\cdot n}$ *is*

$$\boxed{\mathbf{z}' = \mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}_t^{-1}} \qquad \Bigg| \qquad \boxed{z_{m'}^{\cdot n'} = z_m^{\cdot n} C^m_{m'} C^{n'}_n} \qquad 18.28$$

IX. ADVANTAGES OF THE USE OF THE MIXED TENSORS

For design purposes and for numerical calculations it is advantageous to change the equations from z_{mn} to $z_m^{\cdot n}$ or from Y^{nm} to $Y^n_{\cdot m}$, that is from resistances, inductances, and capacities to ratios, namely to decrement factors δ and frequency factors ν. The advantage of their use is manifold:

1. Their value for a given type of apparatus varies little from one design to another, from small to large apparatus, etc., hence the final numerical answer can be estimated more easily, fewer mistakes are made in placing decimal points, etc.

2. The same results are valid for several apparatus of different sizes.

3. When the amount of copper or iron or insulation is varied, the mixed design constants vary proportionally.

4. These ratios are little affected by saturation.

5. In plotting graphs and curves for the quick determination of the performance of a line of apparatus of various sizes, the mixed design constants are the most logical parameters to use since their number is less than the number of usual design constants.

6. In finding the roots of the algebraic equations in p to solve the differential equations, *the roots are functions of the mixed design constants only.*

7. In *graphical* performance calculations the loci can be easily determined if mixed design constants are used for its construction.

It is possible to replace the mixed design constants developed in the previous sections by new and fewer design constants. Their study, however, is not undertaken here.

(The rest of this chapter covers geometrical representations and may be left out without disturbing the developments that follow.)

X. THE UNIT ELLIPSE

(*a*) When a metric tensor is introduced, both the current vector i^m and the flux linkage vector ϕ_m may be denoted by the same base letter i or ϕ, expressing the fact that *both current i^m and flux linkage i_m are two different representations of one and the same physical entity i.* The question now is: What physical entity does the base letter i represent? It should be recalled that in Section III, Chapter VIII, this same question was left unanswered.

In order to answer that let first a geometrical representation of the metric tensor a_{mn} be given by assuming a *plane* with a rectilinear reference axis and a current vector on it as explained in Chapter VIII. *It should be understood that the representation of the current vector i^m as lying in an n-dimensional ordinary space (affine space) is not correct for networks*; it is too general, as explained in Section XI, Chapter VIII, but it will serve the purpose of explaining the geometrical concept of "*magnitude.*"

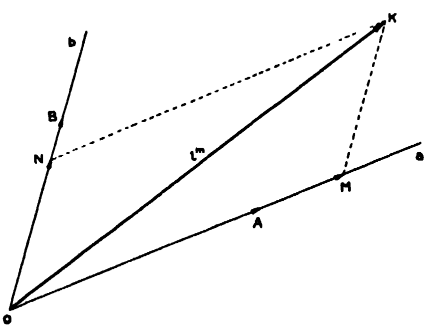


FIG. 18.5—The Contravariant Vector i^m.

Hence let a current vector be given, say

$$i^m = \begin{array}{c|c} a & b \\ \hline 1.34 & 0.772 \end{array} \qquad (m) \tag{18.29}$$

On the plane of Fig. 18.5 it is represented by assuming any two lengths OA and OB along axes a and b as unity, then measuring $OM = 1.34\ OA$ along axis a and $ON = 0.772\ OB$ along axis b, determining thereby point K by a parallelogram. The length OK represents the given current vector i^m, since its *components* OM and ON are 1.34 and 0.772 respectively.

(*b*) The question now arises: *What is the "magnitude" of vector OK* (or what is the "distance" between points O and K)? Theoretically, it is permissible to associate with vector OK any arbitrary number and call it its "magnitude" (or the "distance between points O and K"). Practically, however, the following mechanism is established to attach "magnitude" to any vector drawn on the plane with point O as its origin.

Draw any arbitrary ellipse on the plane with its center at O. *By definition, any contravariant vector drawn from the origin O to the ellipse has unit "magnitude."* If the vector does not touch the ellipse, its "magnitude" is measured with this unit.

For instance, on Fig. 18.6 the magnitude of vector OK is $OK/OP = 1.73$, where point P is the intersection of vector OK with the ellipse.

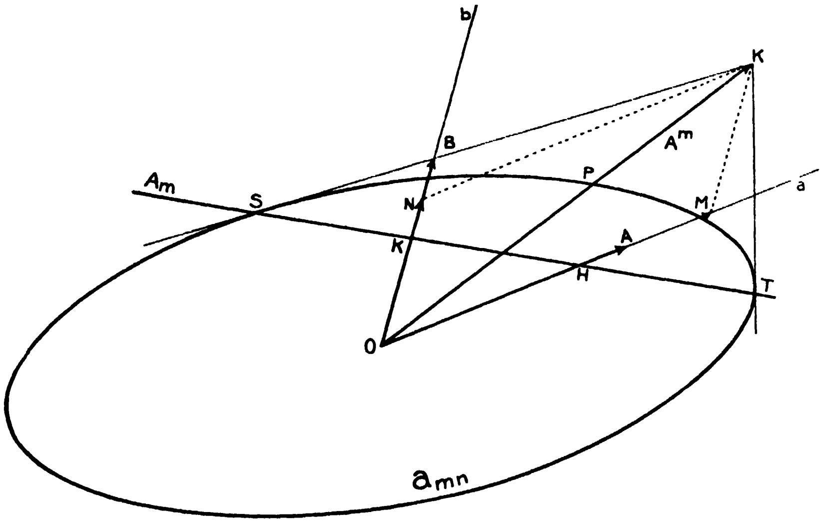


FIG. 18.6—Representation of the Metric Tensor a_{mn}

That is, *in every direction a different length serves as the unit of "magnitude,"* and the same vector OK has a different magnitude if it is rotated around the origin, since it cuts off a different length from the ellipse along each direction.

In an n-dimensional space the ellipse becomes an $n - 1$ dimensional ellipsoid or a "quadric surface."

(*c*) This elaborate definition of the "magnitude" of a vector (or "distance" between two points) reduces to the usual definition in the special case when (Fig. 18.7):

1. The axes are at right angles to each other.
2. The two measuring vectors OA and OB are equal.

3. The ellipse is a circle.
4. The circle passes through points A and B.

(*d*) It is emphasized that *there is no relation between the measuring vectors OM and ON assumed and the ellipse assumed*; *they are entirely independent of each other.* The assumption of the measuring vectors determine the "components" of i^m, and the assumption of the ellipse determines the "magnitude" of i^m.

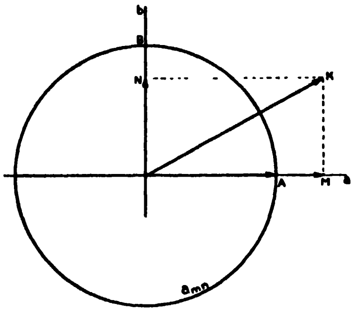


FIG. 18.7.—The Metric Tensor in a Euclidean Space

It is also emphasized that *it is not necessary to define the "magnitude" of a vector when the "components" of the vector are given.* For instance, in the general case of the impedance tensor z_{mn} the metric tensor a_{mn} may not be defined, but nevertheless e_m and i^m have an existence geometrically and physically.

Hence, *in using the equations* $e_m = z_{mn}i^n$, *the concept of "magnitude" does not enter into the study at all.*

XI. THE MAGNITUDE OF A VECTOR

(*a*) *The unit ellipse, serving as a scale for the determination of "magnitude," may also serve as the geometrical representation of the metric tensor* a_{mn}. That is, if the components of the metric tensor are given the unit ellipse can be constructed. If

$$a_{mn} = \begin{array}{c|cc} m \diagdown n & a & b \\ \hline a & 0.6 & 0.547 \\ b & 0.547 & 1.32 \end{array} \qquad 18.30$$

then *the unit quadric surface is defined as*

$$\boxed{a_{mn}i^m i^n = 1} \qquad 18.31$$

where i^m may assume any values. For the given value of a_{mn} this equation is

$$0.6(i^a)^2 + 2 \times 0.547\, i^a i^b + 1.32(i^b)^2 = 1$$

If various values are assumed for i^a and the equation is solved for i^b, the point describes the ellipse of Fig. 18.6. Hence *the metric tensor*

a_{mn} determines a quadric surface with the aid of equation 18.31 *just as a covariant vector e_m determines a plane with the aid of equation* 8.3 ($e_m i^m = 1$).

(*b*) *If a vector A^m is given, its "magnitude" is defined with the aid of the metric tensor as*

$$\boxed{(\text{Magnitude of } A^m)^2 = |A|^2 = a_{mn}A^mA^n} \qquad 18.32$$

This formula is a generalization for rectilinear axes of the Pythagorean theorem for rectangular axes.

For instance, if A^m is the vector given in equation 18.29 and shown as OK in Fig. 18.5, then the square of its magnitude is $\mathbf{A}\cdot\mathbf{a}\cdot\mathbf{A}$ or

$$\underrightarrow{\begin{array}{|c|c|}\hline 1.34 & 0.772 \\ \hline\end{array}} \cdot \begin{array}{|c|c|}\hline 0.6 & 0.547 \\ \hline 0.547 & 1.32 \\ \hline\end{array}\downarrow = \begin{array}{|c|}\hline 1.226 \\ \hline 1.752 \\ \hline\end{array}$$

and

$$\underrightarrow{\begin{array}{|c|c|}\hline 1.226 & 1.752 \\ \hline\end{array}} \cdot \begin{array}{|c|}\hline 1.34 \\ \hline 0.772 \\ \hline\end{array}\downarrow = 3 \qquad 18.33$$

Hence, *the magnitude of A^m is* $\sqrt{3} = 1.73$. Fig. 18.6 also gives for $OK/OP = 1.73$.

Since $a_{mn}A^m = A_n$, the magnitude of A^m is also

$$\boxed{(\text{Magnitude of } A^m)^2 = |A|^2 = A_mA^m = a^{mn}A_mA_n} \qquad 18.34$$

XII. THE POLE AND POLAR OF AN ELLIPSE

The contravariant vector A^m represents a point K on Fig. 18.6. If it is multiplied by a_{mn}, it becomes a covariant vector $a_{mn}A^m = A_n$,

$$A_n = \underrightarrow{\begin{array}{|c|c|}\hline 0.6 & 0.547 \\ \hline 0.547 & 1.32 \\ \hline\end{array}} \cdot \begin{array}{|c|}\hline 1.34 \\ \hline 0.772 \\ \hline\end{array}\downarrow = \overset{n\searrow}{\begin{array}{|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} \\ \hline 1.227 & 1.752 \\ \hline\end{array}} \qquad 18.35$$

It is represented by the line ST on Fig. 18.6 whose intercepts are $OK = 1/1.227$ and $OH = 1/1.752$.

There is an interesting relation between the *line* of A_m and the *point* A^m. *If the point A^m and the ellipse a_{mn} are given, the line A_m is found by drawing tangents from point A^m to the ellipse a_{mn}.* The tangent lines are shown as KS and KT. The line A_m connects the points of contact of the tangent lines.

The point K is called the "*pole*," and the line ST is called the "*polar*" of the ellipse. If one is given, the other can be uniquely constructed. Hence, *the point A^m is the "pole" and the line A_m is the corresponding "polar" of the ellipse a_{mn}.*

In an n-dimensional space (when A^m has n components) a_{mn} is an $n - 1$ dimensional quadric surface (ellipsoid), A^m is a point, and A_m is an $n - 1$ dimensional plane, the "polar plane" of the "pole" with respect to the quadric surface. When the ellipsoid (a_{mn}) is given, for every point A^m of the n-dimensional space belongs one, and only one, $n - 1$ dimensional polar plane A_m, and vice versa.

XIII. THE STORED MAGNETIC ENERGY

(*a*) The contravariant vector i^m represents the currents flowing in the meshes. Its "magnitude" is

$$(\text{Magnitude of } i^m)^2 = |i|^2 = a_{mn}i^m i^n = 2T \qquad 18.36$$

Hence, *the square of the "magnitude" of the current vector i^m is equal to twice the stored magnetic energy in the network.*

The covariant vector $i_m = \phi_m$ represents the flux linkages of the meshes. Its magnitude is

$$\boxed{|\phi|^2 = a^{mn}\phi_m\phi_n = \phi_m i^m} \qquad 18.37$$

That is, the square of the magnitude of the flux linkage vector ϕ_m is also equal to twice the stored magnetic energy.

Hence, *the current vector i^m and the flux linkage vector ϕ_m are two different types of representation of one and the same physical entity, the "stored magnetic energy" or "kinetic energy" of the system.*

(*b*) *The fundamental physical entity existing in a network—or anywhere in nature—is "energy."* Every form of energy has the property that it is *measured* by instruments in two different manifestations:

1. As an "*intensity factor*," like current i^m or velocity v^m, represented as the *contravariant components of energy.*

2. As an "*extensity factor*," like flux linkage ϕ_m or momentum M_m, represented as the *covariant components of energy.*

The product of an intensity and an extensity factor is energy, that is, *the product of a contravariant vector and a covariant vector is energy* like $2T = i^m\phi_m = v^m M_m$.

Hence when the "components" of the current vector i^m alone are known it is impossible to determine how much stored magnetic energy those components represent. Their existence points only to the *existence* of a physical entity, namely to that of the magnetic energy. The

same applies if the flux linkage vector ϕ_m alone is known. If, however, the metric tensor a_{mn} is also known then from the current vector alone, or from the flux linkage vector alone, the amount of stored magnetic energy (the "magnitude" of i^m or ϕ_m) may be determined.

Expressed in another way, *with many mesh network there is associated an actually existing physical entity, the stored magnetic energy i.* This entity has two types of components in the network, that can be measured by two different types of instruments. The contravariant components i^m are measured by an ammeter, the covariant components i_m by a flux meter.

There is no such physically existing quantity in a network as the "resultant current vector," or the "resultant flux linkage vector." There is, however, a physically existing quantity, the "resultant stored magnetic energy" i, which has "contravariant components" i^m *and "covariant components"* $i_m = \phi_m$. The *components* of i may assume different values as the reference frame varies, but the *magnitude* of i is independent of the reference frame assumed.

XIV. THE THERMODYNAMICS OF NETWORKS

(*a*) For the sake of simplicity let an orthogonal network with only stored magnetic energy be assumed, say a multiwinding transformer network in which the winding resistances are ignored.

In any *orthogonal* network of n coils the k meshes with k impressed voltages $\mathbf{e}$ may be assumed as *input* terminals, while the $n - k$ junction-pairs supplying $n - k$ load currents $\mathbf{I}$ to outside loads may be assumed as *output* terminals. That is, *any orthogonal network may be considered as a generalized transformer in which the meshes are the input terminals and the junction-pairs the output terminals.* Energy flows *into* the meshes and *out of* the junction-pairs, and the network serves as a device transforming the magnitude, phase, and *number* of voltages and currents.

Hence an orthogonal network transforms one form of electrical energy into another form.

(*b*) *In an orthogonal network the metric tensor* $\mathbf{a} = a_{\alpha\beta}$ (representing the self- and mutual inductances) *may assume two extreme values:*

1. When the junction-pairs are *open* and no current $\mathbf{I}$ is supplied to the load, $\mathbf{a}$ is equal to that of the mesh network $\mathbf{a}_1$ and it has k rows and columns.

2. When the junction-pairs are *short-circuited* and no terminal voltage $\mathbf{E}$ appears across the loads, $\mathbf{a}$ becomes

$$\mathbf{a}' = \mathbf{a}_1 - \mathbf{a}_2 \cdot \mathbf{a}_4^{-1} \cdot \mathbf{a}_3 \qquad 18.38$$

In the first case the stored magnetic energy in the network is a *minimum* and the flux lines follow paths with the *maximum possible permeances*. In the second case the stored magnetic energy is a *maximum* and the flux lines follow paths with the *minimum possible permeances*. When the network supplies a load the stored energy assumes an intermediate value between the two extreme values. The flow of energy is taking place at constant temperature ("isothermal" process).

The minimum possible stored energy is called the "bound energy" of the network, and the difference between the maximum possible and the minimum possible stored energy is called its "free energy" (or "thermodynamic potential").

Any energy that departs from the system leaves it at the expense of the free energy. The "free energy" represents the ability of the network to supply energy to an outside load.

(*c*) The "*efficiency*" of a network to supply outside loads may be represented by the single number

$$\boxed{\eta = \frac{\text{free energy}}{\text{maximum stored energy}} = \frac{\text{maximum energy} - \text{minimum energy}}{\text{maximum stored energy}}} \qquad 18.39$$

In a two-winding transformer this ratio becomes the "coupling coefficient." Hence, *η also represents the amount of coupling existing between the meshes and the junction-pairs.*

(*d*) These interesting and important thermodynamical considerations are not continued at this point.

XV. THE DUAL METRIC TENSOR

(*a*) In junction networks the previous reasoning may be repeated word for word except that *the "dual" quantities are interchanged.*

The capacitance tensor C^{uv} plays the role of the dual metric tensor A^{uv} that raises and lowers indices of tensors that occur in its equation of current. Hence the mixed tensors are

$$G^{uv}A_{vw} = G^{u}_{\cdot w}, \quad C^{uv}A_{vw} = I^{u}_{w}; \quad K^{uv}A_{vw} = K^{u}_{\cdot w} \qquad 18.40$$

The mixed conductance tensor $G^{u}_{\cdot v}$ represents

$$\frac{\text{conductance}}{\text{open-circuit capacitance}} = \frac{G}{C'} = S'G \qquad 18.41$$

and the mixed susceptance tensor $K^{u}_{\cdot v}$ represents

$$\frac{\text{susceptance}}{\text{open-circuit capacitance}} = \frac{K}{C'} = \frac{1}{LC'} \qquad 18.42$$

(*b*) The contravariant voltage variable E^u

$$\boxed{E_u A^{uv} = E^v = Q^v} \tag{18.43}$$

is equivalent to the electric charges Q^u, so that $E_u = Q_u$.

However, now $I_u = I^v A_{vu}$ and $\Phi^u = \Phi_v A^{vu}$ have no physical interpretation, while in mesh networks e^m and q_m have none.

In terms of mixed tensors the equation of current is

$$I^u = (G^{u}_{\cdot v} + \Gamma^{u}_{v}p + K^{u}_{\cdot v}/p)Q^v = Y^{u}_{\cdot v}Q^v \tag{18.44}$$

(*c*) *The square of the "magnitude" of the voltage vector* E_u (or charge vector Q_u) *is equal to twice the stored electrostatic energy in the network*

$$\boxed{|E|^2 = |Q|^2 = A^{uv}E_uE_v = A_{uv}Q^uQ^v = E_uQ^u = 2V} \tag{18.45}$$

Hence *the voltage vector E_u and the charge vector Q^u are two different types of representation of one and the same physical entity, the "stored electrostatic energy" or "potential energy" of the system.*

(*d*) It should be noted that from a theoretical point of view there is no reason to prefer the use of $a_{mn} = l_{mn}$ instead of $A^{uv} = C^{uv}$. In all books on tensor analysis only a_{mn} is considered as the sole metric tensor since the dual point of view is not introduced. However, the dual point of view in network (and in topological) studies necessitates the introduction of a dual metric tensor, as well as other dual concepts and equations.

XVI. "UNDERLYING" SPACES AND "LOCAL" SPACES

(*a*) In Chapter VIII the contravariant variable i^m was geometrically represented (as a first approximation) by a point in an n-dimensional affine space whose projections (measured from a common origin) represented the components of i^m. However, when i^m becomes a variable dq^m/dt derived from a basic variable q^m, then even this approximate representation needs further extension. *Only the basic variable q^m is represented by a vector drawn from the fixed origin to a variable point.*

In order to visualize better the representation of the derived variable i^m, assume that the point q^m lies on a curved surface, like that of a sphere, having a fixed origin from which all components of q^m are measured along some curvilinear set of axes. In order to represent $i^m = dq^m/dt$, *assume a tangent plane at point q^m* (Fig. 18.8). *As point q^m moves, this tangent plane also moves with it.* In the tangent plane

at each point q^m a set of reference axes may be drawn, each axis being tangent to an axis on the underlying curved surface. That is, point q^m, the point of contact of the plane and the surface, serves as the origin of the reference frame on the tangent plane. In the general case both surface and plane are n-dimensional.

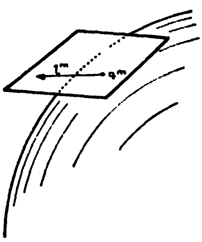


FIG. 18.8.—A Curved *Underlying* Space and a Flat *Local* Space

(*b*) Now, *the derived variable i^m is represented by a point on the tangent plane* just as the basic variable q^m is represented by a point on the underlying surface. Since point q^m serves as the origin of i^m, the origin of q^m is fixed, while *the origin of i^m is not fixed.* As the value of q^m varies, the local space moves on the underlying space, carrying along i^m.

The curved surface, the locus of the basic variable q^m, is called the "*underlying space*," and the plane, the locus of the derived variable dq^m or dq^m/dt, is called the "*local space*" or "tangent space." In network studies the underlying space is not curved but flat, so that in two dimensions it becomes a plane and the two spaces. coincide. *In such a case i^m is represented by a vector drawn between two points $q^m = OA$ and $q^m + dq^m/dt = OB$,* as shown in Fig. 18.9.

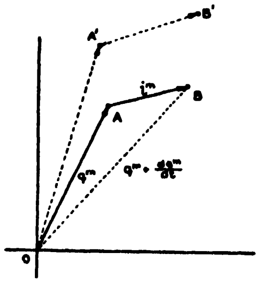


FIG. 18.9—Vector q^m with Fixed Origin Vector i^m with Variable Origin

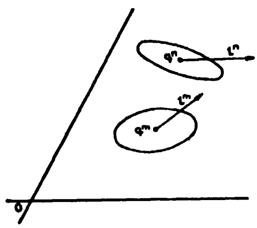


FIG. 18.10.—Variation of the Metric Tensor from Point to Point

As the value of q^m changes to OA', i^m changes to $A'B'$.

(*c*) The variable point q^m serves as origin not only to dq^m or to dq^m/dt but to every other tensor defined. In other words, *all tensors previously introduced are assumed to lie in the tangent space* whose point of contact with the underlying space varies as q^m varies.

For instance, the center of the ellipsoid representing the metric tensor a_{mn} also moves with point q^m as shown in Fig. 18.10. If a_{mn} is a function of q^m, then *at every point of the underlying space the ellipsoid a_{mn} has a different shape.* That is, in general the measuring unit of

the "length" of a vector not only is different along different directions, but it also varies from point to point of the underlying space.

(*d*) In this Chapter the question was investigated, "*What is the magnitude of a vector?*" This question called into existence the first fundamental invariant of Tensor Analysis, the "metric tensor", $a_{\alpha\beta}$.

Since the position of all vectors changes as q^m changes, also a vector such as AB on Fig. 18.9, moves to $A'B'$. The movement of vectors from place to place brings up the question, "*When is the vector $A'B'$ parallel to vector AB?*" This question calls into existence, the second fundamental invariant of Tensor Analysis, the "*affine connection,*" $\Gamma^{\gamma}_{\alpha\beta}$ that however is not introduced in this volume. The *resistance tensor* $r_{\alpha\beta}$ will also play a part in the definition of parallelism of two vectors.

CHAPTER XIX

COMPOUND NETWORKS

I. THE BASIC EQUATIONS OF ACTIVE, ASYMMETRICAL NETWORKS

(*a*) After the equation of performance of a network has been set up, usually it is subjected to further manipulations. For purposes of manipulation the single tensor equation is subdivided into two, three, four, or more tensor equations depending on the problem at hand. In particular the following subdivisions are made:

1. The equation of voltage of a *mesh* network $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$ is divided as

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{i}^2 + \mathbf{z}_{13}\cdot\mathbf{i}^3 + \cdots \mathbf{z}_{1n}\cdot\mathbf{i}^n \\
\mathbf{e}_2 &= \mathbf{z}_{21}\cdot\mathbf{i}^1 + \mathbf{z}_{22}\cdot\mathbf{i}^2 + \mathbf{z}_{23}\cdot\mathbf{i}^3 + \cdots \mathbf{z}_{2n}\cdot\mathbf{i}^n \\
\mathbf{e}_3 &= \mathbf{z}_{31}\cdot\mathbf{i}^1 + \mathbf{z}_{32}\cdot\mathbf{i}^2 + \mathbf{z}_{33}\cdot\mathbf{i}^3 + \cdots \mathbf{z}_{3n}\cdot\mathbf{i}^n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbf{e}_n &= \mathbf{z}_{n1}\cdot\mathbf{i}^1 + \mathbf{z}_{n2}\cdot\mathbf{i}^2 + \mathbf{z}_{n3}\cdot\mathbf{i}^3 + \cdots \mathbf{z}_{nn}\cdot\mathbf{i}^n
\end{aligned} \qquad 19.1$$

This set of n tensor equations is analogous to the set of n ordinary equations representing the equation of voltage of an n-mesh network.

2. The equation of current of a *junction* network $\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}$ is divided as

$$\begin{aligned}
\mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 + \mathbf{Y}^{12}\cdot\mathbf{E}_2 + \mathbf{Y}^{13}\cdot\mathbf{E}_3 + \cdots \mathbf{Y}^{1n}\cdot\mathbf{E}_n \\
\mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 + \mathbf{Y}^{22}\cdot\mathbf{E}_2 + \mathbf{Y}^{23}\cdot\mathbf{E}_3 + \cdots \mathbf{Y}^{2n}\cdot\mathbf{E}_n \\
\mathbf{I}^3 &= \mathbf{Y}^{31}\cdot\mathbf{E}_1 + \mathbf{Y}^{32}\cdot\mathbf{E}_2 + \mathbf{Y}^{33}\cdot\mathbf{E}_3 + \cdots \mathbf{Y}^{3n}\cdot\mathbf{E}_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbf{I}^n &= \mathbf{Y}^{n1}\cdot\mathbf{E}_1 + \mathbf{Y}^{n2}\cdot\mathbf{E}_2 + \mathbf{Y}^{n3}\cdot\mathbf{E}_3 + \cdots \mathbf{Y}^{nn}\cdot\mathbf{E}_n
\end{aligned} \qquad 19.2$$

This set of n tensor equations is analogous to the set of n ordinary equations representing the equations of a network with n junction-pairs.

3. The equation of voltage of an *orthogonal* network $\mathbf{E} + \mathbf{e} = \mathbf{z}\cdot(\mathbf{i} + \mathbf{I})$ is first subdivided into the orthogonal equations of voltage

$$\begin{aligned}
\mathbf{E}_m + \mathbf{e}_m &= \mathbf{z}_{mm}\cdot(\mathbf{i}^m + \mathbf{I}^m) + \mathbf{z}_{mj}\cdot(\mathbf{i}^j + \mathbf{I}^j) \\
\mathbf{E}_j + \mathbf{e}_j &= \mathbf{z}_{jm}\cdot(\mathbf{i}^m + \mathbf{I}^m) + \mathbf{z}_{jj}\cdot(\mathbf{i}^j + \mathbf{I}^j)
\end{aligned} \qquad 19.3$$

each of which may be divided into two or more equations as

$$
\begin{aligned}
\mathbf{E}_{m_1} + \mathbf{e}_{m_1} &= \mathbf{z}_{m_1 m_1}\cdot(\mathbf{i}^{m_1} + \mathbf{I}^{m_1}) + \mathbf{z}_{m_1 m_2}\cdot(\mathbf{i}^{m_2} + \mathbf{I}^{m_2}) + \cdots\cdot \\
&\quad + \mathbf{z}_{m_1 j_1}\cdot(\mathbf{i}^{j_1} + \mathbf{I}^{j_1}) + \mathbf{z}_{m_1 j_2}\cdot(\mathbf{i}^{j_2} + \mathbf{I}^{j_2}) + \cdots\cdot \\
\mathbf{E}_{m_2} + \mathbf{e}_{m_2} &= \mathbf{z}_{m_2 m_1}\cdot(\mathbf{i}^{m_1} + \mathbf{I}^{m_1}) + \mathbf{z}_{m_2 m_2}\cdot(\mathbf{i}^{m_2} + \mathbf{I}^{m_2}) + \cdots\cdot \\
&\quad + \mathbf{z}_{m_2 j_1}\cdot(\mathbf{i}^{j_1} + \mathbf{I}^{j_1}) + \mathbf{z}_{m_2 j_2}\cdot(\mathbf{i}^{j_2} + \mathbf{I}^{j_2}) + \cdots\cdot \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbf{E}_{j_1} + \mathbf{e}_{j_1} &= \mathbf{z}_{j_1 m_1}\cdot(\mathbf{i}^{m_1} + \mathbf{I}^{m_1}) + \mathbf{z}_{j_1 m_2}\cdot(\mathbf{i}^{m_2} + \mathbf{I}^{m_2}) + \cdots\cdot \\
&\quad + \mathbf{z}_{j_1 j_1}\cdot(\mathbf{i}^{j_1} + \mathbf{I}^{j_1}) + \mathbf{z}_{j_1 j_2}\cdot(\mathbf{i}^{j_2} + \mathbf{I}^{j_2}) + \cdots\cdot \\
\mathbf{E}_{j_2} + \mathbf{e}_{j_2} &= \mathbf{z}_{j_2 m_1}\cdot(\mathbf{i}^{m_1} + \mathbf{I}^{m_1}) + \mathbf{z}_{j_2 m_2}\cdot(\mathbf{i}^{m_2} + \mathbf{I}^{m_2}) + \cdots\cdot \\
&\quad + \mathbf{z}_{j_2 j_1}\cdot(\mathbf{i}^{j_1} + \mathbf{I}^{j_1}) + \mathbf{z}_{j_2 j_2}\cdot(\mathbf{i}^{j_2} + \mathbf{I}^{j_2}) + \cdots\cdot \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\qquad 19.4
$$

This set of $m + n$ invariant equations is a generalization of the ordinary equations of voltage of an orthogonal network with m meshes and n junction-pairs.

4. The equation of current of an *orthogonal* network $\mathbf{i} + \mathbf{I} = \mathbf{Y}\cdot(\mathbf{E} + \mathbf{e})$ is first subdivided into the orthogonal equations of current

$$
\begin{aligned}
\mathbf{I}^m + \mathbf{i}^m &= \mathbf{Y}^{mm}\cdot(\mathbf{E}_m + \mathbf{e}_m) + \mathbf{Y}^{mj}\cdot(\mathbf{E}_j + \mathbf{e}_j) \\
\mathbf{I}^j + \mathbf{i}^j &= \mathbf{Y}^{jm}\cdot(\mathbf{E}_m + \mathbf{e}_m) + \mathbf{Y}^{jj}\cdot(\mathbf{E}_j + \mathbf{e}_j)
\end{aligned}
\qquad 19.5
$$

each of which may be divided into two or more equations as

$$
\begin{aligned}
\mathbf{i}^{m_1} + \mathbf{I}^{m_1} &= \mathbf{Y}^{m_1 m_1}\cdot(\mathbf{E}_{m_1} + \mathbf{e}_{m_1}) + \mathbf{Y}^{m_1 m_2}\cdot(\mathbf{E}_{m_2} + \mathbf{e}_{m_2}) + \cdots\cdot \\
&\quad + \mathbf{Y}^{m_1 j_1}\cdot(\mathbf{E}_{j_1} + \mathbf{e}_{j_1}) + \mathbf{Y}^{m_1 j_2}\cdot(\mathbf{E}_{j_2} + \mathbf{e}_{j_2}) + \cdots\cdot \\
\mathbf{i}^{m_2} + \mathbf{I}^{m_2} &= \mathbf{Y}^{m_2 m_1}\cdot(\mathbf{E}_{m_1} + \mathbf{e}_{m_1}) + \mathbf{Y}^{m_2 m_2}\cdot(\mathbf{E}_{m_2} + \mathbf{e}_{m_2}) + \cdots\cdot \\
&\quad + \mathbf{Y}^{m_2 j_1}\cdot(\mathbf{E}_{j_1} + \mathbf{e}_{j_1}) + \mathbf{Y}^{m_2 j_2}\cdot(\mathbf{E}_{j_2} + \mathbf{e}_{j_2}) + \cdots\cdot \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbf{i}^{j_1} + \mathbf{I}^{j_1} &= \mathbf{Y}^{j_1 m_1}\cdot(\mathbf{E}_{m_1} + \mathbf{e}_{m_1}) + \mathbf{Y}^{j_1 m_2}\cdot(\mathbf{E}_{m_2} + \mathbf{e}_{m_2}) + \cdots\cdot \\
&\quad + \mathbf{Y}^{j_1 j_1}\cdot(\mathbf{E}_{j_1} + \mathbf{e}_{j_1}) + \mathbf{Y}^{j_1 j_2}\cdot(\mathbf{E}_{j_2} + \mathbf{e}_{j_2}) + \cdots\cdot \\
\mathbf{i}^{j_2} + \mathbf{I}^{j_2} &= \mathbf{Y}^{j_2 m_1}\cdot(\mathbf{E}_{m_1} + \mathbf{e}_{m_1}) + \mathbf{Y}^{j_2 m_2}\cdot(\mathbf{E}_{m_2} + \mathbf{e}_{m_2}) + \cdots\cdot \\
&\quad + \mathbf{Y}^{j_2 j_1}\cdot(\mathbf{E}_{j_1} + \mathbf{e}_{j_1}) + \mathbf{Y}^{j_2 j_2}\cdot(\mathbf{E}_{j_2} + \mathbf{e}_{j_2}) + \cdots\cdot \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\qquad 19.6
$$

This set of $m + n$ tensor equations is again analogous to the ordinary equations of current of an orthogonal network having n meshes and n junction-pairs.

(*b*) *These four sets of tensor equations represent the basic equations of active, asymmetrical networks, serving as a starting point for their analysis and synthesis.* Each tensor equation refers to a set of axes whose functions remain identical during the analysis.

It cannot be sufficiently emphasized that these four sets of equations are not *matric* equations. They are *tensor* equations representing the performance of a large variety of networks, and not just one particular network. *In the previous pages the mechanism has been established by which it is possible to find the components of these tensors for any particular network of n coils if they are known for any other network of n coils. The various groups of transformations that leave these tensor equations invariant are all subgroups of the "group of linear transformations" or "affine transformations"* G_{lt}. Some of the group of transformations are defined in Section VI, Chapter XI.

II. COMPOUND NETWORKS

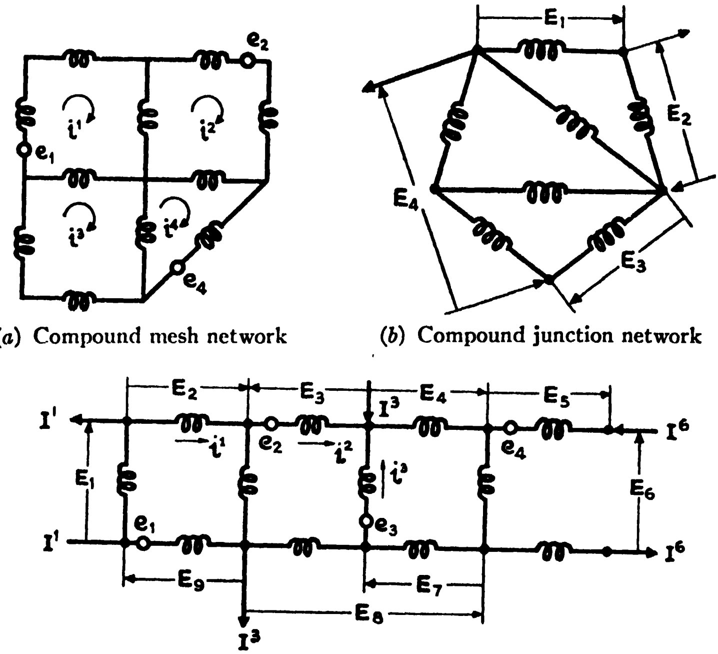
(*a*) A set of tensor equations is analogous in form to a set of ordinary equations representing some physical system. In order to visualize a set of tensor equations it seems logical to set up physical systems in which each building block itself is a system and so each of its elements is represented by a tensor, instead of a single number. Such fictitious physical systems in which each building block represents a whole system instead of a single element will be called here "compound systems" (in analogy to "compound tensors" in which each component is a tensor instead of a single number).

In order to represent physically the set of tensor equations of the previous section, *a fictitious "compound network" is introduced in which the self- and mutual impedances of the coils are tensors of valence two and the currents and voltages in the individual coils are vectors, instead of single quantities. Such compound coils will be drawn with heavy lines to distinguish them from ordinary coils. Each compound coil represents a whole network.*

The fictitious compound networks, whose equations are given in the previous section, are shown in Fig. 19.1 for the case when the number of tensor equations is four. That is, in a compound mesh network currents and voltages are considered only in meshes, in a compound junction-network only across junction-pairs, and in a compound orthogonal network the currents and voltages are considered both in the meshes and across the junction-pairs.

Each compound coil represents a whole network whose reference frame may be the actual circuits, or hypothetical axes like symmetrical

components. The components of each **z** may be actual inductances or any other hypothetical constants like leakage reactances. That is, *the individual reference axes of each compound coil may be changed arbitrarily by an "individual transformation tensor," without making any changes in the compound network. Each compound coil possesses its own individual transformation tensor.*



(*a*) Compound mesh network (*b*) Compound junction network

(*c*) Compound orthogonal network

FIG. 19.1.—The Three Types of Compound Networks

The use of compound networks is introduced in order to analyze complex networks and to set up, manipulate, and solve their tensor equations with the same facility as analogous ordinary networks are analyzed.

(*b*) *Whatever theories, laws, equations, etc., have been developed in the previous sections for ordinary networks are all valid for compound networks by simply replacing single quantities by appropriate tensors, and single tensors by appropriate compound tensors.*

For instance, it is possible to *interconnect* compound coils (each coil representing a whole network) with a *compound transformation tensor* **C** *in which each component is a tensor of valence two instead of an integer)*, or to use several compound transformation tensors in succession. Or

it is possible to *eliminate* certain meshes or junction-pairs which are not needed, or to eliminate magnetizing currents, and so on. The method of analysis and the manipulation of compound networks are analogous to those of ordinary networks, with certain precautions.

III. THE FUNCTIONAL SUBDIVISION OF NETWORKS

(*a*) There are several ways of considering a whole network as a compound network. The most obvious way is to divide the network *physically* into component parts, that is, to assume the network of being built up from small units arranged in shunt or in series. A small unit may consist of, say, a three-phase transformer, or a generator, or a transmission line, etc. Another, less evident but far more important, point of view is to divide the meshes and junction-pairs into smaller units according to their *functions*.

(*b*) The junction-pairs of a network may perform various functions. In particular:

1. Some junction-pairs may serve as terminals for *impressed* voltages **E** or *impressed* currents **I**. Such junction-pairs may be called *input* terminals.

2. Some may serve as terminals for *loads* or outside networks that do not appear in the equations or in the network diagram. In that case **I** represents the currents flowing into the loads and **E** the differences of potential appearing across the loads. Such junction-pairs may be called *output* terminals.

3. Some junction-pairs may be under *control*. For instance, their difference of potential **E** may be maintained constant by a voltage regulator or it may follow some other predetermined values, etc.

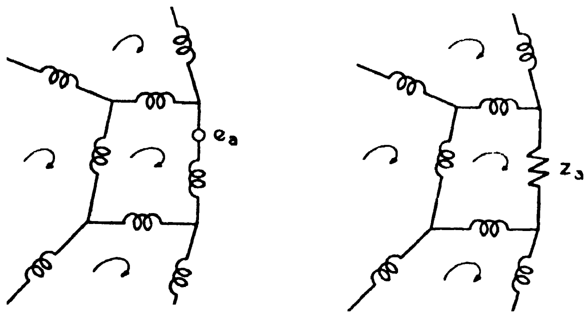
4. Some junction-pairs may be subjected to certain *changes*. A coil with impedance **Z** may be connected across them or their impressed voltage may vary or they may be short-circuited, etc.

5. Some junction-pairs are permanently *open-circuited* and no junction-currents flow through them. The knowledge of the differences of potential **E** appearing across them is often not needed, hence the corresponding axes may be eliminated from **Y**.

(*c*) The various meshes of a network may perform analogous functions. However, *the function of a mesh usually is not as clear-cut as that of a junction-pair.* For instance, a mesh voltage e_a may play the part of an input terminal voltage only if around the whole mesh (consisting of several coils) one impressed voltage exists (Fig. 19.2*a*) and also if that impressed voltage e_a exists in a branch which is not common to other meshes. Similar remarks apply to other mesh quantities such as load in a mesh, Fig. 19.2*b*, etc.

(*d*) In the same network any number and type of junction-pairs and meshes may occur, differing functionally from one another. As a consequence *the single tensor equation of performance is subdivided into as many tensor equations as there are functionally different types of junction-pairs and meshes.*

The *functional* subdivision of networks is treated in greater detail in Chapters XXII and XXIII. In the next few sections certain general network theorems are developed to reduce the number of tensor equations needed in the analysis.



(*a*) Mesh-voltage as terminal voltage

(*b*) Mesh-coil as terminal load

FIG. 19.2.—Mesh Quantities as Terminal Quantities

IV. THE ELIMINATION OF VARIABLES

(*a*) *The equation of performance of a network is divided into as many tensor equations as there are functionally different types of junction-pairs and meshes.* However, it often happens that some of the junction-pairs or meshes are *not needed* in the analysis. For instance, junction-pairs through which no currents **I** flow, or meshes in which no impressed coil voltages **e** exist, are *inactive* as far as analysis is concerned and the corresponding tensor equations act only as excess baggage during the analysis and in the final equations. *Such superfluous equations should be eliminated first before the analysis begins.*

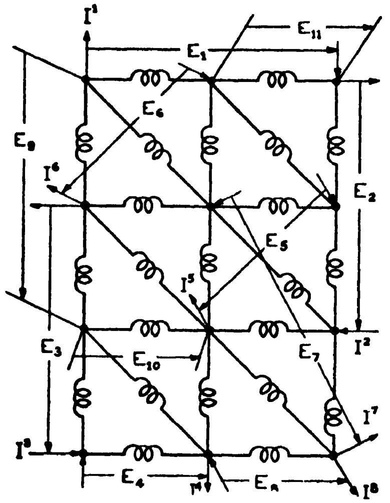
It should be remembered that, when the equation of voltage (**Z**) is set up, the inactive *junction-pairs* are ignored, but the inactive *meshes* cannot be ignored. To ignore them, the equations themselves have to be reduced by the reduction formulas of Chapter X. Similarly, when the equation of current (**Y**) is set up, only the inactive *meshes* can be ignored. The inactive *junction-pairs* may be ignored only by reducing the already established equations.

(*b*) ***Compound networks and their tensor equations may be simplified in the same manner as ordinary networks and their equations.*** That is, their number of meshes may be reduced by a mesh-star transformation, their number of junction-pairs by a star-mesh transformation, etc. These reductions are performed by eliminating one or more variables from their set of tensor equations by the reduction formulas given in Chapter X.

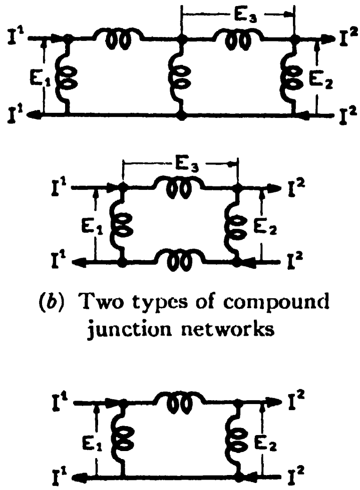
These reductions of superfluous axes should be performed, if possible, before the manipulation of the equations is undertaken.

V. REDUCTION OF COMPOUND JUNCTION NETWORKS

(*a*) As an example let a junction network such as Fig. 19.3 contain three types of junction-pairs. Across the first set of junction-pairs the



(*a*) Given junction network



(*b*) Two types of compound junction networks

(*c*) Equivalent π-network

FIG. 19.3.—Replacement of an Ordinary Junction Network by a Compound Junction Network

voltages $\mathbf{E}_1$ (E_1, E_2, E_3) are impressed, the second set is connected across some loads having a difference of potential $\mathbf{E}_2$ (E_4, E_5, E_6, E_7, E_8) across them. The remaining junction-pairs $\mathbf{E}_3$ (E_9, E_{10}, E_{11}) are permanently open-circuited and play no part in the problem, but their presence must be considered in setting up the equations of currents.

Accordingly, the equation of current of the system $\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}$ containing eleven ordinary equations is to be subdivided into three tensor equations as

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^1 \cdot \mathbf{E}_1 - \mathbf{Y}^2 \cdot \mathbf{E}_2 - \mathbf{Y}^3 \cdot \mathbf{E}_3 \\ \mathbf{I}^2 &= \mathbf{Y}^4 \cdot \mathbf{E}_1 - \mathbf{Y}^5 \cdot \mathbf{E}_2 - \mathbf{Y}^6 \cdot \mathbf{E}_3 \\ 0 &= \mathbf{Y}^7 \cdot \mathbf{E}_1 - \mathbf{Y}^8 \cdot \mathbf{E}_2 - \mathbf{Y}^9 \cdot \mathbf{E}_3 \end{aligned} \tag{19.7}$$

Assuming the impressed voltage $\mathbf{E}_1$ as positive, the differences of potential across the load $\mathbf{E}_2$ and across the open junction-pairs $\mathbf{E}_3$ are negative. Also $\mathbf{I}^3 = 0$. The input current is $\mathbf{I}^1$, and the load current is $\mathbf{I}^2$. The first tensor equation represents three ordinary equations, the second represents five, and the third three ordinary equations. These tensor equations are valid for any other junction network having the same three types of junction-pairs.

The compound junction network representing the three tensor equations is shown in Fig. 19.3*b* in two different forms. *They have three junction-pairs (one junction-pair for each equation), hence four junctions.* The number of compound meshes to be used is immaterial since they do not appear in the tensor equations. For purposes of analysis the simple compound network of Fig. 19.3*b* replaces the complex actual network of Fig. 19.3*a*. (It may be mentioned that the single tensor equation $\mathbf{I} = \mathbf{Y} \cdot \mathbf{E}$ is represented by a single open-circuited coil having one junction-pair.)

(*b*) Now if only the input quantities $\mathbf{E}_1$, $\mathbf{I}^1$ and the output quantities $\mathbf{E}_2$, $\mathbf{I}^2$ are needed in the analysis, then *the variable* $\mathbf{E}_3$ *may be eliminated, thereby reducing the three tensor equations to two.* As a result of the elimination, the compound junction network of Fig. 19.3*b* is simplified to that of Fig. 19.3*c* containing two junction-pairs in place of three. The coils of the simplified compound network are arranged in a π.

(*c*) The equations of the new network of Fig. 19.3*c* are found by eliminating $\mathbf{E}_3$ from the third equation of equation 19.7 by the reduction formulae 14.63 and 14.64 giving

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{1'} \cdot \mathbf{E}_1 - \mathbf{Y}^{2'} \cdot \mathbf{E}_2 \\ \mathbf{I}^2 &= \mathbf{Y}^{3'} \cdot \mathbf{E}_1 - \mathbf{Y}^{4'} \cdot \mathbf{E}_2 \end{aligned} \tag{19.8}$$

where the set of *open-circuit admittances* are

$$\begin{array}{l|l} \mathbf{Y}^{1'} = \mathbf{Y}^1 - \mathbf{Y}^3 . \mathbf{Y}^{9-1} . \mathbf{Y}^7 & \mathbf{Y}^{3'} = \mathbf{Y}^4 - \mathbf{Y}^6 . \mathbf{Y}^{9-1} . \mathbf{Y}^7 \\ \mathbf{Y}^{2'} = \mathbf{Y}^2 - \mathbf{Y}^3 . \mathbf{Y}^{9-1} . \mathbf{Y}^8 & \mathbf{Y}^{4'} = \mathbf{Y}^5 - \mathbf{Y}^6 . \mathbf{Y}^{9-1} . \mathbf{Y}^8 \end{array} \tag{19.9}$$

representing a π-network (a network with two junction-pairs) ***having only input and output terminals.*** The permanently open-circuited junction-pairs have disappeared from the picture. Their place is taken by the primed admittances $\mathbf{Y}^{1'}$, $\mathbf{Y}^{2'}$, $\mathbf{Y}^{3'}$, $\mathbf{Y}^{4'}$ representing the self- and mutual admittances of the input and output terminals measured with the remaining junction-pairs open-circuited.

(*d*) If the superfluous junction-pairs are not permanently open but have currents $\mathbf{I}^3$ impressed upon them, their equation still can be eliminated and the system reduced to a π network. However, the impressed currents $\mathbf{I}^1$ and $\mathbf{I}^2$ change to a new value $\mathbf{I}^{1\prime}$ and $\mathbf{I}^{2\prime}$ found by the reduction formulas as

$$\begin{aligned} \mathbf{I}^{1\prime} &= \mathbf{I}^1 - \mathbf{Y}^3 \cdot \mathbf{Y}^{9-1} \cdot \mathbf{I}^3 \\ \mathbf{I}^{2\prime} &= \mathbf{I}^2 - \mathbf{Y}^6 \cdot \mathbf{Y}^{9-1} \cdot \mathbf{I}^3 \end{aligned} \qquad 19.10$$

It should be remembered that, in order to eliminate k junction-pairs from a system, it is necessary to calculate the inverse of a matrix $\mathbf{Y}^9$ having k rows and column.

The analysis of the simplified equations 19.8 is much simpler than that of the original equations 19.7.

(*e*) *Similar steps are followed if two or more tensor equations are to be eliminated instead of one and if the original set contains more than three equations.*

VI. REDUCTION OF COMPOUND MESH NETWORKS

(*a*) Instead of a junction-network let now a mesh network such as Fig. 19.4*a* be considered containing a set of input meshes with current $\mathbf{i}^1(i^1, i^2)$ and output meshes with current $\mathbf{i}^2(i^3, i^4)$ and also a third set of meshes with current $\mathbf{i}^3(i^5 - i^{12})$ that do not play any part in the analysis. The compound network is shown in Fig. 19.4*b*, containing three meshes.

The three tensor equations, representing the set of ordinary equations $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$ of all mesh networks with three types of meshes, are

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_1 \cdot \mathbf{i}^1 + \mathbf{z}_2 \cdot \mathbf{i}^2 + \mathbf{z}_3 \cdot \mathbf{i}^3 \\ \mathbf{e}_2 &= \mathbf{z}_4 \cdot \mathbf{i}^1 + \mathbf{z}_5 \cdot \mathbf{i}^2 + \mathbf{z}_6 \cdot \mathbf{i}^3 \\ \mathbf{e}_3 &= \mathbf{z}_7 \cdot \mathbf{i}^1 + \mathbf{z}_8 \cdot \mathbf{i}^2 + \mathbf{z}_9 \cdot \mathbf{i}^3 \end{aligned} \qquad 19.11$$

The third set of meshes is eliminated by the reduction formula 10.28 or by the same procedure used with ordinary equations, giving

$$\begin{aligned} \mathbf{e}_1' &= \mathbf{z}_1' \cdot \mathbf{i}^1 + \mathbf{z}_2' \cdot \mathbf{i}^2 \\ \mathbf{e}_2' &= \mathbf{z}_3' \cdot \mathbf{i}^1 + \mathbf{z}_4 \cdot \mathbf{i}^2 \end{aligned} \qquad 19.12$$

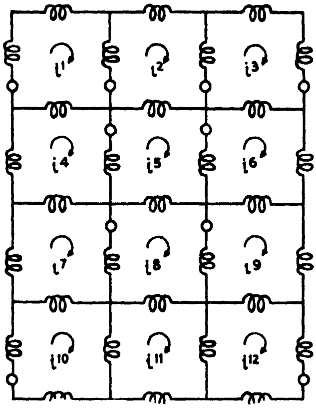
where

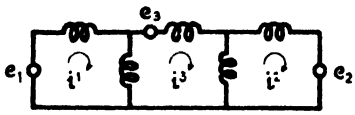
$$\begin{aligned} \mathbf{e}_1' &= \mathbf{e}_1 - \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{e}_3 \\ \mathbf{e}_2' &= \mathbf{e}_2 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{e}_3 \end{aligned} \qquad 19.13$$

$$\begin{aligned} \mathbf{z}_1' &= \mathbf{z}_1 - \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_7 \\ \mathbf{z}_2' &= \mathbf{z}_2 - \mathbf{z}_3 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_8 \end{aligned} \quad \Bigg| \quad \begin{aligned} \mathbf{z}_3' &= \mathbf{z}_4 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_7 \\ \mathbf{z}_4' &= \mathbf{z}_5 - \mathbf{z}_6 \cdot \mathbf{z}_9^{-1} \cdot \mathbf{z}_8 \end{aligned} \qquad 19.14$$

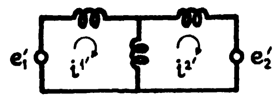
The primed impedances $\mathbf{z}_1'$, $\mathbf{z}_2'$, $\mathbf{z}_3'$, and $\mathbf{z}_4'$ are the self- and mutual impedances of the input and output terminals measured while currents flow in the remaining meshes. The new impressed voltages $\mathbf{e}_1'$ and $\mathbf{e}_2'$ involve the voltage $\mathbf{e}_3$ impressed in the eliminated meshes. When the eliminated meshes contain no impressed voltages, then $\mathbf{e}_1'$ and $\mathbf{e}_2'$ are equal to $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively.

(*b*) Similar steps are followed if the original set contains more than three tensor equations and if more than one equation is eliminated.





(*b*) Compound mesh network

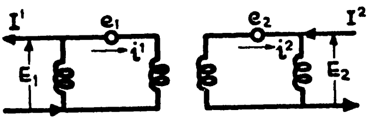


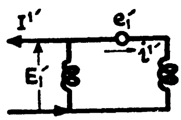
(*a*) Given mesh network (*c*) Equivalent T-network

FIG. 19.4.—Replacement of an Ordinary Mesh Network by a Compound Network

VII. REDUCTION OF COMPOUND ORTHOGONAL NETWORKS

(*a*) Let an orthogonal network with say *two* types of meshes and *two* types of junction-pairs be given as shown in Fig. 19.5*a* Their





(*a*) Before reduction (*b*) After reduction

FIG. 19.5.—Simplification of a Compound Orthogonal Network

equations are given in equation 19.4 as

$$\left.\begin{aligned}
\mathbf{e}_1 &= \mathbf{z}_1\cdot\mathbf{i}^1 + \mathbf{z}_2\cdot\mathbf{i}^2 + \mathbf{z}_3\cdot\mathbf{I}^1 + \mathbf{z}_4\cdot\mathbf{I}^2 \\
\mathbf{e}_2 &= \mathbf{z}_5\cdot\mathbf{i}^1 + \mathbf{z}_6\cdot\mathbf{i}^2 + \mathbf{z}_7\cdot\mathbf{I}^1 + \mathbf{z}_8\cdot\mathbf{I}^2 \\
\mathbf{E}_1 &= \mathbf{z}_9\cdot\mathbf{i}^1 + \mathbf{z}_{10}\cdot\mathbf{i}^2 + \mathbf{z}_{11}\cdot\mathbf{I}^1 + \mathbf{z}_{12}\cdot\mathbf{I}^2 \\
\mathbf{E}_2 &= \mathbf{z}_{13}\cdot\mathbf{i}^1 + \mathbf{z}_{14}\cdot\mathbf{i}^2 + \mathbf{z}_{15}\cdot\mathbf{I}^1 + \mathbf{z}_{16}\cdot\mathbf{I}^2
\end{aligned}\right\} \qquad 19.15$$

(*b*) Let it be assumed that one of the meshes and one of the junction-pairs are superfluous for the analysis. Hence, *let the second and fourth tensor equations be eliminated,* leaving only an equivalent network with *one* mesh and *one* junction-pair as shown in Fig. 19.5*b*.

It should be noted that the eliminated mesh has impressed voltages $\mathbf{e}_2$ and the eliminated junction-pair supplies a load with $-\mathbf{I}^2$ (or has an impressed current $\mathbf{I}^2$.)

Eliminating $\mathbf{I}^2$ from the last equation

$$\mathbf{I}^2 = -\mathbf{z}_{16}^{-1}\cdot(\mathbf{z}_{13}\cdot\mathbf{i}^1 + \mathbf{z}_{14}\cdot\mathbf{i}^2 + \mathbf{z}_{15}\cdot\mathbf{I}^1 - \mathbf{E}_2)$$

Substituting into the remaining three equations

$$\begin{aligned}
\mathbf{e}_1 &= (\mathbf{z}_1 - \mathbf{z}_4\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{z}_{13})\cdot\mathbf{i}^1 + (\mathbf{z}_2 - \mathbf{z}_4\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{z}_{14})\cdot\mathbf{i}^2\\
&\quad + (\mathbf{z}_3 - \mathbf{z}_4\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{z}_{15})\cdot\mathbf{I}^1 + \mathbf{z}_4\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{E}_2\\
\mathbf{e}_2 &= (\mathbf{z}_5 - \mathbf{z}_8\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{z}_{13})\cdot\mathbf{i}^1 + (\mathbf{z}_6 - \mathbf{z}_8\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{z}_{14})\cdot\mathbf{i}^2\\
&\quad + (\mathbf{z}_7 - \mathbf{z}_8\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{z}_{15})\cdot\mathbf{I}^1 + \mathbf{z}_8\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{E}_2\\
\mathbf{E}_1 &= (\mathbf{z}_9 - \mathbf{z}_{12}\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{z}_{13})\cdot\mathbf{i}^1 + (\mathbf{z}_{10} - \mathbf{z}_{12}\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{z}_{14})\cdot\mathbf{i}^2\\
&\quad + (\mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{z}_{15})\cdot\mathbf{I}^1 + \mathbf{z}_{12}\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{E}_2
\end{aligned}$$

It may be written as

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{z}_1'\cdot\mathbf{i}^1 + \mathbf{z}_2'\cdot\mathbf{i}^2 + \mathbf{z}_3'\cdot\mathbf{I}^1 + \mathbf{z}_4\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{E}_2\\
\mathbf{e}_2 &= \mathbf{z}_5'\cdot\mathbf{i}^1 + \mathbf{z}_6'\cdot\mathbf{i}^2 + \mathbf{z}_7'\cdot\mathbf{I}^1 + \mathbf{z}_8\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{E}_2 \qquad 19.16\\
\mathbf{E}_1 &= \mathbf{z}_9'\cdot\mathbf{i}^1 + \mathbf{z}_{10}'\cdot\mathbf{i}^2 + \mathbf{z}_{11}'\cdot\mathbf{I}^1 + \mathbf{z}_{12}\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{E}_2
\end{aligned}$$

(*c*) Eliminating $\mathbf{i}^2$ from the second equation

$$\mathbf{i}^2 = -\mathbf{z}_6'^{-1}\,(\mathbf{z}_5'\cdot\mathbf{i}^1 + \mathbf{z}_7'\cdot\mathbf{I}^1 + \mathbf{z}_8\cdot\mathbf{z}_{16}^{-1}\cdot\mathbf{E}_2 - \mathbf{e}_2)$$

Substituting into the other two equations

$$\begin{aligned}
&\mathbf{e}_1 - \mathbf{z}_2'\cdot\mathbf{z}_6'^{-1}\cdot\mathbf{e}_2 - (\mathbf{z}_4\cdot\mathbf{z}_{16}^{-1} - \mathbf{z}_2'\cdot\mathbf{z}_6'^{-1}\cdot\mathbf{z}_8\cdot\mathbf{z}_{16}^{-1})\cdot\mathbf{E}_2\\
&\quad = (\mathbf{z}_1' - \mathbf{z}_2'\cdot\mathbf{z}_6'^{-1}\cdot\mathbf{z}_5')\cdot\mathbf{i}^1 + (\mathbf{z}_3' - \mathbf{z}_2'\cdot\mathbf{z}_6'^{-1}\cdot\mathbf{z}_7')\cdot\mathbf{I}^1\\
&\mathbf{E}_1 - \mathbf{z}_{10}'\cdot\mathbf{z}_6'^{-1}\cdot\mathbf{e}_2 - (\mathbf{z}_{12}\cdot\mathbf{z}_{16}^{-1} - \mathbf{z}_{10}'\cdot\mathbf{z}_6'^{-1}\cdot\mathbf{z}_8\cdot\mathbf{z}_{16}^{-1})\cdot\mathbf{E}_2\\
&\quad = (\mathbf{z}_9' - \mathbf{z}_{10}'\cdot\mathbf{z}_6'^{-1}\cdot\mathbf{z}_5')\cdot\mathbf{i}^1 + (\mathbf{z}_{11}' - \mathbf{z}_{10}'\cdot\mathbf{z}_6'^{-1}\cdot\mathbf{z}_7')\cdot\mathbf{I}^1
\end{aligned}$$

These two equations may be written as

$$\begin{aligned}
\mathbf{e}_1'' &= \mathbf{z}_1''\cdot\mathbf{i}^1 + \mathbf{z}_2''\cdot\mathbf{I}^2\\
\mathbf{E}_1'' &= \mathbf{z}_3''\cdot\mathbf{i}^1 + \mathbf{z}_4''\cdot\mathbf{I}^2
\end{aligned} \qquad 19.17$$

where the double-primed impedances represent the self- and mutual impedances of the retained mesh and junction-pair, measured in the presence of the eliminated mesh and junction-pair. Similarly $\mathbf{e}_1''$ and

$\mathbf{E}_1''$ are the changed terminal voltages differing from the actual $\mathbf{e}_1$ and $\mathbf{E}_1$ because of the existence of the eliminated $\mathbf{e}_2$ and $\mathbf{E}_2$.

VIII. TERMINOLOGY OF COMPOUND NETWORKS

The concept of compound networks in which each coil represents a whole network having been introduced, it follows that *the whole impedance terminology of ordinary networks can be transferred to compound networks by replacing ordinary numbers with appropriate tensors.*

Some of the impedance terminology hitherto introduced into the study of compound networks are:

1. *Self- and mutual impedances* $\mathbf{z}_{11}$, $\mathbf{z}_{12}$, and admittances $\mathbf{Y}^{11}$, $\mathbf{Y}^{12}$ (also called "driving-point" and "transfer" impedances and admittances in ordinary networks).

2. *Short-circuit self-impedance* $\mathbf{z}_{11}'$ (equal to $\mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{21}$) and open-circuit self-admittance $\mathbf{Y}'^{11}$ (equal to $\mathbf{Y}^{11} - \mathbf{Y}^{12}\cdot\mathbf{Y}^{22-1}\cdot\mathbf{Y}^{21}$).

3. Short-circuit *mutual* (transfer) impedances $\mathbf{z}_{12}'$ and open-circuit mutual admittances $\mathbf{Y}'^{12}$ existing between several sets of terminals, such as given in equations (19.12) and (19.9).

Other concepts are, for instance:

4. *Leakage coefficients* λ = (mutual impedance)/(self-impedance) or (mutual admittance)/(self-admittance), like $\mathbf{z}_{12}\cdot\mathbf{z}_{11}^{-1}$ or $\mathbf{Y}^{12}\cdot\mathbf{Y}^{11-1}$. They are tensors of valence two in which each component is the ratio of two impedances or admittances.

Such leakage coefficients have already been introduced in equations 19.10 and 19.13 where it is shown that *any impressed voltage* $\mathbf{e}$ *or impressed current* $\mathbf{I}$ *of a set of terminals may be transferred to another set of terminals by multiplying them with their respective leakage coefficients.* The product represents the open-circuit voltage appearing across the second set of terminals.

For instance, in equation 19.13 the impressed voltage $\mathbf{e}_3$ may be, transferred to the input terminals as $-\mathbf{z}_3\cdot\mathbf{z}_9^{-1}\cdot\mathbf{e}_3$, where $\mathbf{z}_3\cdot\mathbf{z}_9^{-1}$ is its leakage coefficient. Similarly in equation 19.10 the impressed current $\mathbf{I}_3$ is transferred to the output terminals as $-\mathbf{Y}^6\cdot\mathbf{Y}^{9-1}\cdot\mathbf{I}_3$, where $\mathbf{Y}_6\cdot\mathbf{Y}^{9-1}$ is its leakage coefficient. The product represents the short-circuit current at the output terminals.

5. *Leakage impedances* $\mathbf{z}_1 = \mathbf{z}_{11} - \mathbf{z}_{12} =$ self-impedance minus mutual impedance and leakage admittances $\mathbf{Y}^1 = \mathbf{Y}^{11} - \mathbf{Y}^{12}$. For instance, the tensor equations of a symmetrical two-mesh compound network

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{i}^2 \\ \mathbf{e}_2 &= \mathbf{z}_{12}\cdot\mathbf{i}^1 + \mathbf{z}_{22}\cdot\mathbf{i}^2 \end{aligned} \qquad 19.18$$

may be represented by the T-network of Fig. 19.6*b* having no mutual impedances between the three coils where $\mathbf{z}_1 = \mathbf{z}_{11} - \mathbf{z}_{12}$ and $\mathbf{z}_2 = \mathbf{z}_{22} - \mathbf{z}_{12}$ analogously to the equivalent T-network of a two-winding transformer.

(*a*) Two-mesh network (*b*) Equivalent T-network

FIG. 19.6

IX. THE PHYSICAL SUBDIVISION OF NETWORKS

(*a*) In the previous sections a network was assumed to be composed of compound coils where the coils differed from one another in *functional* behavior. Another way of assuming a network to be composed of compound coils is to subdivide the network *physically* into smaller component networks, each represented by a compound coil, so that the actual network consists of these component networks connected in shunt or series, etc.

An example of such a division of a network is a three-phase transmission network built up by a collection of transmission lines, transformers, generators, loads, etc.

Another example is a *vacuum-tube circuit* consisting of amplifiers, modulators, transformers, static networks, etc., interconnected in various ways. Other examples are *communication networks* consisting of sending and receiving apparatus, transmission lines, filters, corrective networks, interconnected four-terminal networks, etc.; *power drives* consisting of various types of rotating machines interconnected with static networks; also control systems, relay systems, etc. Practically all electrical and electromechanical systems can be divided *physically* into numerous smaller units each of which can be analyzed separately, then reassembled into the original system.

Certain general principles are identical for the analysis of all complex systems; some of them will be illustrated by a more detailed analysis of three-phase transmission systems.

(*b*) *The method of analysis of compound networks parallels the analysis of ordinary networks with the difference that each ordinary quantity is replaced by a tensor, and each tensor by a compound tensor.* That is, if the compound network is, say, a *mesh* network, then:

1. A compound primitive network is set up and its geometric objects $\mathbf{z}$, $\mathbf{e}$, and $\mathbf{i}$ (compound tensors) are established.

2. The current flow in the compound network is established.

3. A compound transformation tensor **C** is set up (showing the manner of interconnection of the compound coils) by equating the old and the new currents flowing in each coil.

4. The geometric objects of the actual network are established as $\mathbf{z}' = \mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$, $\mathbf{e}' = \mathbf{C}_t^* \cdot \mathbf{e}$, $\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}'$. The individual coil quantities are found by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$, $\mathbf{e}_c = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$, $\mathbf{y}_c = \mathbf{C} \cdot \mathbf{y}' \cdot \mathbf{C}_t^*$, and so on.

Of course a compound network may be analyzed as a mesh, as a junction, or as an orthogonal network, according to the problem.

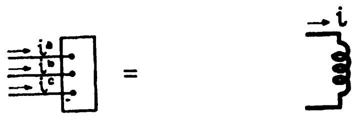
(*c*) It will be found that the second step, namely the establishment of the flow of the currents through the compound network, is not such a simple procedure as it is in ordinary networks, since the manner of interconnection of the individual coils of each compound coil also has to be considered.

X. THREE-PHASE APPARATUS AS COMPOUND COILS

(*a*) In three-phase compound networks it will be assumed for the sake of simplicity that *the impedance tensor* **z** *of each compound coil before interconnection is expressed along the three individual coils* (along the individual primitive network), that is, each **z** has three rows and columns or its multiple. (Of course, in general compound networks the individual coils may be expressed along any arbitrary axes.) An *individual* transformation tensor is also set up for each apparatus to change its primitive network to the actual apparatus.

(*b*) The impedance tensor of *unbalanced* three-phase generators or transmission lines or loads may be expressed as

$$\mathbf{z} = \begin{array}{c|ccc} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \hline \mathbf{a} & Z_1 & Z_2 & Z_3 \\ \mathbf{b} & Z_4 & Z_5 & Z_7 \\ \mathbf{c} & Z_7 & Z_8 & Z_9 \end{array} \qquad 19.19$$



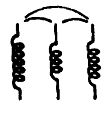
(*a*) Three-phase apparatus (*b*) Compound coil

FIG. 19.7

This tensor in the general case is not symmetrical. As a compound network it is represented by a single coil as shown in Fig. 19.7*b*. Some special cases are shown in the first column of Table 19.1.

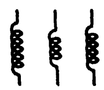
(*c*) The impedance tensors of three-phase *multiwinding transformers* have three, six, nine, twelve, etc., rows and columns, depending on their degrees of freedom. Their components may represent actual or leakage reactances as shown in Section XI, Chapter XI. In terms of leakage reactances the impedance tensor of a *two-winding*, three-phase

1



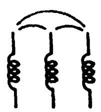
$$z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z_{aa} & Z_{ab} & Z_{ac} \\ b & Z_{ba} & Z_{bb} & Z_{bc} \\ c & Z_{ca} & Z_{cb} & Z_{cc} \\ \hline \end{array} \qquad z = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z_{00} & Z_{01} & Z_{02} \\ 1 & Z_{10} & Z_{11} & Z_{12} \\ 2 & Z_{20} & Z_{21} & Z_{22} \\ \hline \end{array}$$

2



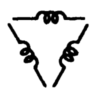
$$z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z_a & & \\ b & & Z_b & \\ c & & & Z_c \\ \hline \end{array} \qquad z = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z_0 & Z_2 & Z_1 \\ 1 & Z_1 & Z_0 & Z_2 \\ 2 & Z_2 & Z_1 & Z_0 \\ \hline \end{array}$$

3



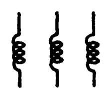
$$z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z & X_1 & X_2 \\ b & X_2 & Z & X_1 \\ c & X_1 & X_2 & Z \\ \hline \end{array} \qquad z = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z_0 & & \\ 1 & & Z_1 & \\ 2 & & & Z_2 \\ \hline \end{array}$$

4



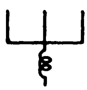
$$z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z & X & X \\ b & X & Z & X \\ c & X & X & Z \\ \hline \end{array} \qquad z = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z+2X & & \\ 1 & & Z-X & \\ 2 & & & Z-X \\ \hline \end{array}$$

5



$$z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z & & \\ b & & Z & \\ c & & & Z \\ \hline \end{array} \qquad z = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & Z & & \\ 1 & & Z & \\ 2 & & & Z \\ \hline \end{array}$$

6



$$z = \begin{array}{c|ccc|} & a & b & c \\ \hline a & Z_g & Z_g & Z_g \\ b & Z_g & Z_g & Z_g \\ c & Z_g & Z_g & Z_g \\ \hline \end{array} \qquad z = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & 3Z_g & & \\ 1 & & & \\ 2 & & & \\ \hline \end{array}$$

Table 19.1.—Individual Impedance Tensors z of Various Types of Three-phase Coils
First column—z along phase axes Second column—z along sequence axes

transformer consisting of *three single-phase transformers* is, if magnetizing currents are neglected,

$$z = \begin{array}{c|cc|} & 1 & 2 \\ \hline 1 & & z_{1-2} \\ 2 & z_{1-2} & \\ \hline \end{array} = \begin{array}{c|ccc|ccc|} & a_1 & b_1 & c_1 & a_2 & b_2 & c_2 \\ \hline a_1 & & & & z'_{1-2} & & \\ b_1 & & & & & z''_{1-2} & \\ c_1 & & & & & & z'''_{1-2} \\ \hline a_2 & z'_{1-2} & & & & & \\ b_2 & & z''_{1-2} & & & & \\ c_2 & & & z'''_{1-2} & & & \\ \hline \end{array} \qquad 19.20$$

That is, *each compound leakage reactance has a diagonal matrix.* If the three single-phase transformers are different, the three components along the diagonal are also different.

The impedance tensor of three-phase *three-winding* transformers in terms of leakage reactances is

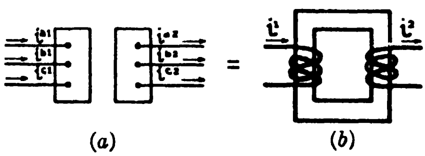
$$\mathbf{z} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{c|c|c} & z_{1-2} & z_{1-3} \\ \hline z_{1-2} & & z_{2-3} \\ \hline z_{1-3} & z_{2-3} & \end{array}\right] \end{array} \qquad 19.21$$

each component being a diagonal tensor of valence two.

When the *three phases are on the same magnetic network*, mutual inductances exist between the phases and the impedance tensor of a two-winding transformer is, if the magnetizing current is to be neglected

$$\mathbf{z} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & z_{1-1} & z_{1-2} \\ 2 & z_{1-2} & z_{2-2} \end{array} = \begin{array}{c|cccccc} & a_1 & b_1 & c_1 & a_2 & b_2 & c_2 \\ \hline a_1 & & z^{11}_{a-b} & z^{11}_{a-c} & z^{12}_{a-a} & z^{12}_{a-b} & z^{12}_{a-c} \\ b_1 & z^{11}_{a-b} & & z^{11}_{b-c} & z^{12}_{b-a} & z^{12}_{b-b} & z^{12}_{b-c} \\ c_1 & z^{11}_{a-c} & z^{11}_{b-c} & & z^{12}_{c-a} & z^{12}_{c-b} & z^{12}_{c-c} \\ a_2 & z^{12}_{a-a} & z^{12}_{b-a} & z^{12}_{c-a} & & z^{22}_{a-b} & z^{22}_{a-c} \\ b_2 & z^{12}_{a-b} & z^{12}_{b-b} & z^{12}_{c-b} & z^{22}_{a-b} & & z^{22}_{b-c} \\ c_2 & z^{12}_{a-c} & z^{12}_{b-c} & z^{12}_{c-c} & z^{22}_{a-c} & z^{22}_{b-c} & \end{array} \qquad 19.22$$

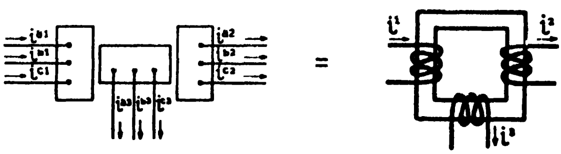
In the primitive compound network a two-winding three-phase transformer is represented by two coils and an n-winding three-phase transformer by n coils, as shown in Fig. 19.8 and 19.9.



(a) (b)

(a) Two-winding three-phase transformer
(b) Compound two-winding transformer

FIG. 19.8



(a) (b)

(a) Three-winding three-phase transformer
(b) Compound three-winding transformer

FIG. 19.9

(*d*) When mutual inductances exist between any two three-phase apparatuses, say between two transmission lines, their impedance tensor in the primitive compound network is

$$\mathbf{z} = \begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & \mathbf{z}_{11} & \mathbf{z}_{12} \\ \hline 2 & \mathbf{z}_{12} & \mathbf{z}_{22} \\ \hline \end{array} = \begin{array}{c|c|c|c|c|c|c|} & \mathbf{a}_1 & \mathbf{b}_1 & \mathbf{c}_1 & \mathbf{a}_2 & \mathbf{b}_2 & \mathbf{c}_2 \\ \hline \mathbf{a}_1 & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 \\ \hline \mathbf{b}_1 & Z_2 & Z_7 & Z_8 & Z_9 & Z_{10} & Z_{11} \\ \hline \mathbf{c}_1 & Z_3 & Z_8 & Z_{12} & Z_{13} & Z_{14} & Z_{15} \\ \hline \mathbf{a}_2 & Z_4 & Z_9 & Z_{13} & Z_{16} & Z_{17} & Z_{18} \\ \hline \mathbf{b}_2 & Z_5 & Z_{10} & Z_{14} & Z_{17} & Z_{19} & Z_{20} \\ \hline \mathbf{c}_2 & Z_6 & Z_{11} & Z_{15} & Z_{18} & Z_{20} & Z_{21} \\ \hline \end{array} \qquad 19.23$$

(*e*) When three-phase apparatus are represented by compound coils, the interconnection of the various three-phase apparatus in series and parallel is represented by analogous interconnection of the compound coils in series and parallel. *The compound network diagram of unbalanced three-phase systems is identical with the customary single-line diagram of balanced three-phase systems.* The slight difference is that here the meshes are closed in order to follow the method of analysis of ordinary mesh networks.

XI. THE COMPOUND TRANSFORMATION TENSOR

(*a*) First let a simple example be considered in which the individual coils of each apparatus are not interconnected among themselves, that is, no *individual* transformation tensors exist (Or rather all individual transformation tensors are unit tensors). Such a network is shown in Fig. 19.10*b*. Its compound network and its primitive mesh network are shown in Figs. 19.10*c* and *d*.

The indices **1** – **5** of the compound network are *compound indices* (Section II*b*, Chapter IX), each index representing three individual indices. For instance, **1** stands for $\mathbf{a}_1$, $\mathbf{b}_1$, and $\mathbf{c}_1$.

The impedance tensor of the primitive network is $\mathbf{z} =$

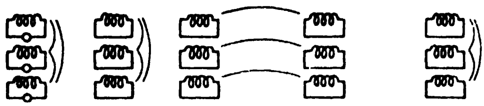
$$\begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \mathbf{z}_{11} & & & & \\ \hline 2 & & \mathbf{z}_{22} & & & \\ \hline 3 & & & \mathbf{z}_{33} & \mathbf{z}_{34} & \\ \hline 4 & & & \mathbf{z}_{34} & \mathbf{z}_{44} & \\ \hline 5 & & & & & \mathbf{z}_{55} \\ \hline \end{array} \qquad 19.24$$

and its impressed voltage vector is $\mathbf{e} =$

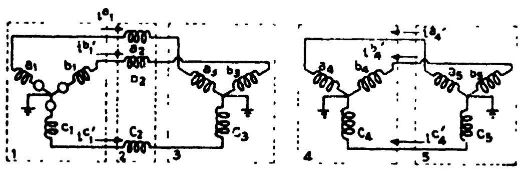
$$\begin{array}{|c|c|c|c|c|} 1 & 2 & 3 & 4 & 5 \\ \hline \mathbf{e}_1 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad 19.25$$

(*b*) *Now let* **C** *be established by considering the compound network of Fig.* 19.10*c* representing the original network of Fig. 19.10*b*.

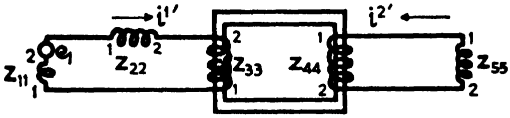
The actual compound network has two meshes, hence two new mesh currents $\mathbf{i}^{1'}$ and $\mathbf{i}^{2'}$ are assumed. The same current flows through all coils in series. Its primitive network is shown in Fig. 19.10*d*.



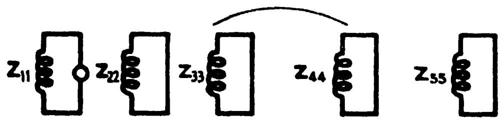
(*a*) The primitive actual network



(*b*) The actual network



(*c*) The compound network



(*d*) The primitive compound network

FIG. 19.10

Equating the old and the new currents flowing in each compound coil

$$
\begin{aligned}
\mathbf{i}^1 &= \mathbf{i}^{1'} \\
\mathbf{i}^2 &= \mathbf{i}^{1'} \\
\mathbf{i}^3 &= -\mathbf{i}^{1'} \\
\mathbf{i}^4 &= \mathbf{i}^{2'} \\
\mathbf{i}^5 &= -\,\mathbf{i}^{2'}
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cc} 1' & 2' \end{array} \\
\left[\begin{array}{c|c}
\mathrm{I} & \\
\mathrm{I} & \\
-\mathrm{I} & \\
 & \mathrm{I} \\
 & -\mathrm{I}
\end{array}\right]
\end{array}
\tag{19.26}
$$

the coefficients of the new currents represent the transformation tensor **C**.

(*c*) In order to check the correctness of the compound **C**, *let the*

ordinary **C** *be established also*, showing the manner of interconnection of the fifteen coils of Fig. 19.10*b* into six meshes as

$$\mathbf{C} = \begin{array}{c|cccccc} & a_1' & b_1' & c_1' & a_2' & b_2' & c_2' \\ \hline a_1 & 1 & & & & & \\ b_1 & & 1 & & & & \\ c_1 & & & 1 & & & \\ a_2 & 1 & & & & & \\ b_2 & & 1 & & & & \\ c_2 & & & 1 & & & \\ a_3 & -1 & & & & & \\ b_3 & & -1 & & & & \\ c_3 & & & -1 & & & \\ a_4 & & & & 1 & & \\ b_4 & & & & & 1 & \\ c_4 & & & & & & 1 \\ a_5 & & & & -1 & & \\ b_5 & & & & & -1 & \\ c_5 & & & & & & -1 \end{array} \qquad \mathbf{C} = \begin{array}{c|cc} & 1' & 2' \\ \hline 1 & \mathbf{I} & \\ 2 & \mathbf{I} & \\ 3 & -\mathbf{I} & \\ 4 & & \mathbf{I} \\ 5 & & -\mathbf{I} \end{array} \qquad 19.27$$

$$\mathbf{I} = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array}$$

Now **C** may be subdivided into a compound tensor in which each component is zero or the unit tensor **I**. *The compound* **C** *shows that the systems* **1**, **2**, *and* **3** *are connected in series, also the systems* **4** *and* **5** *are in series as shown physically in Fig.* 19.10*c*.

(*d*) The impedance tensor and impressed voltage vector of the whole network are then by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$ and $\mathbf{C}_t \cdot \mathbf{e}$ respectively

$$\mathbf{z}' = \begin{array}{c|c|c} & 1' & 2' \\ \hline 1' & \mathbf{z}_{11} + \mathbf{z}_{22} + \mathbf{z}_{33} & \mathbf{z}_{34} \\ \hline 2' & \mathbf{z}_{34} & \mathbf{z}_{44} + \mathbf{z}_{55} \end{array} \qquad 19.28 \qquad \mathbf{e}' = \begin{array}{|c|c|} 1' & 2' \\ \hline \mathbf{e}_1 & 0 \end{array} \qquad 19.29$$

The currents are found by $\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}' = \mathbf{y}' \cdot \mathbf{e}'$. The self- and mutual admittances of the individual systems (or compound coils) are found from $\mathbf{y}'$ by $\mathbf{C} \cdot \mathbf{y}' \cdot \mathbf{C}_t$. The currents in the *individual* systems in terms of the known $\mathbf{i}'$ are found by $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$, and the *voltages* in the individual systems are found by $\mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$.

The second row and column of $\mathbf{z}'$ *may be eliminated by the reduction formula* leaving

$$\mathbf{z}' = \begin{array}{c|c|} & 1' \\ \hline 1' & \mathbf{z}_{11} + \mathbf{z}_{22} + \mathbf{z}_{33} + \mathbf{z}_{34}\cdot(\mathbf{z}_{44} + \mathbf{z}_{55})^{-1}\cdot\mathbf{z}_{34} \\ \hline \end{array} \qquad 19.30$$

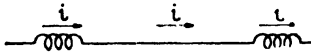
(*e*) It is emphasized that three-phase networks may be analyzed just as any other mesh networks without introducing compound tensors, etc., that is, by using *three times as many variables*. The use of compound tensors and networks simply *speeds up the analysis*, reduces the number of variables to one-third, and clarifies the physical picture, but it does not reduce the number of final slide-rule operations needed to get a numerical answer, except where repetitions occur.

The great advantage of using compound tensors consists of the complete freedom allowed for changing the individual reference frame of one or more of the three-phase apparatus in a routine automatic manner, without disturbing the rest of the set-up. This question of freedom in introducing symmetrical components or other three-phase axes will be treated in greater detail in the next chapter.

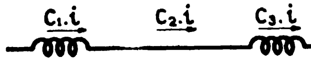
XII. THE FLOW OF COMPOUND CURRENTS THROUGH COMPOUND NETWORKS

(*a*) In this section the way in which the *individual* axes of each apparatus may be changed by an "*individual transformation tensor*" will be investigated.

When an *ordinary coil* z_1 is connected in series with another coil z_2, the same current i flows through z_1 and z_2, also through the lead connecting them. However, with *compound coils* the impedance tensors $\mathbf{z}_1$ and $\mathbf{z}_2$ may be expressed along different set of axes, the current $\mathbf{i}$ along a third set, so that, in $\mathbf{z}_1$, $\mathbf{C}_1\cdot\mathbf{i}$ (instead of $\mathbf{i}$) flows; in $\mathbf{z}_2$, $\mathbf{C}_2\cdot\mathbf{i}$ flows; and in the leads connecting them $\mathbf{C}_3\cdot\mathbf{i}$ flows as shown in Fig. 19.11. A change of individual reference frame occurs as the current enters or leaves certain types of compound coils.



(*a*) The flow of ordinary current



(*b*) The flow of compound current

FIG. 19.11

(*b*) *There are two different ways in which a current may enter a coil:*

1. The current is assumed to originate in the coil; that is, it is assumed as a new variable starting its course in this coil.

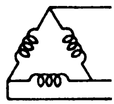
2. The current enters from another coil through the terminals.

***Each time a current appears in a compound coil or leaves a coil, it may be subjected to a transformation of axes with the aid of a* C.**

(*c*) Hence a current **i** acquires an *individual* **C** in three cases:

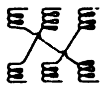
1. When appearing in a coil as a new variable, the coil has different types of axes from those of the assumed current.
2. When arriving at a coil at its terminals the coil again has different types of axes.
3. When it leaves the coil at its terminals the leads have different axes from those of the current leaving the coil.

That is, *a current acquires a* **C** *each time it encounters a different type of reference frame from its own.*



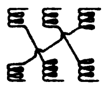
$$C_\Delta = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a' & b' & c' \end{array} \\ \left[\begin{array}{ccc} & 1 & -1 \\ -1 & & 1 \\ 1 & -1 & \end{array}\right] \end{array} \qquad C_\Delta = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0' & 1' & 2' \end{array} \\ \left[\begin{array}{ccc} & & \\ & a^2-a & \\ & & a-a^2 \end{array}\right] \end{array}$$

1



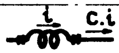
$$C_d = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a' & b' & c' \end{array} \\ \left[\begin{array}{ccc} & 1 & \\ & & 1 \\ 1 & & \end{array}\right] \end{array} \qquad C_d = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0' & 1' & 2' \end{array} \\ \left[\begin{array}{ccc} 1 & & \\ & a^2 & \\ & & a \end{array}\right] \end{array}$$

2

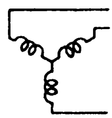


$$C_z = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a' & b' & c' \end{array} \\ \left[\begin{array}{ccc} & & 1 \\ 1 & & \\ & 1 & \end{array}\right] \end{array} \qquad C_z = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0' & 1' & 2' \end{array} \\ \left[\begin{array}{ccc} 1 & & \\ & a & \\ & & a^2 \end{array}\right] \end{array}$$

3

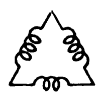


a) Junction tensors



$$C_u = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{cc} a' & b' \end{array} \\ \left[\begin{array}{cc} 1 & \\ & 1 \\ -1 & -1 \end{array}\right] \end{array} \qquad C_u = \begin{array}{c} \\ 0' \\ 1' \\ 2' \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \left[\begin{array}{cc} & \\ 1 & \\ & 1 \end{array}\right] \end{array}$$

4



$$C_{c\Delta} = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} a' \\ \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] \end{array} \qquad C_{c\Delta} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} 0' \\ \left[\begin{array}{c} 1 \\ \\ \end{array}\right] \end{array}$$

5

b) Coil tensors

TABLE 19.2.—Individual Transformation Tensors **C** of Various Types of Three-phase Connections

First column—**C** along phase axes
Second column—**C** along sequence axes

(*d*) When it acquires a **C** as a new variable the **C** will be called a "coil transformation tensor" or "*coil tensor.*" When it acquires a **C** in leaving or entering a coil at its junctions it will be called a "*junction tensor.*" (In general, they may be called "individual transformation tensors" or "three-phase tensors.") A more detailed study of these two types of tensors is undertaken separately in the following sections.

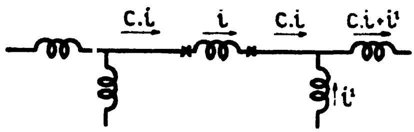
It should be expressly noted that, as the current **i** is assumed to flow through the compound network, Kirchhoff's laws are satisfied at all compound junctions. *The coil and junction tensors do not change the magnitude of the currents; they change only the individual reference frames of each apparatus, and the continuity of the current flow is maintained throughout the whole compound network*, just as in any actual network.

Hence the only difference between the analysis of ordinary networks and compound networks is the manner in which the flow of currents is established in the network. That is, when i is replaced by **i**, its individual reference frame has to be established also with the aid of an individual transformation tensor.

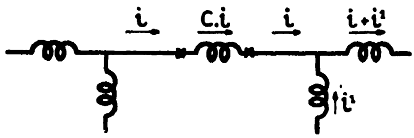
XIII. THE JUNCTION TENSOR

(*a*) If a current **i**′ is subjected to a change of axes **C**·**i**′ as it enters or leaves the terminals of a coil, the two terminal junctions of the coil will be represented by small crosses as shown in Fig. 19.12.

Examples of some frequently occurring junction tensors, shown in Table 19.2, are worked out presently. It is emphasized that there are many other three-phase apparatus that require a coil tensor. *Once these tensors have been established for a particular apparatus they will not have to be calculated again. Whenever the apparatus is used in any three-phase systems, these tensors are picked out of a table and are used over again without any change.*



(*a*) Current originates in the coil

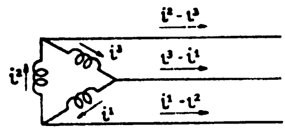


(*b*) Current enters the coil

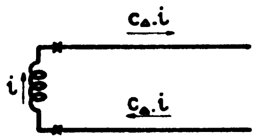
FIG. 19.12.—Compound Junctions

(*b*) The most important example of a compound junction changing the reference axes is the junction connecting a *delta* to the lines.

If the current in the three coils of the delta is $\mathbf{i}'$, then the current in the leads is $\mathbf{i} = \mathbf{C}_\Delta \cdot \mathbf{i}'$, where



(*a*) Currents in delta



(*b*) Compound delta

FIG. 19.13.—Delta Connections

$$\mathbf{i}' = \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \boxed{\begin{array}{c|c|c} i^1 & i^2 & i^3 \end{array}} \end{array} \qquad 19.31$$

$$\mathbf{i} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \boxed{\begin{array}{c|c|c} i^2 - i^3 & i^3 - i^1 & i^1 - i^2 \end{array}} \end{array} \qquad 19.32$$

$$\mathbf{C}_\Delta = \begin{array}{c|ccc} & 1' & 2' & 3' \\ \hline 1 & & 1 & -1 \\ 2 & -1 & & 1 \\ 3 & 1 & -1 & \end{array} \qquad 19.33$$

$\mathbf{C}_\Delta$ *changes the currents in the delta to the currents in the line.*

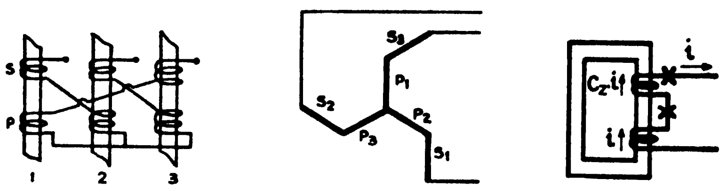
When a compound coil represents a delta-connected apparatus, *one of the new compound variables should be assumed to originate in the delta.*

(*c*) It may be assumed that *all compound junctions introduce a change of reference frame,* but the transformation tensor of an uncrossed junction is the unit tensor **I**.

XIV. JUNCTION TENSORS AS PERMUTATIONS

(*a*) Since a compound junction may introduce a change of reference frame **C**, *this change of reference frame may be of a great variety.* In the previous example **C** represents an actual *interconnection;* now another example is shown where **C** represents a *permutation.*

In three-phase networks usually corresponding phase-windings are



(*a*) Actual network (*b*) Representation (*c*) Componnd network

FIG. 19.14.—Zig-zag Connection

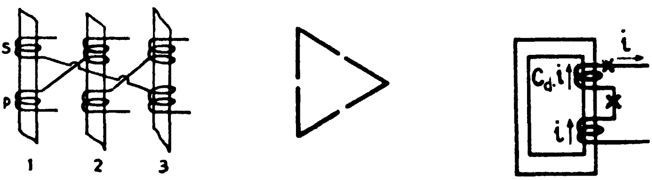
interocnnected. In multiwinding transformers often the phases are innterconected, as shown in Fig. 19.14 for a *zigzag* connection, where

the three coils of the secondary are connected *in series* with the three coils of the primary, but not phase per phase.

If $\mathbf{i}'$ flows through the secondary coil, then through the primary coil $\mathbf{C}_z \cdot \mathbf{i}' = \mathbf{i}$ will flow, where

$$\begin{array}{c} \qquad \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \mathbf{i}' = \begin{array}{|c|c|c|} \hline i^1 & i^2 & i^3 \\ \hline \end{array} \\ \qquad \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \mathbf{i} = \begin{array}{|c|c|c|} \hline i^3 & i^1 & i^2 \\ \hline \end{array} \end{array} \qquad \mathbf{C}_z = \begin{array}{c|c|c|c|} & s_1 & s_2 & s_3 \\ \hline p_1 & & & 1 \\ \hline p_2 & 1 & & \\ \hline p_3 & & 1 & \\ \hline \end{array} \tag{19.34}$$

$\mathbf{C}_z$ shows the manner of interconnection of the phases.



(*a*) Actual connection (*b*) Representation (*c*) Compound coils

FIG. 19.15.—Zig-zag Connection

(*b*) Instead of connecting the *third* phase to the first phase, the *second* phase may be connected to the first as in double-delta shown in Fig. 19.15 where

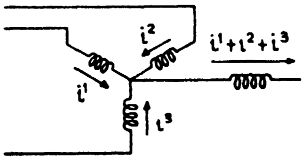
$$\begin{array}{c} \qquad \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \mathbf{i}' = \begin{array}{|c|c|c|} \hline i^1 & i^2 & i^3 \\ \hline \end{array} \\ \qquad \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \mathbf{i} = \begin{array}{|c|c|c|} \hline i^2 & i^3 & i^1 \\ \hline \end{array} \end{array} \qquad \mathbf{C}_d = \begin{array}{c|c|c|c|} & s_1 & s_2 & s_3 \\ \hline p_1 & & 1 & \\ \hline p_2 & & & 1 \\ \hline p_3 & 1 & & \\ \hline \end{array} \tag{19.35}$$

If $\mathbf{i}'$ flows in the *primary* instead of in the secondary then the above two tensors $\mathbf{C}_z$ and $\mathbf{C}_d$ interchange roles.

XV. JUNCTION TENSOR OF A GROUND IMPEDANCE

(*a*) The compound junction may introduce a *singular transformation* reducing the number of axes. (With a delta the number of axes is not changed.) An example is a *ground impedance*, in the absence of fault currents. At the star point three outside currents $\mathbf{i}'$ are changed

to one current i (flowing through the ground impedance) by $\mathbf{C}_g \cdot \mathbf{i}'$ where



(*a*) Ground impedance

(*b*) Compound coils

FIG. 19.16.—Ground

$$\mathbf{i}' = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{|c|c|c|} \hline i^1 & i^2 & i^3 \\ \hline \end{array} \end{array}$$

$$\mathbf{i} = \begin{array}{c} g \\ \begin{array}{|c|} \hline i^1 + i^2 + i^3 \\ \hline \end{array} \end{array} \qquad 19.36$$

$$\mathbf{C}_g = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ g & \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array} \end{array}$$

It should be noted that with a delta the current i′ is assumed to be known *in the coil* of the delta. In case of the ground impedance (in the absence of fault currents) the current i′ is assumed to be known *outside* the ground impedance.

(*b*) It is interesting that *a current* i′ *originating in a delta becomes zero in flowing through a ground impedance* since there its value becomes $\mathbf{C}_g \cdot \mathbf{C}_\Delta \cdot \mathbf{i}'$ ($\mathbf{C}_\Delta$ is added when i′ leaves the delta and $\mathbf{C}_g$ is added when it enters the ground as shown in Fig. 19.17) and

$$\boxed{\mathbf{C}_g \cdot \mathbf{C}_\Delta = 0} \qquad 19.37$$

since

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array} \; \begin{array}{|c|c|c|} \hline & 1 & -1 \\ \hline -1 & & 1 \\ \hline 1 & -1 & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \end{array}$$

(*c*) A ground impedance and a star are always connected *in series* in the compound network as shown in Fig. 19.17. *The two compound*

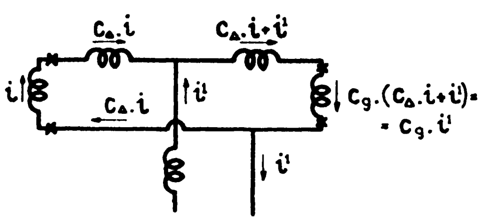


FIG. 19.17.—Delta Current Through Ground

coils in series may be replaced by one compound coil whose impedance tensor is the sum of the two coils in series.

Solving the circuit of the two coils in series (Fig. 19.18) as an

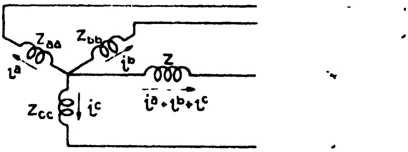


FIG. 19.18.—Grounded Star

ordinary circuit (or as a compound circuit), **z** of the primitive network is

$$\mathbf{z} = \begin{array}{c|cccc|} & a & b & c & g \\ \hline a & Z_{aa} & Z_{ab} & Z_{ac} & \\ b & Z_{ba} & Z_{bb} & Z_{bc} & \\ c & Z_{ca} & Z_{cb} & Z_{cc} & \\ \hline g & & & & Z_g \\ \hline \end{array} = \begin{array}{c|cc|} & m & g \\ \hline m & \mathbf{z}_1 & \\ g & & \mathbf{z}_2 \\ \hline \end{array} \qquad 19.38$$

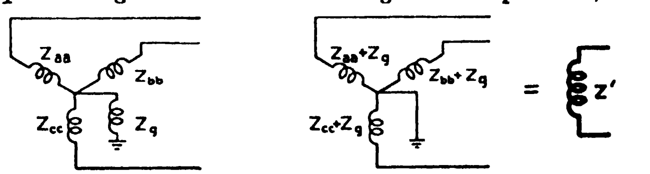
The transformation tensor is

$$\begin{aligned} i^a &= i^{a'} \\ i^b &= \qquad i^{b'} \\ i^c &= \qquad\qquad i^{c'} \\ i^g &= i^{a'} + i^{b'} + i^{c'} \end{aligned} \qquad \mathbf{C} = \begin{array}{c|ccc|} & a' & b' & c' \\ \hline a & 1 & & \\ b & & 1 & \\ c & & & 1 \\ \hline g & 1 & 1 & 1 \\ \hline \end{array} = \begin{array}{c|c|} & m' \\ \hline m & \mathbf{I} \\ g & \mathbf{C}_g \\ \hline \end{array} \qquad 19.39$$

The resultant impedance tensor is by $\mathbf{C}_t \cdot \mathbf{z} \cdot \mathbf{C}$

$$\mathbf{z}' = \mathbf{z}_1 + \mathbf{C}_{gt} \cdot \mathbf{z}_2 \cdot \mathbf{C}_g = \begin{array}{c|ccc|} & a' & b' & c' \\ \hline a' & Z_{aa} + Z_g & Z_{ab} + Z_g & Z_{ac} + Z_g \\ b' & Z_{ba} + Z_g & Z_{bb} + Z_g & Z_{bc} + Z_g \\ c' & Z_{ca} + Z_g & Z_{cb} + Z_g & Z_{cc} + Z_g \\ \hline \end{array} \qquad 19.40$$

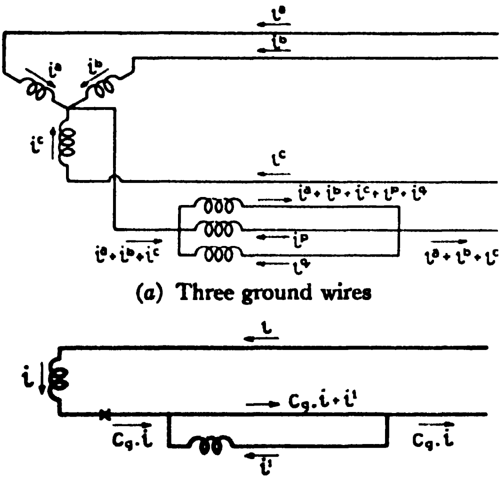
Hence a star impedance and a ground impedance may be replaced by one compound coil whose impedance tensor is **z**′. *This compound coil represents an equivalent grounded star without ground impedance, whose self-*



(*a*) Grounded star (*b*) Equivalent star (*c*) Compound coil

FIG. 19.19

and mutual impedances have been increased by Z_g as the result of the presence of the ground impedance as shown in Fig. 19.19.



(a) Three ground wires

(b) Compound network of ground

FIG. 19.20

(d) When there are *several ground wires* in parallel as shown in Fig. 19.20a then *one* of them is selected as the ground impedance with a junction tensor $\mathbf{C}_g$ and the rest are considered as one compound coil placed *in shunt* with the ground impedance.

A new variable $\mathbf{i}^1$ with components i^p and i^q is assumed in the shunt flowing around in the additional compound mesh (Fig. 19.20b). This shunt current $\mathbf{i}^1$ is not assumed to flow in the rest of the system.

XVI. EXAMPLE FOR THE USE OF JUNCTION TENSORS

(a) As an example of junction tensors consider the network of Fig. 19.21a containing seven three-phase apparatus.

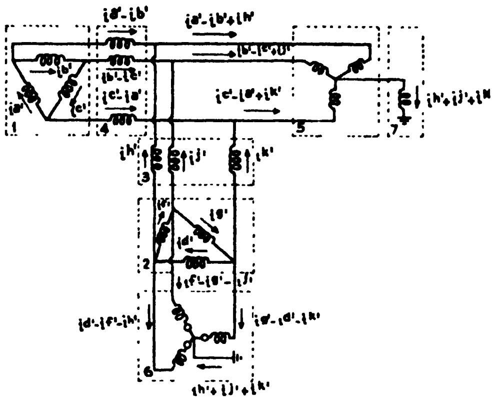
The compound primitive network of Fig. 19.21b contains *seven* coils corresponding to the seven three-phase apparatus that are interconnected. Its geometric objects are

$$\mathbf{e} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \boxed{\begin{array}{c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & e & 0 \end{array}} \end{array} \tag{19.41}$$

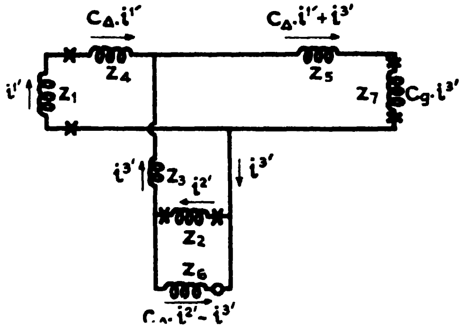
$$\mathbf{z} = \begin{array}{c|c|c|c|c|c|c|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & z_1 & & & & & & \\ \hline 2 & & z_2 & & & & & \\ \hline 3 & & & z_3 & & & & z_{37} \\ \hline 4 & & & & z_4 & & & \\ \hline 5 & & & & & z_5 & & \\ \hline 6 & & & & & & z_6 & \\ \hline 7 & & & z_{37} & & & & z_7 \end{array} \tag{19.42}$$

assuming mutual inductance between the transmission line **3** and the ground wire **7**.

(*b*) There are three closed meshes in the compound network, hence three new currents are assumed. Two of the new currents $\mathbf{i}^{1'}$ and $\mathbf{i}^{2'}$ must be assumed to flow in the delta connected apparatus **1** and **2**, while the third current $\mathbf{i}^{3'}$ may be assumed arbitrarily in any branch, say through **3**. The corresponding nine currents of the original network are shown in Fig. 19.21*a* as $i^{a'}$, $i^{b'}$, $i^{c'}$ in network **1**, $i^{d'}$, $i^{f'}$, $i^{g'}$ in network **2** and $i^{h'}$, $i^{j'}$, $i^{k'}$ in network **3**. *It should be noted that these nine currents are assumed in all the coils of the three apparatus.*



(*a*) Three-phase network



(*b*) Compound network

FIG. 19.21

The currents flowing in *all the coils* of the compound network are also shown in Fig. 19.21*b*. They are all expressed in terms of the three new currents $\mathbf{i}^{1'}$, $\mathbf{i}^{2'}$, and $\mathbf{i}^{3'}$, and also in terms of the junction transformation tensors $\mathbf{C}_\Delta$ and $\mathbf{C}_g$. In the ground impedance z_7 the current component $\mathbf{C}_g \cdot \mathbf{C}_\Delta \cdot \mathbf{i}^{1'}$ is zero.

(*c*) Now the transformation tensor **C** of the compound network is set up by equating the old and the new currents flowing in each compound coil as

$$
\begin{aligned}
\mathbf{i}^1 &= \mathbf{i}^{1'} \\
\mathbf{i}^2 &= \qquad\quad \mathbf{i}^{2'} \\
\mathbf{i}^3 &= \qquad\qquad\qquad \mathbf{i}^{3'} \\
\mathbf{i}^4 &= \mathbf{C}_\Delta \cdot \mathbf{i}^{1'} \\
\mathbf{i}^5 &= \mathbf{C}_\Delta \cdot \mathbf{i}^{1'} \qquad\quad + \mathbf{i}^{3'} \\
\mathbf{i}^6 &= \qquad \mathbf{C}_\Delta \cdot \mathbf{i}^{2'} - \mathbf{i}^{3'} \\
\mathbf{i}^7 &= \qquad\qquad \mathbf{C}_g \cdot \mathbf{i}^{3'}
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccc} 1' & 2' & 3' \end{array} \\
\begin{array}{|c|c|c|}
\hline
\mathbf{I} & & \\ \hline
 & \mathbf{I} & \\ \hline
 & & \mathbf{I} \\ \hline
\mathbf{C}_\Delta & & \\ \hline
\mathbf{C}_\Delta & & \mathbf{I} \\ \hline
 & \mathbf{C}_\Delta & -\mathbf{I} \\ \hline
 & & \mathbf{C}_g \\ \hline
\end{array}
\end{array}
\qquad 19.43
$$

The correctness of this compound transformation tensor **C** *may be checked by setting up the ordinary transformation tensor from Fig.* 19.21*a*, *having* 19 *rows and* 9 *columns.*

Strictly speaking, the various $\mathbf{C}_\Delta$ *and* **I** *occurring in the compound tensor* **C** *are not equivalent to each other, since each of them has different compound indices and also individual indices, consequently they ought to be distinguished from each other.* Since the only operation performed on **C** is multiplication (**C** itself is not transformed) the distinguishing marks may be dispensed with. When, however, each of the $\mathbf{C}_\Delta$-s are transformed to different types of individual axes it is absolutely necessary to distinguish the $\mathbf{C}_\Delta$-s and **I**-s referring to different apparatus.

(*d*) The resultant impedance tensor $\mathbf{z}'$ is, by $\mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$,

$$\mathbf{z}' = \begin{array}{c|c|c|c} & 1' & 2' & 3' \\ \hline 1' & \mathbf{z}_1 + \mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_4 \cdot \mathbf{C}_\Delta + \mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_5 \cdot \mathbf{C}_\Delta & 0 & \mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_5 \\ \hline 2' & 0 & \mathbf{z}_2 + \mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_6 \cdot \mathbf{C}_\Delta & -\mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_6 \\ \hline 3' & \mathbf{z}_5 \cdot \mathbf{C}_\Delta & -\mathbf{z}_6 \cdot \mathbf{C}_\Delta & \mathbf{z}_3 + \mathbf{z}_5 + \mathbf{z}_6 + \mathbf{z}_{37} \cdot \mathbf{C}_g + \mathbf{C}_{gt}^* \cdot \mathbf{z}_{87} + \mathbf{C}_{gt}^* \cdot \mathbf{z}_7 \cdot \mathbf{C}_g \end{array} \qquad 19.44$$

The impressed voltage vector is, by $\mathbf{C}_t^* \cdot \mathbf{e}$,

$$\mathbf{e}' = \begin{array}{c|c|c} 1' & 2' & 3' \\ \hline 0 & \mathbf{C}_{\Delta t}^* \cdot \mathbf{e} & -\mathbf{e} \end{array} \qquad 19.45$$

The currents are found by $\mathbf{i}' = \mathbf{z}'^{-1} \cdot \mathbf{e}'$. The currents in the individual coils are $\mathbf{C} \cdot \mathbf{i}'$, and the induced voltages are $\mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$.

XVII. THE "COIL" TRANSFORMATION TENSOR

(*a*) In the previous example the new variables $\mathbf{i}^{1'}$, $\mathbf{i}^{2'}$, and $\mathbf{i}^{3'}$ assumed in the new network flow in their respective windings in the same way as in the windings of the primitive network. That is, the three components of $\mathbf{i}^{1'}$, namely $i^{a'}$, $i^{b'}$, and $i^{c'}$, flow through the same three coils in the same way as i^a, i^b, and i^c flow in the primitive network.

(*b*) However, often another set of currents may form the components of the new current $\mathbf{i}^{1'}$ that is arbitrarily assumed. For instance, in an *ungrounded star* shown in Fig. 19.22 the new current $\mathbf{i}'$ that is assumed as a variable has only two components $i^{a'}$ and $i^{b'}$ instead of the three $i^{a'}$, $i^{b'}$, $i^{c'}$. *In other words, the new variable* $\mathbf{i}'$ *does not flow in all the windings of the compound coil, only in part of them. Hence in*

the three coils of the compound coil $\mathbf{C}_u \cdot \mathbf{i}' = \mathbf{i}$ ***flows instead of the assumed variable*** $\mathbf{i}'$, ***where***

$$\mathbf{i}' = \begin{array}{c} \mathbf{a}' \quad \mathbf{b}' \\ \boxed{i^{a'} \mid i^{b'}} \end{array}$$

$$\mathbf{i} = \begin{array}{c} \mathbf{a} \qquad \mathbf{b} \qquad\qquad \mathbf{c} \\ \boxed{i^{a'} \mid i^{b'} \mid -i^{a'} - i^{b'}} \end{array} \tag{19.46}$$

$$\mathbf{C}_u = \begin{array}{c|cc} & \mathbf{a}' & \mathbf{b}' \\ \hline \mathbf{a} & 1 & \\ \mathbf{b} & & 1 \\ \mathbf{c} & -1 & -1 \end{array}$$

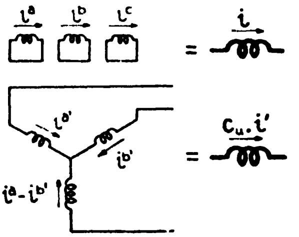


FIG. 19.22.—Ungrounded star

That is, in selecting the new variables one of them often must be the current flowing in the compound coil corresponding to an ungrounded star, just as in case of a delta. However, in a delta the new variable $\mathbf{i}'$ flows through all the coils of the delta (and in the same order) as in the primitive network, but in an ungrounded star the new variable $\mathbf{i}'$ only covers part of the star. Hence, *in all the coils of the star* $\mathbf{C} \cdot \mathbf{i}'$ *flows*. This same current flows in the leads of the star, hence its junctions are not crossed.

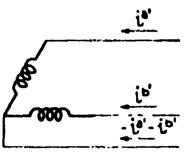


FIG. 19.23.—Open Delta

(*c*) A special case of the ungrounded star is the *open delta* (Fig. 19.23). It may be considered as an ungrounded star whose impedance Z_{cc} is zero.

(*d*) Another example of a coil tensor is a *closed delta* in a transformer bank, Fig. 19.24, where the new variable $\mathbf{i}'$ has only one component $i^{a'}$, hence in the three coils of the compound coil $\mathbf{C}_{c\Delta} \cdot \mathbf{i}' = \mathbf{i}$ flows, where

$$\mathbf{i}' = \begin{array}{c} \mathbf{a}' \\ \boxed{i^{a'}} \end{array} \qquad C_{c\Delta} = \begin{array}{c|c} & \mathbf{a}' \\ \hline \mathbf{a} & 1 \\ \mathbf{b} & 1 \\ \mathbf{c} & 1 \end{array} \tag{19.47}$$

$$\mathbf{i} = \begin{array}{c} \mathbf{a} \qquad \mathbf{b} \qquad \mathbf{c} \\ \boxed{i^{a'} \mid i^{a'} \mid i^{a'}} \end{array}$$

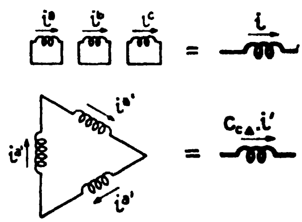


FIG. 19.24.—Closed Delta

Examples of networks with coil tensors are worked out in the next chapter.

If all networks are considered as *complete* networks then all coil-tensors are non-singular.

XVIII. NEGLECTING THE MAGNETIZING CURRENT

(*a*) *Whatever labor-saving device can be introduced in ordinary networks can usually be employed in compound networks also.* For instance, when compound multiwinding transformers are included in the compound network, *their magnetizing currents may be neglected with exactly the same steps as used in ordinary multiwinding transformers in Chapter* XI, namely:

1. The impedance tensor of the compound primitive network $\mathbf{z}$ is set up, using *leakage-impedances* $\mathbf{z}_{1-2}$ for the compound transformer as shown in equation 19.21.

2. The transformation tensor $\mathbf{C}_1$ of the compound network is set up, as in the previous section.

3. The magnetizing currents are neglected by a second transformation tensor $\mathbf{C}_2$.

4. Their product $\mathbf{C}_1 \cdot \mathbf{C}_2$ gives the final transformation tensor $\mathbf{C}$.

5. Then $\mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$ gives the impedance tensor $\mathbf{z}'$ of the new compound network, and so on.

(*b*) *The second transformation tensor* $\mathbf{C}_2$ *neglecting the magnetizing currents is set up as follows:*

1. The equations of constraints are set up as

$$\mathbf{n}_1 \cdot \mathbf{i}^1 + \mathbf{n}_2 \cdot \mathbf{i}^2 + \cdots = 0 \qquad 19.48$$

in terms of the *old* currents where $\mathbf{n}_1$ is a diagonal tensor of valence two, containing the number of turns of each coil of the first network, etc.

2. The *old* currents in the equations of constraint are replaced by the new currents from $\mathbf{i} = \mathbf{C}_1 \cdot \mathbf{i}'$.

3. As many of the currents are eliminated as there are equations of constraints.

4. The relation $\mathbf{i} = \mathbf{C}_2 \cdot \mathbf{i}'$ is set up, giving $\mathbf{C}_2$.

CHAPTER XX

SYMMETRICAL COMPONENTS

I. GENERALITY OF THE COMPOUND NETWORKS

(*a*) When the impedances of the three-phase apparatus are balanced their impedance tensors may be brought to a diagonal form by the sequence tensor **C**, as shown in Chapter XIII. Hence, when *balanced* three-phase apparatus are connected into a network in a *balanced* manner the resultant **z** of the network assumes a simpler form if the individual **z**'s are expressed along the sequence axes.

Now *in setting up the compound network of any three-phase system it does not make any difference whether the individual three-phase apparatus are expressed along the circuit axes* **a**, **b**, **c** *or along the sequence axes* **0**, **1**, **2**. *In either case the compound network and its complete analysis are the same, only the values of the individual* **z**'s *and the individual* **C**'s (*coil tensors, and junction-tensors*) *are different for the two types of axes.* Hence the contents of Chapter XIX are equally valid if the individual apparatus are considered along phase axes or along sequence axes.

(*b*) The impedance tensors of three coils (balanced or unbalanced) expressed along the sequence axes **0**, **1**, **2** have already been given in Chapter XIII. It is only necessary to develop the **z** of other types of three-phase apparatus and the various coil- and junction-tensors along the sequence axes **0**, **1**, **2** in addition to the phase axes **a**, **b**, **c**.

It should be noted that *once the individual impedance tensor* **z** *and the individual transformation tensor* **C** *have been developed for a particular type of apparatus along a particular reference frame, then they will not have to be developed all over when the apparatus is used again in any type of three-phase system.* From then on, the apparatus is treated as a single compound coil, its internal interconnections being entirely forgotten.

Hence with tables available for **z** and **C**, the *analysis* of three-phase systems consists of that of compound mesh (or junction or orthogonal) networks, and the particular three-phase reference frame (say symmetrical components or any other frame) disappears from view. Of course, they reappear in the *routine* manipulations, which, however, can be performed by anyone unacquainted with symmetrical components.

(c) When the network is balanced and sequence axes are used for the individual apparatus, the resultant compound $\mathbf{z}'$ of the whole system has several rows and columns, each component being itself a tensor with three axes along **0**, **1**, and **2** as shown in Chapter XIX. If the resultant matrix $\mathbf{z}'$ is rearranged by putting identical sequence axes side by side (that is, all **0** axes side by side, etc.), it is found that *the tensors containing the same sequence axes are independent of each other and do not contain any mutual terms.* The equivalent networks corresponding to each independent tensor are called "*sequence networks.*" (The establishment of networks equivalent to a tensor of valence two is not undertaken in this volume.)

II. IMPEDANCE TENSORS ALONG SEQUENCE AXES

(a) For *three unequal coils* their $\mathbf{z}$ along the phase axes and sequence axes are given in equations 13.18 and 13.21 respectively.

For *three equal coils with balanced mutual inductances* the equations reduce to simpler form. For reference, some of these simpler forms are shown again in the second column of Table 19.1.

(b) With *balanced three-phase multiwinding transformers* each leakage-reactance tensor $\mathbf{z}_{p-q}$ has a diagonal matrix with identical components, if the magnetizing currents are to be neglected. Hence by $\mathbf{C}^*_{st} \cdot \mathbf{z}_{p-q} \cdot \mathbf{C}_s$

$$\mathbf{z}_{p-q} = \begin{array}{c|ccc} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \hline \mathbf{a} & z_{p-q} & & \\ \mathbf{b} & & z_{p-q} & \\ \mathbf{c} & & & z_{p-q} \end{array} = \begin{array}{c|ccc} & \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \hline \mathbf{0} & z_{p-q} & & \\ \mathbf{1} & & z_{p-q} & \\ \mathbf{2} & & & z_{p-q} \end{array} \qquad 20.1$$

Similarly the turn-ratio tensor is by $\mathbf{C}_s^{-1} \cdot \mathbf{n}_p \cdot \mathbf{C}_s$

$$\mathbf{n}_p = \begin{array}{c|ccc} & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' \\ \hline \mathbf{a} & n_p & & \\ \mathbf{b} & & n_p & \\ \mathbf{c} & & & n_p \end{array} = \begin{array}{c|ccc} & \mathbf{0}' & \mathbf{1}' & \mathbf{2}' \\ \hline \mathbf{0} & n_p & & \\ \mathbf{1} & & n_p\cdot & \\ \mathbf{2} & & & n_p \end{array} \qquad 20.2$$

In the general case $\mathbf{n}_p$ contains complex components along the sequence axes.

(c) For many three-phase apparatus the impedance tensors $\mathbf{z}$ along the sequence axes are *not* calculated by $\mathbf{C}^*_{st} \cdot \mathbf{z} \cdot \mathbf{C}_s$ from those along the phase axes, but are determined separately from test or from the design

constants of the apparatus. For rotating or static apparatus respectively they usually have the following forms

$$\mathbf{z} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & Z_0 & & \\ 1 & & Z_1 & \\ 2 & & & Z_2 \end{array} \qquad \mathbf{z} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & Z_0 & & \\ 1 & & Z_1 & \\ 2 & & & Z_1 \end{array} \qquad 20.3$$

III. JUNCTION AND COIL TENSORS ALONG SEQUENCE AXES

(*a*) Since the *junction tensors* $\mathbf{C}_j$ are non-singular (square) they are transformed to sequence axes by $\mathbf{C}_s^{-1}\cdot\mathbf{C}_j\cdot\mathbf{C}_s$. For the delta and the two types of zigzag connection they are given in Table 19.2*a*. It should be noted that *a zero-sequence current cannot pass out of a delta.*

It is interesting to find that, if the interconnections of three-phase apparatus are balanced, the corresponding transformation tensor expressed along the sequence axes also has a diagonal form. Hence it may be stated that:

1. The sequence tensor $\mathbf{C}_s$ changes the $\mathbf{z}$ of a balanced *impedance* to a diagonal form.

2. It also changes the $\mathbf{C}$ of a balanced *interconnection* to a diagonal form.

(*b*) The *coil tensors* $\mathbf{C}_c$ are singular, and the formula $\mathbf{C}_s^{-1}\cdot\mathbf{C}_c\cdot\mathbf{C}_s$ cannot be used indiscriminately. *One procedure to transform a singular coil tensor to sequence axes is the following*:

1. Express the singular transformation $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ as a set of "equations of constraint" containing phase currents.

2. *In the equations of constraint replace* all phase currents by sequence currents.

3. Replace the equations of constraint (containing now sequence currents) by the transformation $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$.

The coefficients give the desired coil tensor expressed along sequence axes.

(*c*) For instance, let the coil tensor of the *ungrounded star* be (equation 19.46)

$$\begin{aligned} i^a &= i^{a'} \\ i^b &= \qquad\quad i^{b'} \\ i^c &= -i^{a'} - i^{b'} \end{aligned} \qquad \mathbf{C}_u = \begin{array}{c|cc} & \mathbf{a}' & \mathbf{b}' \\ \hline \mathbf{a} & 1 & \\ \mathbf{b} & & 1 \\ \mathbf{c} & -1 & -1 \end{array} \qquad 20.4$$

The transformation $\mathbf{i} = \mathbf{C}_u \cdot \mathbf{i}'$ stands for the following "equation of constraint"

$$i^a + i^b + i^c = 0 \qquad 20.5$$

Replacing the phase currents by sequence currents

$$\frac{1}{\sqrt{3}}\left[(i^0 + i^1 + i^2) + (i^0 + a^2 i^1 + a i^2) + (i^0 + a i^1 + a^2 i^2)\right] = 0$$

or $\sqrt{3}\, i^0 = 0$. Hence the equation of constraint in terms of sequence currents is

$$i^0 = 0 \qquad 20.6$$

This is equivalent to the following transformation $\mathbf{i} = \mathbf{C}_u \cdot \mathbf{i}'$

$$\begin{aligned} i^0 &= 0 \\ i^1 &= i^{1'} \\ i^2 &= \quad i^{2'} \end{aligned} \qquad \mathbf{C}_u = \begin{array}{c|cc} & 1' & 2' \\ \hline 0 & & \\ 1 & 1 & \\ 2 & & 1 \end{array} \qquad 20.7$$

This new $\mathbf{C}_u$ is the sequence equivalent of the phase $\mathbf{C}_u$ of equation 19.46.

It should be noted that *no zero-sequence current can be assumed to flow in an ungrounded star* (or in an open delta).

(*d*) For a *closed delta* the coil tensors along the phase and sequence axes are given in Table 19.2.

It should be expressly noted that the same number of new variables are introduced along the sequence axes as along the phase axes, that is, the coil tensors have the same form when expressed along either axes. In the usual method of analysis practically always more sequence variables are introduced than are actually needed, making the analysis still more involved.

IV. EXAMPLE WITH TWO TYPES OF AXES

(*a*) As an example, let $\mathbf{z}'$ of Fig. 20.1*a* be expressed along both types of axes. *The compound network of Fig.* 20.1*b and its analysis are the same for both types of three-phase axes.*

(*b*) The $\mathbf{z}$ and $\mathbf{e}$ of the primitive network are

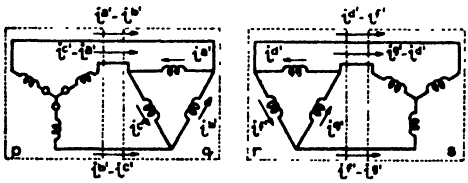
$$\mathbf{z} = \begin{array}{c|cccc} & p & q & r & s \\ \hline p & z_p & & & \\ q & & & z_{q-r} & \\ r & & z_{q-r} & & \\ s & & & & z_s \end{array} \qquad 20.8 \qquad \mathbf{e} = \begin{array}{cccc} p & q & r & s \\ \hline e_p & & & \end{array} \qquad 20.9$$

The transformation tensor is

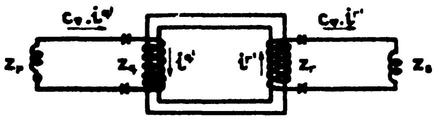
$$
\begin{aligned}
\mathbf{i}^p &= \mathbf{C}_\Delta \cdot \mathbf{i}^{q'} \\
\mathbf{i}^q &= \mathbf{i}^{q'} \\
\mathbf{i}^r &= \mathbf{i}^{r'} \\
\mathbf{i}^s &= \mathbf{C}_\Delta \cdot \mathbf{i}^{r'}
\end{aligned}
\qquad
\mathbf{C}_1 = \begin{array}{c|c|c|}
 & q' & r' \\
\hline
p & \mathbf{C}_\Delta & \\
\hline
q & \mathbf{I} & \\
\hline
r & & \mathbf{I} \\
\hline
s & & \mathbf{C}_\Delta \\
\hline
\end{array} \tag{20.10}
$$

To neglect the magnetizing current, the equation of constraint is set up as

$$\mathbf{n}_q \cdot \mathbf{i}^q + \mathbf{n}_r \cdot \mathbf{i}^r = 0 \tag{20.11}$$



(*a*) Given network



(*b*) Compound network

FIG. 20.1

Replacing the old currents by the new currents from equation 20.10

$$\mathbf{n}_q \cdot \mathbf{i}^{q'} + \mathbf{n}_r \cdot \mathbf{i}^{r'} = 0$$

Eliminating $\mathbf{i}^{q'}$

$$\mathbf{i}^{q'} = -\mathbf{n}_q^{-1} \cdot \mathbf{n}_r \cdot \mathbf{i}^{r'} = -\mathbf{n} \cdot \mathbf{i}^{r'}$$

$$
\mathbf{C}_2 = \begin{array}{c|c|}
 & r' \\
\hline
q' & -\mathbf{n} \\
\hline
r' & \mathbf{I} \\
\hline
\end{array}
\qquad
\mathbf{C}_1 \cdot \mathbf{C}_2 = \mathbf{C} = \begin{array}{c|c|}
 & r' \\
\hline
p & -\mathbf{C}_\Delta \cdot \mathbf{n} \\
\hline
q & -\mathbf{n} \\
\hline
r & \mathbf{I} \\
\hline
s & \mathbf{C}_\Delta \\
\hline
\end{array} \tag{20.12}
$$

The final impedance tensor $\mathbf{z}'$ is by $\mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$

$$
\mathbf{z}' = \begin{array}{c|c|}
 & r' \\
\hline
r' & \mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_p \cdot \mathbf{C}_\Delta \cdot \mathbf{n} - \mathbf{n}_t^* \cdot \mathbf{z}_{q-r} - \mathbf{z}_{q-r} \cdot \mathbf{n} + \mathbf{C}_{\Delta t}^* \cdot \mathbf{Z}_s \cdot \mathbf{C}_\Delta \\
\hline
\end{array} \tag{20.13}
$$

$$
\mathbf{e}' = \begin{array}{|c|}
r' \\
\hline
-\mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t}^* \cdot \mathbf{e}_p \\
\hline
\end{array} \tag{20.14}
$$

If the turn ratio is the same for each phase, $\mathbf{n} = n\mathbf{I}$, and

$$\mathbf{z}' = \begin{array}{c|c|} & \mathbf{r}' \\ \hline \mathbf{r}' & \mathbf{C}^*_{\Delta t}\cdot(n^2\mathbf{z}_p + \mathbf{z}_s)\cdot\mathbf{C}_\Delta - 2n\mathbf{z}_{q-r} \\ \hline \end{array} \qquad 20.15$$

(*c*) Let now the following values be given

$$\mathbf{z}_p = \begin{array}{c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \hline \mathbf{a} & Z_p & & \\ \hline \mathbf{b} & & Z_p & \\ \hline \mathbf{c} & & & Z_p \\ \hline \end{array} = \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 0 & Z_p & & \\ \hline 1 & & Z_p & \\ \hline 2 & & & Z_p \\ \hline \end{array} \qquad 20.16$$

$$\mathbf{z}_s = \begin{array}{c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \hline \mathbf{a} & Z_a & & \\ \hline \mathbf{b} & & Z_b & \\ \hline \mathbf{c} & & & Z_c \\ \hline \end{array} = \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 0 & Z_0 & Z_2 & Z_1 \\ \hline 1 & Z_1 & Z_0 & Z_2 \\ \hline 2 & Z_2 & Z_1 & Z_0 \\ \hline \end{array} \qquad 20.17$$

$$\mathbf{z}_{q-r} = \begin{array}{c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \hline \mathbf{a} & Z_{q-r} & & \\ \hline \mathbf{b} & & Z_{q-r} & \\ \hline \mathbf{c} & & & Z_{q-r} \\ \hline \end{array} = \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 0 & Z_{q-r} & & \\ \hline 1 & & Z_{q-r} & \\ \hline 2 & & & Z_{q-r} \\ \hline \end{array} \qquad 20.18$$

$$\mathbf{C}_\Delta = \begin{array}{c|c|c|c|} & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' \\ \hline \mathbf{a} & & 1 & -1 \\ \hline \mathbf{b} & -1 & & 1 \\ \hline \mathbf{c} & 1 & -1 & \\ \hline \end{array} = \begin{array}{c|c|c|c|} & 0' & 1' & 2' \\ \hline 0 & & & \\ \hline 1 & & a^2 - a & \\ \hline 2 & & & a - a^2 \\ \hline \end{array} \qquad 20.19$$

$$\mathbf{e}_p = \begin{array}{|c|c|c|} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \hline e_p & a^2e_p & ae_p \\ \hline \end{array} = \begin{array}{|c|c|c|} 0 & 1 & 2 \\ \hline & \sqrt{3}e_p & \\ \hline \end{array} \qquad 20.20$$

(*d*) The impedance tensors are then by equation 20.15

$\mathbf{z}'$	$\mathbf{a}'$	$\mathbf{b}'$	$\mathbf{c}'$
$\mathbf{a}'$	$Z_b+Z_c+2n^2Z_p-2nZ_{q-r}$	$-Z_c-Z_p$	$-Z_b-Z_p$
$\mathbf{b}'$	$-Z_c-Z_p$	$Z_a+Z_c+2n^2Z_p-2nZ_{q-r}$	$-Z_n-Z_p$
$\mathbf{c}'$	$-Z_b-Z_p$	$-Z_a-Z_p$	$Z_a+Z_b+2n^2Z_p-2nZ_{q-r}$

20.21

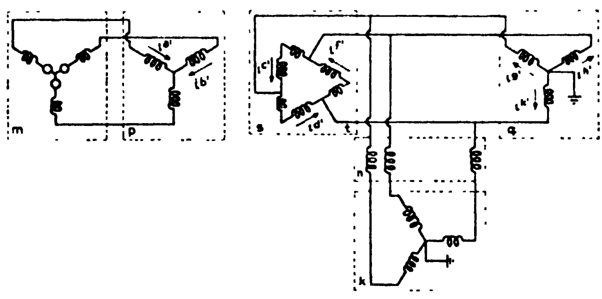
$$\mathbf{z}' = \begin{array}{c|c|c|c|} & 0' & 1' & 2' \\ \hline 0' & -2nZ_{q-r} & & \\ \hline 1' & & 3Z_0 + 3n^2Z_p - 2nZ_{q-r} & -3Z_2 \\ \hline 2' & & -3Z_1 & 3Z_0 + 3n^2Z_p - 2nZ_{q-r} \\ \hline \end{array} \qquad 20.22$$

and the impressed voltage vectors are by equation 20.14

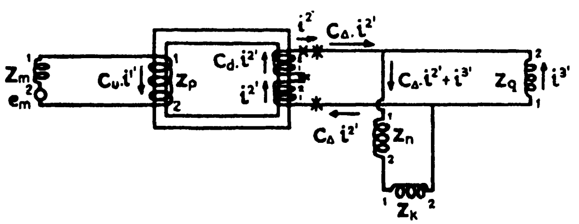
$$\mathbf{e}' = \begin{array}{c|c|} \mathbf{a}' & n(a^2 - a)e_p \\ \hline \mathbf{b}' & n(a - 1)e_p \\ \hline \mathbf{c}' & n(1 - a^2)e_p \\ \end{array} = \begin{array}{c|c|} 0' & \\ \hline 1' & 3(a - a^2)ne_p \\ \hline 2' & \\ \end{array} \qquad 20.23$$

V. COMPOUND MULTIWINDING TRANSFORMERS

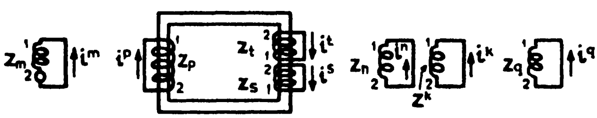
(*a*) Let the network of Fig. 20.2 be analyzed containing an inscribed delta three-winding transformer of Fig. 10.3. The compound network contains a compound three-winding transformer.



(*a*) Given network



(*b*) Compound network



(*c*) The primitive compound network

FIG. 20.2

The primitive compound network shown in Fig. 20.2*b* contains seven coils. Its impedance tensor and impressed voltage vector are

$$\mathbf{z} = \begin{array}{c|ccccccc|} & m & p & s & t & n & k & q \\ \hline m & z_m & & & & & & \\ p & & & z_{p-s} & z_{p-t} & & & \\ s & & z_{p-s} & & z_{s-t} & & & \\ t & & z_{p-t} & z_{s-t} & & & & \\ n & & & & & z_n & & \\ k & & & & & & z_k & \\ q & & & & & & & z_q \\ \hline \end{array} \qquad 20.24$$

$$\mathbf{e} = \begin{array}{|ccccccc|} m & p & s & t & n & k & q \\ \hline e_m & & & & & & \\ \hline \end{array} \qquad 20.25$$

The individual three-phase apparatus $\mathbf{z}_m$, *etc., may be expressed either along the phase axes* **a**, **b**, **c** *or along the sequence axes* **0**, **1**, **2**.

(*b*) In the compound transformer it should be noted that:

1. The secondary and tertiary windings $\mathbf{z}_s$ and $\mathbf{z}_t$ are in series.

2. The compound junctions of the tertiary shows that the phase connections are interchanged by $\mathbf{C}_d$.

3. The secondary and tertiary *together* form a delta, hence the crossed compound junctions of the delta (denoted by three crossed lines on Fig. 20.2*b*) *exist across both of them.*

(*c*) Since there are three compound meshes, *three new variables are assumed.* The first is assumed in one of the ungrounded stars as $\mathbf{C}_u \cdot \mathbf{i}^{1'}$, the second $\mathbf{i}^{2'}$ in one of the two coils of the delta, and the third $\mathbf{i}^{3'}$ is assumed arbitrarily. *The coil and junction tensors may be expressed either along the phase axes or along the sequence axes, depending on the components of the primitive compound network.*

Equating the old and the new currents in each coil

$$\begin{aligned} \mathbf{i}^m &= -\mathbf{C}_u \cdot \mathbf{i}^{1'} \\ \mathbf{i}^p &= \mathbf{C}_u \cdot \mathbf{i}^{1'} \\ \mathbf{i}^s &= + \mathbf{i}^{2'} \\ \mathbf{i}^t &= + \mathbf{C}_d \cdot \mathbf{i}^{2'} \\ \mathbf{i}^n &= + \mathbf{C}_\Delta \cdot \mathbf{i}^{2'} + \mathbf{i}^{3'} \\ \mathbf{i}^k &= + \mathbf{C}_\Delta \cdot \mathbf{i}^{2'} + \mathbf{i}^{3'} \\ \mathbf{i}^q &= \mathbf{i}^{3'} \end{aligned} \qquad \mathbf{C}_1 = \begin{array}{c|ccc|} & 1' & 2' & 3' \\ \hline m & -\mathbf{C}_u & & \\ p & \mathbf{C}_u & & \\ s & & \mathbf{I} & \\ t & & \mathbf{C}_d & \\ n & & \mathbf{C}_\Delta & \mathbf{I} \\ k & & \mathbf{C}_\Delta & \mathbf{I} \\ q & & & \mathbf{I} \\ \hline \end{array} \qquad 20.26$$

(*d*) The transformation tensor $\mathbf{C}_2$ neglecting the magnetizing current of the transformer coils **p**, **s**, and **t** is set up in four steps as follows:

1. The equation of constraint in terms of the *old* currents making the sum of the m.m.f. around the compound transformer of the primitive network of Fig. 20.2*b* equal to zero, is

$$\mathbf{n}_p \cdot \mathbf{i}^p + \mathbf{n}_s \cdot \mathbf{i}^s + \mathbf{n}_t \cdot \mathbf{i}^t = 0 \qquad 20.27$$

where $\mathbf{n}_p$ represents the number of turns of the three primary coils as

$$\mathbf{n}_p = \begin{array}{c|ccc} & p_1 & p_2 & p_3 \\ \hline p_1 & n_{p1} & & \\ p_2 & & n_{p2} & \\ p_3 & & & n_{p3} \end{array} \qquad 20.28$$

Similar matrices apply to $\mathbf{n}_s$ and $\mathbf{n}_t$.

2. Replacing the old currents by the new currents from equation 20.26

$$\mathbf{n}_p \cdot \mathbf{C}_u \cdot \mathbf{i}^{1'} + \mathbf{n}_s \cdot \mathbf{i}^{2'} + \mathbf{n}_t \cdot \mathbf{C}_d \cdot \mathbf{i}^{2'} = 0 \qquad 20.29$$

or

$$\mathbf{n}_p \cdot \mathbf{C}_u \cdot \mathbf{i}^{1'} + (\mathbf{n}_s + \mathbf{n}_t \cdot \mathbf{C}_d) \cdot \mathbf{i}^{2'} = 0$$

expressing the equation of constraint in terms of the *new* currents.

3. Eliminating $\mathbf{i}^{2'}$

$$\mathbf{i}^{2'} = -(\mathbf{n}_s + \mathbf{n}_t \cdot \mathbf{C}_d)^{-1} \cdot \mathbf{n}_p \cdot \mathbf{C}_u \cdot \mathbf{i}^{1'} = -\mathbf{n} \cdot \mathbf{i}^{1'}$$

In eliminating one of the currents care should be taken that its coefficient have an inverse. For instance, here $\mathbf{i}^{1'}$ cannot be eliminated since $\mathbf{C}_u$ (equation 20.4) has no inverse. That is, since $\mathbf{i}^{1'}$ has two components and $\mathbf{i}^{2'}$ has three (as many as there are closed magnetic meshes) $\mathbf{i}^{2'}$ should be eliminated.

4. Hence the equations of transformations $\mathbf{i}' = \mathbf{C}_2 \cdot \mathbf{i}''$ are

$$\begin{aligned} \mathbf{i}^{1'} &= \mathbf{i}^{1''} \\ \mathbf{i}^{2'} &= -\mathbf{n} \cdot \mathbf{i}^{1''} \\ \mathbf{i}^{3'} &= \mathbf{i}^{3''} \end{aligned} \qquad \mathbf{C}_2 = \begin{array}{c|cc} & 1'' & 3'' \\ \hline 1' & \mathbf{I}' & \\ 2' & -\mathbf{n} & \\ 3' & & \mathbf{I} \end{array} \qquad 20.30$$

The product of the two transformation tensors is $\mathbf{C}_1 \cdot \mathbf{C}_2 =$

$$
\mathbf{C} = \begin{array}{c} \\ \mathrm{m} \\ \mathrm{p} \\ \mathrm{s} \\ \mathrm{t} \\ \mathrm{n} \\ \mathrm{k} \\ \mathrm{q} \end{array}
\begin{array}{c} \begin{array}{cc} 1'' & 3'' \end{array} \\
\boxed{\begin{array}{c|c}
-\mathbf{C}_u & \\ \hline
\mathbf{C}_u & \\ \hline
-\mathbf{n} & \\ \hline
-\mathbf{C}_d \cdot \mathbf{n} & \\ \hline
-\mathbf{C}_\Delta \cdot \mathbf{n} & \mathbf{I} \\ \hline
-\mathbf{C}_\Delta \cdot \mathbf{n} & \mathbf{I} \\ \hline
 & \mathbf{I}
\end{array}} \end{array}
\qquad
\mathbf{C}_t^* = \begin{array}{c} \\ 1'' \\ 3'' \end{array}
\begin{array}{c} \begin{array}{ccccccc} \mathrm{m} & \mathrm{p} & \mathrm{s} & \mathrm{t} & \mathrm{n} & \mathrm{k} & \mathrm{q} \end{array} \\
\boxed{\begin{array}{c|c|c|c|c|c|c}
-\mathbf{C}_{ut}^* & \mathbf{C}_{ut}^* & -\mathbf{n}_t^* & -\mathbf{n}_t^* \cdot \mathbf{C}_{dt}^* & -\mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t}^* & -\mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t}^* & \\ \hline
 & & & & \mathbf{I} & \mathbf{I} & \mathbf{I}
\end{array}} \end{array}
\qquad 20.31
$$

(*e*) The final impedance tensor is by $\mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$

$$
\mathbf{z}' = \begin{array}{c} \\ 1'' \\ \\ 3'' \end{array}
\begin{array}{c|c}
1'' & 3'' \\ \hline
\begin{array}{c}
\mathbf{C}_{ut}^* \cdot \mathbf{z}_m \cdot \mathbf{C}_u + \mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_n \cdot \mathbf{C}_\Delta \cdot \mathbf{n} + \mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t} \cdot \mathbf{z}_k \cdot \mathbf{C}_\Delta \cdot \mathbf{n} \\
\mathbf{C}_{ut}^* \cdot \mathbf{z}_{p-t} \cdot \mathbf{C}_d \cdot \mathbf{n} - \mathbf{n}_t^* \cdot \mathbf{C}_{dt}^* \cdot \mathbf{z}_{p-t} \cdot \mathbf{C}_u - \mathbf{n}_t^* \cdot \mathbf{z}_{p-s} \cdot \mathbf{C}_u \\
-\mathbf{C}_{ut}^* \cdot \mathbf{z}_{p-s} \cdot \mathbf{n} + \mathbf{n}_t^* \cdot \mathbf{z}_{s-t} \cdot \mathbf{C}_d \cdot \mathbf{n} + \mathbf{n}_t^* \cdot \mathbf{C}_{dt}^* \cdot \mathbf{z}_{s-t} \cdot \mathbf{n}
\end{array} &
\begin{array}{c}
-\mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_n \\
-\mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_k
\end{array} \\ \hline
-\mathbf{z}_n \cdot \mathbf{C}_\Delta \cdot \mathbf{n} - \mathbf{z}_k \cdot \mathbf{C}_\Delta \cdot \mathbf{n} & \mathbf{z}_n + \mathbf{z}_k + \mathbf{z}_q
\end{array}
\qquad 20.32
$$

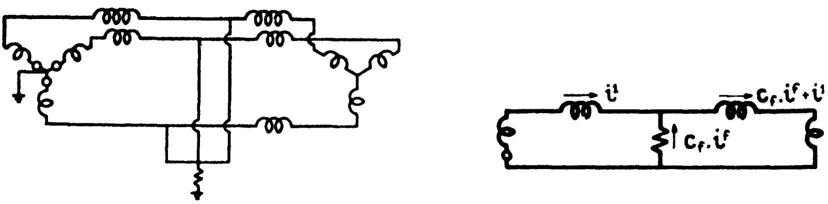
Since $\mathbf{z}_{p-s}$, etc., has a diagonal matrix, its position in a term can be shifted.

The last row and column may be eliminated by the reduction formula $\mathbf{z}_1 - \mathbf{z}_2 \cdot \mathbf{z}_4^{-1} \cdot \mathbf{z}_3$, leaving one component, representing *two* ordinary equations with two unknowns $i^{a\prime}$ and $i^{b\prime}$ ($\mathbf{i}^{1\prime}$).

If the currents $\mathbf{i}'$ are found, the currents in the individual compound coils are $\mathbf{i} = \mathbf{C} \cdot \mathbf{i}'$ and the induced voltages are $\mathbf{e}_c = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'$.

VI. FAULTS

(*a*) When one or more phases of a transmission line are short-circuited or grounded at several points through impedances, ***each fault***



(*a*) Fault on a network (*b*) Compound fault

FIG. 20.3

may be considered as a compound coil connected in shunt across two points of the compound network as shown in Fig. 20.3. In the most general

case it has a coil tensor $\mathbf{C}_f$ and a junction tensor $\mathbf{C}_{fj}$, but in most practical cases it is sufficient to assume a coil tensor $\mathbf{C}_f$. *Each fault coil contributes a new variable* $\mathbf{i}_f$ *since it introduces an extra mesh into the compound network.*

A compound network containing one or more fault coils is analyzed just as any other compound mesh (or junction or orthogonal) network. That is, as many variables $\mathbf{i}$ are assumed as there are meshes, and so on. *The individual apparatus of faults may be expressed either along actual phase axes or along sequence axes.*

(*b*) In order to be able to substitute *several types of faults* across two points, etc., the impedance tensor of the network may be set up *without the presence of faults by replacing the fault coil temporarily by a generator.* The procedure is equivalent to measuring the impedance of the network across the fault (or faults) as shown in Section XV, Chapter X. Then the impedance tensor of the network is connected *in series* with the fault, substituting different types of $\mathbf{z}$ for the various types of faults.

This latter method of attack (of calculating the network and the faults separately) is used especially when symmetrical components can be introduced, namely when the rest of the network is balanced and the faults alone are unbalanced.

VII. THE IMPEDANCE AND COIL TENSORS OF FAULTS

(*a*) Each fault may be considered as a three-phase unbalanced apparatus connected to the system. As such its individual $\mathbf{z}_f$ is that of three isolated coils, some of the coils having zero impedance, as shown along the first five figures of Table 20.1. They are special cases of equations 13.18 and 13.21 respectively.

The ground impedance of a fault forms another compound coil in series with the compound coil of the fault as shown in Fig. 19.16. In order to save labor, the ground impedance Z_g may be assumed to be incorporated in the three isolated coils of the fault as in Fig. 19.19 so that their $\mathbf{z}_f$ also includes Z_g in the manner of equation 19.40 as shown in the last figures of Table 20.1. On the figures each coil of the fault is still isolated; *the effect of the ground coil is only to change the self- and mutual impedances of those fault coils to which it is connected,* so that with the new value of $\mathbf{z}_f$ the presence of the ground impedance may be ignored.

(*b*) The coil tensor of a fault shows the manner in which the fault impedances are connected to the network. The coil tensors of the various types of faults are shown in Table 20.2 for both phase and

sequence axes. In each fault as many new currents are assumed as there are actual currents flowing (mostly *one* actual current exists).

When the fault consists of a delta, then it should be analyzed as any other delta-connected three-phase apparatus.

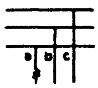
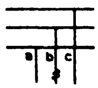
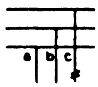
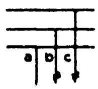
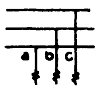
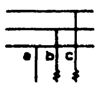
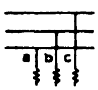
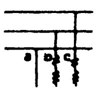
	Fault	z_f along phase axes	z_f along sequence axes
1		$z_f = \begin{array}{c\|ccc} & a & b & c \\ \hline a & Z & & \\ b & & & \\ c & & & \end{array}$	$z_f = \frac{1}{3} \times \begin{array}{c\|ccc} & 0 & 1 & 2 \\ \hline 0 & Z & Z & Z \\ 1 & Z & Z & Z \\ 2 & Z & Z & Z \end{array}$
2		$z_f = \begin{array}{c\|ccc} & a & b & c \\ \hline a & & & \\ b & & Z & \\ c & & & \end{array}$	$z_f = \frac{1}{3} \times \begin{array}{c\|ccc} & 0 & 1 & 2 \\ \hline 0 & Z & a^2Z & aZ \\ 1 & aZ & Z & a^2Z \\ 2 & a^2Z & aZ & Z \end{array}$
3		$z_f = \begin{array}{c\|ccc} & a & b & c \\ \hline a & & & \\ b & & & \\ c & & & Z \end{array}$	$z_f = \frac{1}{3} \times \begin{array}{c\|ccc} & 0 & 1 & 2 \\ \hline 0 & Z & aZ & a^2Z \\ 1 & a^2Z & Z & aZ \\ 2 & aZ & a^2Z & Z \end{array}$
4		$z_f = \begin{array}{c\|ccc} & a & b & c \\ \hline a & & & \\ b & & Z & \\ c & & & Z \end{array}$	$z_f = \frac{1}{3} \times \begin{array}{c\|ccc} & 0 & 1 & 2 \\ \hline 0 & 2Z & -Z & -Z \\ 1 & -Z & 2Z & -Z \\ 2 & -Z & -Z & 2Z \end{array}$
5		$z_f = \begin{array}{c\|ccc} & a & b & c \\ \hline a & Z & & \\ b & & Z & \\ c & & & Z \end{array}$	$z_f = \begin{array}{c\|ccc} & 0 & 1 & 2 \\ \hline 0 & Z & & \\ 1 & & Z & \\ 2 & & & Z \end{array}$
6		$z_f = \begin{array}{c\|ccc} & a & b & c \\ \hline a & & & \\ b & & Z_g & Z_g \\ c & & Z_g & Z_g \end{array}$	$z_f = \frac{1}{3} \times \begin{array}{c\|ccc} & 0 & 1 & 2 \\ \hline 0 & 4Z_g & -2Z_g & -2Z_g \\ 1 & -2Z_g & Z_g & Z_g \\ 2 & -2Z_g & Z_g & Z_g \end{array}$
7		$z_f = \begin{array}{c\|ccc} & a & b & c \\ \hline a & Z_g & Z_g & Z_g \\ b & Z_g & Z_g & Z_g \\ c & Z_g & Z_g & Z_g \end{array}$	$z_f = \begin{array}{c\|ccc} & 0 & 1 & 2 \\ \hline 0 & 3Z_g & & \\ 1 & & & \\ 2 & & & \end{array}$
8		$z_f = \begin{array}{c\|ccc} & a & b & c \\ \hline a & & & \\ b & & Z+Z_g & Z_g \\ c & & Z_g & Z+Z_g \end{array}$	$z_f = \frac{1}{3} \times \begin{array}{c\|ccc} & 0 & 1 & 2 \\ \hline 0 & 2Z+4Z_g & -Z-2Z_g & -Z-2Z_g \\ 1 & -Z-2Z_g & 2Z+Z_g & -Z+Z_g \\ 2 & -Z-2Z_g & -Z+Z_g & 2Z+Z_g \end{array}$

TABLE 20.1.—Individual Impedance Tensors z_f of Various Types of Faults

First column—z_f along phase axes

Second column—z_f along sequence axes

VIII. EXAMPLE OF A SINGLE FAULT

(*a*) On the network of Fig. 20.4*a* let a double line to ground fault occur on phases *b* and *c*. On the compound network the fault is represented by a shunt coil z_f across the line.

Because of the three meshes *three variables* are assumed. In the fault flows $\mathbf{C}_f \cdot \mathbf{i}^{f'}$ (where $\mathbf{i}^{f'}$ has two components $i^{j'}$ and $i^{k'}$), in the delta flows $\mathbf{i}^{b'}$ (with three components), and in the ungrounded star $\mathbf{C}_u \cdot \mathbf{i}^{a'}$ (where $\mathbf{i}^{a'}$ has two components $i^{a'}$ and $i^{b'}$).

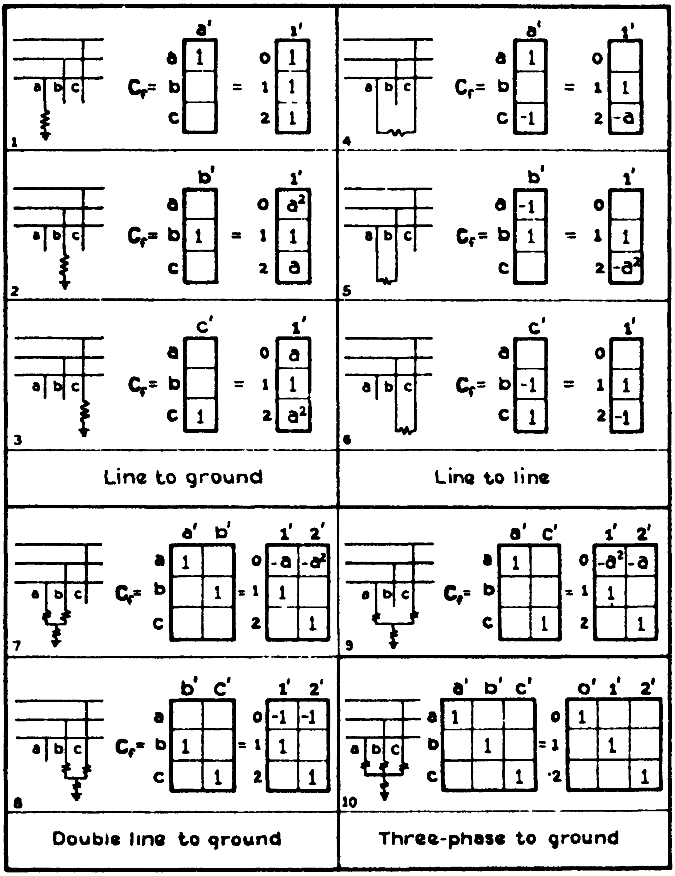


TABLE 20.2.—Individual Transformation Tensors $\mathbf{C}_f$ of Various Types of Faults

First column—$\mathbf{C}_f$ along phase axes
Second column—$\mathbf{C}_f$ along sequence axes

Once the compound network has been established and the flow of currents has been mapped out, *the rest of the analysis follows closely that of any ordinary mesh network.*

(*b*) The impedance tensor and impressed voltage vector of the primitive compound network with five coils are

$\mathbf{z} =$

	p	q	r	f	s
p	$\mathbf{z}_p$				
q			$\mathbf{z}_{q-r}$		
r		$\mathbf{z}_{q-r}$			
f				$\mathbf{z}_f$	
s					$\mathbf{z}_s$

20.33

$\mathbf{e} =$

	p	q	r	f	s
	$\mathbf{e}_p$				

20.34

where the value of $\mathbf{z}_f$, from Table 20.2, is

$\mathbf{z}_f =$

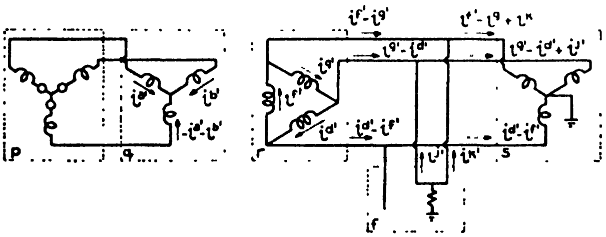
	a	b	c
a			
b		Z_f	Z_f
c		Z_f	Z_f

$= \frac{1}{3}$

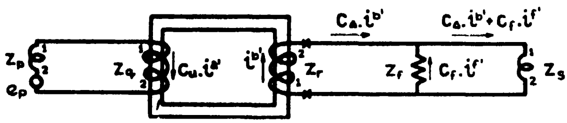
	0	1	2
0	$4Z_f$	$-2Z_f$	$-2Z_f$
1	$-2Z_f$	Z_f	Z_f
2	$-2Z_f$	Z_f	Z_f

20.35

The three-phase impedance tensors $\mathbf{z}_p$ and $\mathbf{z}_s$ have three rows and columns, containing the self- and mutual inductances of the three coils of each three-phase apparatus *without interconnections.*



(*a*) Single fault on a network



(*b*) Compound fault

FIG. 20.4

(*c*) Setting up a relation between the old and the new currents flowing in each coil of Fig. 20.4*b*

$$\begin{aligned}
\mathbf{i}^p &= -\mathbf{C}_u\cdot\mathbf{i}^{a'}\\
\mathbf{i}^q &= \mathbf{C}_u\cdot\mathbf{i}^{a'}\\
\mathbf{i}^r &= \mathbf{i}^{b'}\\
\mathbf{i}^f &= \mathbf{C}_f\cdot\mathbf{i}^{f'}\\
\mathbf{i}^s &= \mathbf{C}_\Delta\cdot\mathbf{i}^{b'} + \mathbf{C}_f\cdot\mathbf{i}^{f'}
\end{aligned}$$

$\mathbf{C}_1 =$

	a′	b′	f′
p	$-\mathbf{C}_u$		
q	$\mathbf{C}_u$		
r		$\mathbf{I}$	
f			$\mathbf{C}_f$
s		$\mathbf{C}_\Delta$	$\mathbf{C}_f$

20.36

where the three-phase transformation tensors $\mathbf{C}_u$, $\mathbf{C}_f$, and $\mathbf{C}_\Delta$ are from Tables 19.2 and 20.1 respectively

$$\mathbf{C}_u = \begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{array}\begin{array}{c} \begin{array}{cc} \mathbf{a}' & \mathbf{b}' \end{array} \\ \left[\begin{array}{c|c} 1 & \\ \hline & 1 \\ \hline -1 & -1 \end{array}\right] \end{array} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array}\begin{array}{c} \begin{array}{cc} \mathbf{1}' & \mathbf{2}' \end{array} \\ \left[\begin{array}{c|c} & \\ \hline 1 & \\ \hline & 1 \end{array}\right] \end{array} \tag{20.37}$$

$$\mathbf{C}_f = \begin{array}{c} \\ \mathbf{h} \\ \mathbf{j} \\ \mathbf{k} \end{array}\begin{array}{c} \begin{array}{cc} \mathbf{j}' & \mathbf{k}' \end{array} \\ \left[\begin{array}{c|c} & \\ \hline 1 & \\ \hline & 1 \end{array}\right] \end{array} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array}\begin{array}{c} \begin{array}{cc} \mathbf{1}' & \mathbf{2}' \end{array} \\ \left[\begin{array}{c|c} -1 & -1 \\ \hline 1 & \\ \hline & 1 \end{array}\right] \end{array} \tag{20.38}$$

$$\mathbf{C}_\Delta = \begin{array}{c} \\ \mathbf{d} \\ \mathbf{f} \\ \mathbf{g} \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{d}' & \mathbf{f}' & \mathbf{g}' \end{array} \\ \left[\begin{array}{c|c|c} & 1 & -1 \\ \hline -1 & & 1 \\ \hline 1 & -1 & \end{array}\right] \end{array} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{0}' & \mathbf{1}' & \mathbf{2}' \end{array} \\ \left[\begin{array}{c|c|c} & & \\ \hline & a^2 - a & \\ \hline & & a - a \end{array}\right] \end{array} \tag{20.39}$$

The coefficients of the new currents give the circuit transformation tensor $\mathbf{C}_1$.

(*d*) In order to neglect the magnetizing current of the transformer, the *equation of constraint* of the transformer *before* interconnection is set up as

$$\mathbf{n}_q \cdot \mathbf{i}^q + \mathbf{n}_r \cdot \mathbf{i}^r = 0 \tag{20.40}$$

Substituting from equation 20.36 the equation of constraint *after* the interconnection is

$$\mathbf{n}_q \cdot \mathbf{C}_u \cdot \mathbf{i}^{a'} + \mathbf{n}_r \cdot \mathbf{i}^{b'} = 0 \tag{20.41}$$

Eliminating $\mathbf{i}^{b'}$

$$\mathbf{i}^{b'} = -\,\mathbf{n}_r^{-1} \cdot \mathbf{n}_q \cdot \mathbf{C}_u \cdot \mathbf{i}^{a'} \tag{20.42}$$

Hence the equation of constraint is replaced by the transformation $\mathbf{i}' = \mathbf{C}_2 \cdot \mathbf{i}''$

$$\begin{aligned} \mathbf{i}^{a'} &= \mathbf{i}^{a''} \\ \mathbf{i}^{b'} &= -\,\mathbf{n}_r^{-1} \cdot \mathbf{n}_q \cdot \mathbf{C}_u \cdot \mathbf{i}^{a''} = -\mathbf{n} \cdot \mathbf{i}^{a''} \\ \mathbf{i}^{f'} &= \mathbf{i}^{f''} \end{aligned} \qquad \mathbf{C}_2 = \begin{array}{c} \\ \mathbf{a}' \\ \mathbf{b}' \\ \mathbf{f}' \end{array}\begin{array}{c} \begin{array}{cc} \mathbf{a}'' & \mathbf{f}'' \end{array} \\ \left[\begin{array}{c|c} \mathbf{I} & \\ \hline -\mathbf{n} & \\ \hline & \mathbf{I} \end{array}\right] \end{array} \tag{20.43}$$

(*e*) The resultant transformation tensor is $\mathbf{C}_1 \cdot \mathbf{C}_2$ =

$$\mathbf{C} = \begin{array}{c|c|c|} & \mathbf{a}'' & \mathbf{f}'' \\ \hline \mathbf{p} & -\mathbf{C}_u & \\ \mathbf{q} & \mathbf{C}_u & \\ \mathbf{r} & -\mathbf{n} & \\ \mathbf{f} & & \mathbf{C}_f \\ \mathbf{s} & -\mathbf{C}_\Delta \cdot \mathbf{n} & \mathbf{C}_f \\ \hline \end{array} \qquad \mathbf{C}_t^* = \begin{array}{c|c|c|c|c|c|} & \mathbf{p} & \mathbf{q} & \mathbf{r} & \mathbf{f} & \mathbf{s} \\ \hline \mathbf{a}'' & -\mathbf{C}_{ut}^* & \mathbf{C}_{ut}^* & -\mathbf{n}_t^* & & -\mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t}^* \\ \mathbf{f}'' & & & & \mathbf{C}_{ft}^* & \mathbf{C}_{ft}^* \\ \hline \end{array} \qquad 20.44$$

(*f*) The impedance tensor of the whole system is by $\mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$

$$\mathbf{z}'' = \begin{array}{c|c|c|} & \mathbf{a}'' & \mathbf{f}'' \\ \hline \mathbf{a}'' & \begin{array}{l} \mathbf{C}_{ut}^* \cdot \mathbf{z}_p \cdot \mathbf{C}_u - \mathbf{C}_{ut}^* \cdot \mathbf{z}_{q-r} \cdot \mathbf{n} \\ - \mathbf{n}_t^* \cdot \mathbf{z}_{q-r} \cdot \mathbf{C}_u + \\ + \mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_s \cdot \mathbf{C}_\Delta \cdot \mathbf{n} \end{array} & - \mathbf{n}_t^* \cdot \mathbf{C}_{\Delta t}^* \cdot \mathbf{z}_s \cdot \mathbf{C}_f \\ \hline \mathbf{f}'' & - \mathbf{C}_{ft}^* \cdot \mathbf{z}_s \cdot \mathbf{C}_\Delta \cdot \mathbf{n} & \mathbf{C}_{ft}^* \cdot (\mathbf{z}_f + \mathbf{z}_s) \cdot \mathbf{C}_f \\ \hline \end{array} \qquad 20.45$$

having a 2-matrix with *four* rows and columns. The axis of $\mathbf{a}''$ may be eliminated by the reduction formulas of Chapter X, leaving only the two fault variables.

The currents are found by $\mathbf{i}'' = \mathbf{z}''^{-1} \cdot \mathbf{e}''$, where $\mathbf{e}'' = \mathbf{C}_t^* \cdot \mathbf{e}$

$$\mathbf{e}'' = \begin{array}{|c|c|} \mathbf{a}'' & \mathbf{f}'' \\ \hline \mathbf{C}_{ut}^* \cdot \mathbf{e}_p & 0 \\ \hline \end{array} \qquad 20.46$$

When the currents $\mathbf{i}''$ have been found, *the voltages existing across the individual coils* of each three-phase apparatus are found by

$$\mathbf{e}_c = \mathbf{z} \cdot \mathbf{C} \cdot \mathbf{i}'' = \begin{array}{c|c|} \mathbf{p} & -\mathbf{z}_p \cdot \mathbf{C}_u \cdot \mathbf{i}^{a''} \\ \mathbf{q} & -\mathbf{z}_{q-r} \cdot \mathbf{n} \cdot \mathbf{i}^{a''} \\ \mathbf{r} & \mathbf{z}_{q-r} \cdot \mathbf{C}_u \cdot \mathbf{i}^{a''} \\ \mathbf{f} & \mathbf{z}_f \cdot \mathbf{C}_f \cdot \mathbf{i}_f'' \\ \mathbf{s} & -\mathbf{z}_s \cdot \mathbf{C}_\Delta \cdot \mathbf{n} \cdot \mathbf{i}^{a''} + \mathbf{z}_s \cdot \mathbf{C}_f \cdot \mathbf{i}^{f''} \\ \hline \end{array} \qquad 20.47$$

where $\mathbf{z} \cdot \mathbf{C}$ has already been calculated in equation 20.45. The *currents* in the individual coils are found by $\mathbf{i}_c = \mathbf{C} \cdot \mathbf{i}''$. The self- and mutual admittances of the individual compound coils are found by $\mathbf{C} \cdot \mathbf{y}'' \cdot \mathbf{C}_t^*$.

IX. PRELIMINARY CALCULATIONS FOR SIMULTANEOUS FAULTS

(*a*) Let the network of Fig. 20.5*a* be given on which two faults are assumed to occur at points f_1 and f_2. There are eight closed electric meshes and *two* magnetic meshes in the primitive compound network of Fig. 20.5*b*. Since several types of faults are to be assumed at these points *the faults are replaced temporarily by three-phase generators* $\mathbf{e}_{f1}$ and $\mathbf{e}_{f2}$. The method of attack is the same as that for any other compound network.

After finding $\mathbf{z}'$ of the compound network containing six rows and columns (two of the eight original variables being eliminated by neglecting the magnetizing currents of the two transformers), *four more rows and columns will be eliminated to leave in* $\mathbf{z}'$ *only the rows and columns of the faults* $\mathbf{f}_1$ *and* $\mathbf{f}_2$. *The resultant* $\mathbf{z}'$ *with two rows and columns will be put in series with various types of fault impedances* $\mathbf{z}_f$ *in place of* $\mathbf{e}_f$, *and* $\mathbf{e}_{f2}$.

(*b*) The impedance tensor $\mathbf{z}$ of the primitive network is

$\mathbf{z} =$

	p	q	r	s	t	f₂	u	v	w	k	h	f₁	m
p	z_p												
q			z_{q-r}	z_{q-s}									
r		z_{q-r}		z_{r-s}									
s		z_{q-s}	z_{r-s}										
t					z_t								z_{tm}
f₂													
u								z_{u-v}	z_{u-w}				
v							z_{u-v}		z_{v-w}				
w							z_{u-w}	z_{v-w}					
k										z_k			
h											z_h		
f₁													
m					z_{tm}								z_m

20.48

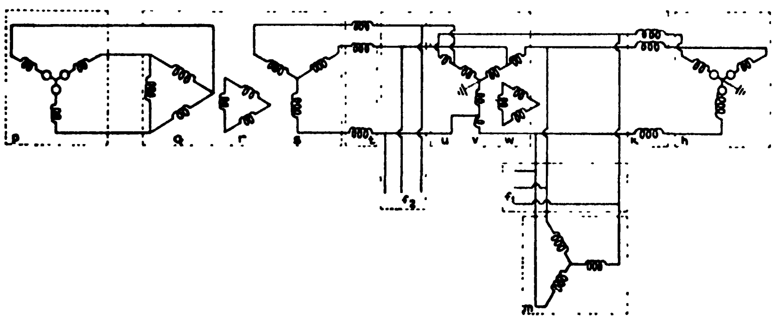
It should be noted that the branches representing the faults $\mathbf{f}_1$ and $\mathbf{f}_2$ have no impedances, still the corresponding rows and columns

are reserved for them. Also a mutual impedance $\mathbf{z}_{tm}$ is assumed to exist between the two transmission lines $\mathbf{t}$ and $\mathbf{m}$.

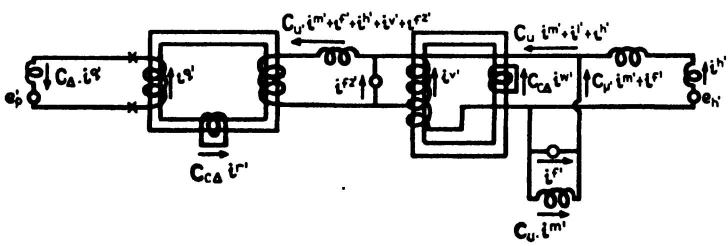
The impressed voltage vector is

$$\mathbf{e} = \begin{array}{c} \begin{array}{ccccccccccccc} p & q & r & s & t & f_2 & u & v & w & k & h & f_1 & m \end{array} \\ \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c} e_p & & & & & e_{f2} & & & & & e_h & e_{f1} & \end{array}\right] \end{array} \qquad 20.49$$

(If a compound network with several simultaneous faults is analyzed as a compound *orthogonal* network, then the additional fault-generators do not introduce additional apparent coils in the primitive network. The fault-generators appear then as components of $\mathbf{E}'$ instead of the components of $\mathbf{e}$. In either case the resultant impedance tensors $\mathbf{z}'$ have the same number of rows and columns.)



(*a*) Network with two faults



(*b*) Compound network

FIG. 20.5.—Multiple faults

(*c*) In order to set up the transformation tensor $\mathbf{C}_1$ *as many new variables are assumed as there are meshes, namely eight.* Two of them, i^{f1} and i^{f2}, are assumed to flow through the fault generators (the coil tensors $\mathbf{C}_f$ will be assumed when the fault impedances will replace the

generators). The others are assumed as usual in deltas and in ungrounded stars.

Finding the currents flowing through each coil as shown in Fig. 20.5*b*, the relation $\mathbf{i} = \mathbf{C}_1 \cdot \mathbf{i}'$ is set up by equating the old and the new currents flowing in each coil as

$$
\begin{aligned}
\mathbf{i}^p &= \mathbf{C}_\Delta \cdot \mathbf{i}^{q'} \\
\mathbf{i}^q &= \mathbf{i}^{q'} \\
\mathbf{i}^r &= \mathbf{C}_{c\Delta} \cdot \mathbf{i}^{r'} \\
\mathbf{i}^s &= \mathbf{i}^{f_2'} + \mathbf{i}^{v'} + \mathbf{i}^{h'} + \mathbf{i}^{f_1'} + \mathbf{C}_u \cdot \mathbf{i}_m{}' \\
\mathbf{i}^t &= \mathbf{i}^{f_2'} + \mathbf{i}^{v'} + \mathbf{i}^{h'} + \mathbf{i}^{f_1'} + \mathbf{C}_u \cdot \mathbf{i}_m{}' \\
\mathbf{i}^{f_2} &= \mathbf{i}^{f_2'} \\
\mathbf{i}^u &= \mathbf{i}^{h'} + \mathbf{i}^{f_1'} + \mathbf{C}_u \cdot \mathbf{i}_m{}' \\
\mathbf{i}^v &= \mathbf{i}^{v'} \\
\mathbf{i}^w &= \mathbf{C}_{'\Delta} \cdot \mathbf{i}^{w'} \\
\mathbf{i}^k &= \mathbf{i}^{h'} \\
\mathbf{i}^h &= \mathbf{i}^{h'} \\
\mathbf{i}^{f_1} &= \mathbf{i}^{f_1'} \\
\mathbf{i}^m &= \mathbf{C}_u \cdot \mathbf{i}^{m'}
\end{aligned}
$$

$$
\mathbf{C}_1 = \begin{array}{c|c|c|c|c|c|c|c|c|}
 & q' & r' & f_2' & v' & w' & h' & f'_1 & m' \\
\hline
p & \mathbf{C}_\Delta & & & & & & & \\
q & \mathbf{I} & & & & & & & \\
r & & \mathbf{C}_{,\Delta} & & & & & & \\
s & & & \mathbf{I} & \mathbf{I} & & \mathbf{I} & \mathbf{I} & \mathbf{C}_u \\
t & & & \mathbf{I} & \mathbf{I} & & \mathbf{I} & \mathbf{I} & \mathbf{C}_u \\
f_2 & & & \mathbf{I} & & & & & \\
u & & & & & & \mathbf{I} & \mathbf{I} & \mathbf{C}_u \\
v & & & & \mathbf{I} & & & & \\
w & & & & & \mathbf{C}_{c\Delta} & & & \\
k & & & & & & \mathbf{I} & & \\
h & & & & & & \mathbf{I} & & \\
f & & & & & & & \mathbf{I} & \\
m & & & & & & & & \mathbf{C}_u \\
\hline
\end{array} \qquad (20.50)
$$

The coefficients of the *new* currents give $\mathbf{C}_1$.

(*d*) To neglect the magnetizing currents, the equations of constraint

of the two transformers in terms of the *old* currents are

$$\begin{aligned} \mathbf{n}_q\cdot\mathbf{i}^q + \mathbf{n}_r\cdot\mathbf{i}^r + \mathbf{n}_s\cdot\mathbf{i}^s &= 0 \\ \mathbf{n}_u\cdot\mathbf{i}^u + \mathbf{n}_v\cdot\mathbf{i}^v + \mathbf{n}_w\cdot\mathbf{n}^w &= 0 \end{aligned} \tag{20.51}$$

Replacing the old currents by the new currents

$$\begin{aligned} \mathbf{n}_q\cdot\mathbf{i}^{q'} + \mathbf{n}_r\cdot\mathbf{C}_{c\Delta}\cdot\mathbf{i}^{r'} + \mathbf{n}_s\cdot(\mathbf{i}^{f_2'} + \mathbf{i}^{v'} + \mathbf{i}^{h'} + \mathbf{i}^{f_1'} + \mathbf{C}_u\cdot\mathbf{i}^{m'}) &= 0 \\ \mathbf{n}_u\cdot(\mathbf{i}^{h'} + \mathbf{i}^{f_1'} + \mathbf{C}_u\cdot\mathbf{i}^{m'}) + \mathbf{n}_v\cdot\mathbf{i}^{v'} + \mathbf{n}_w\cdot\mathbf{C}_{c\Delta}\cdot\mathbf{i}^{w'} &= 0 \end{aligned} \tag{20.52}$$

Eliminating say $\mathbf{i}^{q'}$ in the first and $\mathbf{i}^{h'}$ in the second equation

$$\begin{aligned} \mathbf{i}^{q'} &= -\,\mathbf{n}_q^{-1}\cdot[\mathbf{n}_r\cdot\mathbf{C}_{c\Delta}\cdot\mathbf{i}^{r'} + \mathbf{n}_s\cdot(\mathbf{i}^{f_2'} + \mathbf{i}^{v'} + \mathbf{i}^{h'} + \mathbf{i}^{f_1'} + \mathbf{C}_u\cdot\mathbf{i}^{m'})] \\ \mathbf{i}^{h'} &= -\,\mathbf{n}_u^{-1}\cdot(\mathbf{n}_v\cdot\mathbf{i}^{v'} + \mathbf{n}_w\cdot\mathbf{C}_{c\Delta}\cdot\mathbf{i}^{w'}) - \mathbf{i}^{f'} - \mathbf{C}_u\cdot\mathbf{i}^{m'} \end{aligned}$$

Substituting $\mathbf{i}^{h'}$ into the first equation

$$\mathbf{i}^{q'} = -\,\mathbf{n}_q^{-1}\cdot\mathbf{n}_r\cdot\mathbf{C}_{c\Delta}\cdot\mathbf{i}^{r'} - \mathbf{n}_q^{-1}\cdot\mathbf{n}_s\cdot[\mathbf{i}^{f_2'} + (\mathbf{I} - \mathbf{n}_u^{-1}\cdot\mathbf{n}_v)\cdot\mathbf{i}^{v'} - \mathbf{n}_u^{-1}\cdot\mathbf{n}_w\cdot\mathbf{C}_{c\Delta}\cdot\mathbf{i}^{w'}]$$

Defining

$$\mathbf{n}_1 = -\,\mathbf{n}_q^{-1}\cdot\mathbf{n}_r,\quad \mathbf{n}_2 = -\,\mathbf{n}_q^{-1}\cdot\mathbf{n}_s,\quad \mathbf{n}_3 = -\,(\mathbf{n}_q^{-1}\cdot\mathbf{n}_s - \mathbf{n}_q^{-1}\cdot\mathbf{n}_s\cdot\mathbf{n}_u^{-1}\cdot\mathbf{n}_v),$$

$$\mathbf{n}_4 = \mathbf{n}_q^{-1}\cdot\mathbf{n}_s\cdot\mathbf{n}_u^{-1}\cdot\mathbf{n}_w,\quad \mathbf{n}_6 = -\,\mathbf{n}_u^{-1}\cdot\mathbf{n}_v,\quad \mathbf{n}_7 = -\,\mathbf{n}_u^{-1}\cdot\mathbf{n}_w$$

and substituting for $\mathbf{i}^{q'}$ and $\mathbf{i}^{h'}$

$$\begin{aligned} \mathbf{i}^{q'} &= \mathbf{n}_1\mathbf{C}_{c\Delta}\cdot\mathbf{i}^{r'} + \mathbf{n}_2\cdot\mathbf{i}^{f_2'} + \mathbf{n}_3\cdot\mathbf{i}^{v'} + \mathbf{n}_4\cdot\mathbf{C}_{c\Delta}\cdot\mathbf{i}^{w'} \\ \mathbf{i}^{h'} &= \mathbf{n}_6\cdot\mathbf{i}^{v'} + \mathbf{n}_7\cdot\mathbf{C}_{c\Delta}\cdot\mathbf{i}^{w'} - \mathbf{i}^{f_1'} - \mathbf{C}_u\cdot\mathbf{i}^{m'} \end{aligned}$$

Leaving the other currents unchanged, the two equations of constraint are expressed as a transformation $\mathbf{i}' = \mathbf{C}_2\cdot\mathbf{i}''$, where

$\mathbf{C}_2 =$ (20.53)

	r''	f_2''	v''	w''	f_1''	m''
q'	$\mathbf{n}_1\cdot\mathbf{C}_{c\Delta}$	$\mathbf{n}_2$	$\mathbf{n}_3$	$\mathbf{n}_4\cdot\mathbf{C}_{c\Delta}$		
r'	$\mathbf{I}$					
f_2'		$\mathbf{I}$				
v'			$\mathbf{I}$			
w'				$\mathbf{I}$		
h'			$\mathbf{n}_6$	$\mathbf{n}_7\cdot\mathbf{C}_{c\Delta}$	$-\mathbf{I}$	$-\mathbf{C}_u$
f_1'					$\mathbf{I}$	
m'						$\mathbf{I}$

(*e*) The resultant transformation tensor is by $\mathbf{C}_1 \cdot \mathbf{C}_2$ =

$\mathbf{C} =$

	$\mathbf{r}''$	$\mathbf{f}_2''$	$\mathbf{v}''$	$\mathbf{w}''$	$\mathbf{f}_1''$	$\mathbf{m}'$
$\mathbf{p}$	$\mathbf{C}_\Delta \cdot \mathbf{n}_1 \cdot \mathbf{C}_{c\Delta}$	$\mathbf{C}_\Delta \cdot \mathbf{n}_2$	$\mathbf{C}_\Delta \cdot \mathbf{n}_3$	$\mathbf{C}_\Delta \cdot \mathbf{n}_4 \cdot \mathbf{C}_{c\Delta}$		
$\mathbf{q}$	$\mathbf{n}_1 \cdot \mathbf{C}_{c\Delta}$	$\mathbf{n}_2$	$\mathbf{n}_3$	$\mathbf{n}_4 \cdot \mathbf{C}_{c\Delta}$		
$\mathbf{r}$	$\mathbf{C}_{c\Delta}$					
$\mathbf{s}$		$\mathbf{I}$	$\mathbf{I} + \mathbf{n}_6$	$\mathbf{n}_7 \cdot \mathbf{C}_{c\Delta}$		
$\mathbf{t}$		$\mathbf{I}$	$\mathbf{I} + \mathbf{n}_6$	$\mathbf{n}_7 \cdot \mathbf{C}_{c\Delta}$		
$\mathbf{f}_2$		$\mathbf{I}$				
$\mathbf{u}$			$\mathbf{n}_6$	$\mathbf{n}_7 \cdot \mathbf{C}_{c\Delta}$		
$\mathbf{v}$			$\mathbf{I}$			
$\mathbf{w}$				$\mathbf{C}_{c\Delta}$		
$\mathbf{k}$			$\mathbf{n}_6$	$\mathbf{n}_7 \cdot \mathbf{C}_{c\Delta}$	$-\mathbf{I}$	$-\mathbf{C}_u$
$\mathbf{h}$			$\mathbf{n}_6$	$\mathbf{n}_7 \cdot \mathbf{C}_{c\Delta}$	$-\mathbf{I}$	$-\mathbf{C}_u$
$\mathbf{f}_1$					$\mathbf{I}$	
$\mathbf{m}$						$\mathbf{C}_u$

20.54

(*f*) The impedance tensor $\mathbf{z}'$ of the network is by $\mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$ where $\mathbf{z}$ is given in equation 20.48

$\mathbf{z}' =$

	$\mathbf{r}''$	$\mathbf{f}_2''$	$\mathbf{v}''$	$\mathbf{w}''$	$\mathbf{f}_1''$	$\mathbf{m}''$
$\mathbf{r}''$	z_{rr}	z_{rf_2}	z_{rv}	z_{rw}	z_{rf_1}	z_{rm}
$\mathbf{f}''$	z_{rf_2}	$z_{f_2f_2}$	z_{f_2v}	z_{f_2w}	$z_{f_2f_1}$	z_{f_2m}
$\mathbf{v}''$	z_{rv}	z_{f_2v}	z_{vv}	z_{vw}	z_{vf_1}	z_{vm}
$\mathbf{w}''$	z_{rw}	z_{f_2w}	z_{vw}	z_{ww}	z_{wf_1}	z_{wm}
$\mathbf{f}_1'$	z_{rf_1}	$z_{f_2f_1}$	z_{vf_1}	z_{wf_1}	$z_{f_1f_1}$	z_{f_1m}
$\mathbf{m}''$	z_{rm}	z_{f_2m}	z_{vm}	z_{wm}	z_{f_1m}	z_{mm}

=

	$\mathbf{f}_1''$	$\mathbf{f}_2''$	$\mathbf{r}''$	$\mathbf{v}''$	$\mathbf{w}''$	$\mathbf{m}''$
$\mathbf{f}_1''$	$z_{f_1f_1}$	$z_{f_1f_2}$	z_{f_1r}	z_{f_1v}	z_{f_1w}	z_{f_1m}
$\mathbf{f}_2''$	$z_{f_2f_1}$	$z_{f_2f_2}$	z_{f_2r}	z_{f_2v}	z_{f_2w}	z_{f_2m}
$\mathbf{r}''$	z_{f_1r}	z_{f_2r}	z_{rr}	z_{rv}	z_{rw}	z_{rm}
$\mathbf{v}''$	z_{f_1v}	z_{f_2v}	z_{rv}	z_{vv}	z_{vw}	z_{vm}
$\mathbf{w}''$	z_{f_1w}	z_{f_2w}	z_{rw}	z_{vw}	z_{ww}	z_{wm}
$\mathbf{m}''$	z_{f_1m}	z_{f_2m}	z_{rm}	z_{vm}	z_{wm}	z_{mm}

20.55

The impressed voltage vector is by $\mathbf{C}_t^* \cdot \mathbf{e}$ =

$\mathbf{e}' =$

$\mathbf{r}''$	$\mathbf{f}_2''$	$\mathbf{v}''$	$\mathbf{w}''$	$\mathbf{f}_1''$	$\mathbf{m}''$
$\overset{*}{\mathbf{C}}_{c\Delta t} \cdot \overset{*}{\mathbf{n}}_{1t} \cdot \overset{*}{\mathbf{C}}_{\Delta t} \cdot \mathbf{e}_p$	$\overset{*}{\mathbf{n}}_{2t} \cdot \overset{*}{\mathbf{C}}_{\Delta t} \cdot \mathbf{e}_p + \mathbf{e}_{f2}$	$\overset{*}{\mathbf{n}}_{3t} \cdot \overset{*}{\mathbf{C}}_{\Delta t} \cdot \mathbf{e}_p + \overset{*}{\mathbf{n}}_{6t} \cdot \mathbf{e}_h$	$\overset{*}{\mathbf{C}}_{c\Delta t} \cdot \overset{*}{\mathbf{n}}_{4t} \cdot \overset{*}{\mathbf{C}}_{\Delta t} \cdot \mathbf{e}_p + \overset{*}{\mathbf{C}}_{c\Delta t} \cdot \overset{*}{\mathbf{n}}_{7t} \cdot \mathbf{e}_h$	$+\mathbf{e}_{f1} - \mathbf{e}_h$	$-\overset{*}{\mathbf{C}}_{ut} \cdot \mathbf{e}_h$

20.56

$\mathbf{e}' =$

$\mathbf{r}''$	$\mathbf{f}_2''$	$\mathbf{v}''$	$\mathbf{w}''$	$\mathbf{f}_1''$	$\mathbf{m}''$
$\mathbf{e}_r$	$\mathbf{e}_{f_2}$	$\mathbf{e}_v$	$\mathbf{e}_w$	$\mathbf{e}_{f_1}$	$\mathbf{e}_m$

=

$\mathbf{f}_1''$	$\mathbf{f}_2''$	$\mathbf{r}''$	$\mathbf{v}''$	$\mathbf{w}''$	$\mathbf{m}''$
$\mathbf{e}_{f_1}$	$\mathbf{e}_{f_2}$	$\mathbf{e}_r$	$\mathbf{e}_v$	$\mathbf{e}_w$	$\mathbf{e}_m$

20.57

(*g*) *From* $\mathbf{z}'$ *and* $\mathbf{e}'$ *all rows and columns may be eliminated except* $\mathbf{f}_{1''}$ and $\mathbf{f}_{2''}$ *by the reduction formulas of Chapter X.* First the order of the axes should be changed so that $\mathbf{f}_{1''}$ and $\mathbf{f}_{2''}$ should be the first two axes, as shown, in which case $\mathbf{z}'$ and $\mathbf{e}'$ may be written as *doubly compound tensors*

$$\mathbf{z}' = \begin{array}{c|c|c|} & \mathbf{d} & \mathbf{g} \\ \hline \mathbf{d} & \mathbf{z}_{dd} & \mathbf{z}_{dg} \\ \hline \mathbf{g} & \mathbf{z}_{dg} & \mathbf{z}_{gg} \\ \hline \end{array} \qquad 20.58 \qquad \mathbf{e}' = \begin{array}{|c|c|} \mathbf{d} & \mathbf{g} \\ \hline \mathbf{e}_d & \mathbf{e}_g \\ \hline \end{array} \qquad 20.59$$

Eliminating the second row and column by the reduction formulas

$$\mathbf{z}'' = \mathbf{z}_{dd} - \mathbf{z}_{dg}\cdot\mathbf{z}_{gg}^{-1}\cdot\mathbf{z}_{dg} \qquad 20.60$$

$$\mathbf{e}'' = \mathbf{e}_d - \mathbf{z}_{dg}\cdot\mathbf{z}_{gg}^{-1}\cdot\mathbf{e}_g \qquad 20.61$$

where $\mathbf{z}''$ has altogether six rows and columns representing the self- and mutual impedances of the network as viewed from the three-phase lines at the two faults and $\mathbf{e}''$ *has six components, representing the open-circuit voltages appearing across the two faults,* that is, the equivalent impressed voltages. Hence

$$\mathbf{z}'' = \mathbf{z}_n = \begin{array}{c|c|c|} & \mathbf{f}_1'' & \mathbf{f}_2'' \\ \hline \mathbf{f}_1'' & \mathbf{z}_{11} & \mathbf{z}_{12} \\ \hline \mathbf{f}_2'' & \mathbf{z}_{12} & \mathbf{z}_{22} \\ \hline \end{array} \qquad \mathbf{e}'' = \mathbf{e}_n = \begin{array}{|c|c|} \mathbf{f}_1'' & \mathbf{f}_2'' \\ \hline \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \end{array} \qquad 20.62$$

X. SIMULTANEOUS FAULTS IN BALANCED NETWORKS

(*a*) *Whenever* $\mathbf{z}_n$ *and* $\mathbf{e}_n$ *of a network as viewed from the faults are separately calculated, then the network may be represented by a doubly compound coil* $\mathbf{z}_n$ *and the two (or more) faults as another doubly compound coil* $\mathbf{z}_f$ *in series with it,* as shown in Fig. 20.6. The impedance of a doubly compound coil is a tensor in which each component itself is a tensor.

FIG. 20.6.—Multiple faults as doubly compound coils

(*b*) The impedance tensor $\mathbf{z}'$ of the network and the faults may be found from Fig. 20.6 in the usual manner as follows:

The impedance tensor of the primitive network is

$$\mathbf{z} = \begin{array}{c|c|c|} & \mathbf{n} & \mathbf{f} \\ \hline \mathbf{n} & \mathbf{z}_n & \\ \hline \mathbf{f} & & \mathbf{z}_f \\ \hline \end{array} \qquad 20.63 \qquad \mathbf{e} = \begin{array}{|c|c|} \mathbf{n} & \mathbf{f} \\ \hline \mathbf{e}_n & \\ \hline \end{array} \qquad 20.64$$

The transformation tensor **C** of the network and the faults is found from Fig. 20.6 as

$$\mathbf{i}^n = \mathbf{C}_f \cdot \mathbf{i}^{f'} \qquad \mathbf{C} = \begin{array}{c|c} & \mathbf{f}' \\ \hline \mathbf{n} & \mathbf{C}_f \\ \mathbf{f} & \mathbf{C}_f \end{array} \tag{20.65}$$
$$\mathbf{i}^f = \mathbf{C}_f \cdot \mathbf{i}^{f'}$$

The impedance tensor of the resultant system is by $\mathbf{C}_t^* \cdot \mathbf{z} \cdot \mathbf{C}$

$$\boxed{\mathbf{z}' = \mathbf{C}_{ft}^* \cdot (\mathbf{z}_n + \mathbf{z}_f) \cdot \mathbf{C}_f} \tag{20.66}$$

and the impressed voltage vector is by $\mathbf{C}_t^* \cdot \mathbf{e}$

$$\boxed{\mathbf{e}' = \mathbf{C}_{ft}^* \cdot \mathbf{e}_n} \tag{20.67}$$

where each tensor is a compound tensor expressed along the phase or along the sequence axes.

These two equations are valid for the calculation of any number of aults.

(*c*) In case of *two faults* these equations become

$$\mathbf{z}_n = \begin{array}{|c|c|} \hline \mathbf{z}_{11} & \mathbf{z}_{12} \\ \hline \mathbf{z}_{12} & \mathbf{z}_{22} \\ \hline \end{array} \qquad \mathbf{z}_f = \begin{array}{|c|c|} \hline \mathbf{z}_{f_1} & \\ \hline & \mathbf{z}_{f_2} \\ \hline \end{array} \tag{20.68}$$

$$\mathbf{C}_f = \begin{array}{|c|c|} \hline \mathbf{C}_{f_1} & \\ \hline & \mathbf{C}_{f_2} \\ \hline \end{array} \qquad \mathbf{e}_n = \begin{array}{|c|c|} \hline \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \end{array}$$

Performing the operations indicated in equations 20.66 and 20.67

$$\mathbf{z}' = \begin{array}{c|c|c|} & \mathbf{f}_1 & \mathbf{f}_2 \\ \hline \mathbf{f}_1 & \mathbf{C}_{f_1t}^* \cdot (\mathbf{z}_{11} + \mathbf{z}_{f_1}) \cdot \mathbf{C}_{f_1} & \mathbf{C}_{f_1t}^* \cdot \mathbf{z}_{12} \cdot \mathbf{C}_{f_2} \\ \hline \mathbf{f}_2 & \mathbf{C}_{f_2t}^* \cdot \mathbf{z}_{12} \cdot \mathbf{C}_{f_1} & \mathbf{C}_{f_2t}^* \cdot (\mathbf{z}_{22} + \mathbf{z}_{f_2}) \cdot \mathbf{C}_{f_2} \\ \hline \end{array} \tag{20.69}$$

$$\mathbf{e}' = \begin{array}{|c|c|} \mathbf{f}_1 & \mathbf{f}_2 \\ \hline \mathbf{C}_{f_1t}^* \cdot \mathbf{e}_1 & \mathbf{C}_{f_2t}^* \cdot \mathbf{e}_2 \\ \hline \end{array} \tag{20.70}$$

It should be noted that $\mathbf{z}'$ *is not a symmetrical tensor.*

XI. EXAMPLE OF A SINGLE FAULT IN A BALANCED NETWORK

Let the impedance tensor $\mathbf{z}_n$ and also $\mathbf{e}_n$ of a balanced network as viewed from a fault be

$$\mathbf{z}_n = \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 0 & Z_0 & & \\ \hline 1 & & Z_1 & \\ \hline 2 & & & Z_2 \\ \hline \end{array} \qquad \mathbf{e}_n = \begin{array}{|c|c|c|} 0 & 1 & 2 \\ \hline & e_1 & \\ \hline \end{array} \tag{20.71}$$

Let the fault be, say, a double line to ground fault as shown in

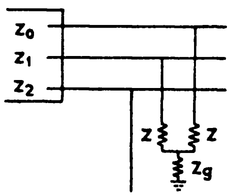


FIG. 20.7.—Double line to ground fault

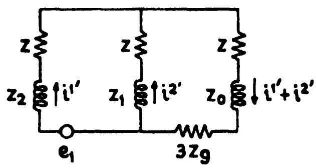


FIG. 20.8.—Equivalent circuit of the network of Fig. 20.7

Fig. 20.7. From Tables 20.2 and 20.1 the impedance tensor and the coil tensor of the fault are

$$\mathbf{z} = \tfrac{1}{3} \times \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 0 & 2Z + 4Z_g & -Z - 2Z_g & -Z - 2Z_g \\ \hline 1 & -Z - 2Z_g & 2Z + Z_g & -Z + Z_g \\ \hline 2 & -Z - 2Z_g & -Z + Z_g & 2Z + Z_g \\ \hline \end{array} \qquad \mathbf{C}_f = \begin{array}{c|c|c|} & 1' & 2' \\ \hline 0 & -1 & -1 \\ \hline 1 & 1 & \\ \hline 2 & & 1 \\ \hline \end{array} \tag{20.72}$$

By equations 20.66 and 20.67 $\mathbf{z}'$ and $\mathbf{e}'$ of the whole system are

$$\mathbf{z}' = \begin{array}{c|c|c|} & 1' & 2' \\ \hline 1' & Z_0 + Z_1 + 2Z + 3Z_g & Z_0 + Z + 3Z_g \\ \hline 2' & Z_0 + Z + 3Z_g & Z_0 + Z_2 + 2Z + 3Z_g \\ \hline \end{array} \qquad \mathbf{e}' = \begin{array}{|c|c|} 1' & 2' \\ \hline e_1 & \\ \hline \end{array} \tag{20.73}$$

The equations $\mathbf{e}' = \mathbf{z}' \cdot \mathbf{i}'$ may be represented by the equivalent circuit of Fig. 20.8 having two meshes.

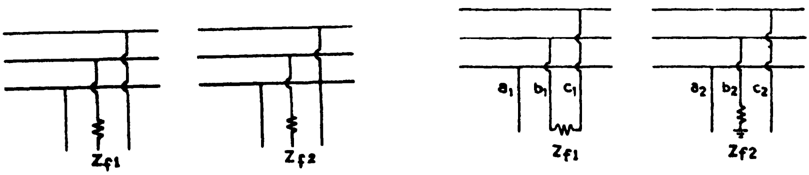
XII. EXAMPLE OF A DOUBLE FAULT IN A BALANCED NETWORK

Let the impedance tensor $\mathbf{z}_n$ and also $\mathbf{e}_n$ of a balanced network as viewed from two faults be

$$
\mathbf{z}_n = \begin{array}{c|c|c|} & \mathbf{f}_1 & \mathbf{f}_2 \\ \hline \mathbf{f}_1 & \mathbf{z}_{11} & \mathbf{z}_{12} \\ \hline \mathbf{f}_2 & \mathbf{z}_{12} & \mathbf{z}_{22} \\ \hline \end{array}
=
\begin{array}{c|c|c|c|c|c|c|}
 & 0_1 & 1_1 & 2_1 & 0_2 & 1_2 & 2_2 \\ \hline
0_1 & Z_0 + X_0 & & & X_0 & & \\ \hline
1_1 & & Z_1 + X_1 & & & X_1 & \\ \hline
2_1 & & & Z_2 + X_2 & & & X_2 \\ \hline
0_2 & X_0 & & & Z'_0 + X_0 & & \\ \hline
1_2 & & X_1 & & & Z'_1 + X_1 & \\ \hline
2_2 & & & X_2 & & & Z'_2 + X_2 \\ \hline
\end{array}
\qquad 20.74
$$

$$
\mathbf{e}_n = \begin{array}{|c|c|} \mathbf{f}_1 & \mathbf{f}_2 \\ \hline e_1 & e_2 \\ \hline \end{array}
=
\begin{array}{|c|c|c|c|c|c|}
0_1 & 1_1 & 2_1 & 0_2 & 1_2 & 2_2 \\ \hline
 & e_1 & & & e_2 & \\ \hline
\end{array}
\qquad 20.75
$$

Let one of the faults be, say, a line to line and the other a line to ground fault as shown in Fig. 20.9. From Tables 20.2 and 20.1 the



(a) Before the fault (b) After the fault

FIG. 20.9.—Example of a double fault

impedance tensor and the coil tensor of the faults are (before faults occur)

$$
\mathbf{z} = \begin{array}{c|c|c|} & \mathbf{f}_1 & \mathbf{f}_2 \\ \hline \mathbf{f}_1 & \mathbf{z}_{f1} & \\ \hline \mathbf{f}_2 & & \mathbf{z}_{f2} \\ \hline \end{array}
=
\begin{array}{c|c|c|c|c|c|c|}
 & 0_1 & 1_1 & 2_1 & 0_2 & 1_2 & 2_2 \\ \hline
0_1 & Z_{f1}/3 & a^2Z_{f1}/3 & aZ_{f1}/3 & & & \\ \hline
1_1 & aZ_{f1}/3 & Z_{f1}/3 & a^2Z_{f1}/3 & & & \\ \hline
2_1 & a^2Z_{f1}/3 & aZ_{f1}/3 & Z_{f1}/3 & & & \\ \hline
0_2 & & & & Z_{f2}/3 & a^2Z_{f2}/3 & aZ_{f2}/3 \\ \hline
1_2 & & & & aZ_{f2}/3 & Z_{f2}/3 & a^2Z_{f2}/3 \\ \hline
2_2 & & & & a^2Z_{f2}/3 & aZ_{f2}/3 & Z_{f2}/3 \\ \hline
\end{array}
\qquad 20.76
$$

The fault impedances are connected to their network by their coil tensors

$$\mathbf{C}_f = \begin{array}{c|cc} & 1_{1'} & 1_{2'} \\ \hline f_1 & C_{f1} & \\ f_2 & & C_{f2} \end{array} = \begin{array}{c|cc} & 1_{1'} & 1_{2'} \\ \hline 0_1 & 0 & \\ 1_1 & 1 & \\ 2_1 & -1 & \\ 0_2 & & a^2 \\ 1_2 & & 1 \\ 2_2 & & a \end{array} \qquad 20.77$$

By equations 20.66 and 20.67 (or by equations 20.69 and 20.70), $\mathbf{z}'$ and $\mathbf{e}'$ of the whole system are

$$\mathbf{z}' = \begin{array}{c|cc} & 1_{1'} & 1_{2'} \\ \hline 1_{1'} & Z_1 + Z_2 + X_1 + X_2 + Z_{f1} & X_1 - aX_2 \\ 1_{2'} & X_1 - a^2X_2 & Z_0' + Z_1' + Z_2' + X_0 + X_1 + X_2 + Z_{f2} \end{array} \qquad 20.78$$

$$\mathbf{e}' = \begin{array}{cc} 1_{1'} & 1_{2'} \\ \hline \mathbf{e}_1 & \mathbf{e}_2 \end{array} \qquad 20.79$$

XIII. MIXED REFERENCE AXES

(*a*) It was shown that in setting up the compound network for three-phase systems it is immaterial whether the individual apparatus are all expressed along the phase axes **a**, **b**, **c** or all along the sequence axes **0**, **1**, **2**. This difference in the reference axes of the component **z**'s is taken care of by the coil and junction tensors, a separate set for each type of **z**.

(*b*) However, it is not necessary that all apparatus of the same system should be expressed along the same type of axes. *The compound network and its method of analysis are valid without any change even if the various three-phase apparatus are expressed along different types of axes. Some may be expressed along the phase axes, others along the sequence axes. Even some of the component* **z** *may have two types of indices, the first set being phase axes, the second being sequence axes, or vice versa.*

This difference in the individual reference axes of the component **z**'s is taken care of again by setting up a different set of *individual*

transformation tensors (coil and junction tensors) having two types of indices, one of the set being the phase, the other the sequence axes. That is, *in addition to the two sets of coil and junction tensors given in Tables* 19.2 *and* 20.2 *two additional sets need to be set up.* In the first set the vertical indices are **0, 1, 2** and the horizontal ones are **a, b, c**; in the second set the position of the indices is interchanged. A similar set of tables may be set up for the **z**'s of various types of apparatus and faults.

(*c*) *Similarly, it is possible to introduce for some or for all of the three-phase apparatus other types of reference frames besides* **a, b, c** *and* **0, 1, 2**. *The solution of the compound network is equally valid for all of them.* However, the individual impedance and transformation tensors have to be expressed along the new individual reference axes.

It is also possible to replace some or all of the three-phase apparatus by *two-phase* or four-phase, etc., apparatus, without changing the compound network.

(*d*) It is emphasized that, since each compound coil of the compound network may be expressed individually along several types of reference frames, therefore *each component* $\mathbf{z}_1$, $\mathbf{z}_2$, etc., *of the compound impedance tensor* $\mathbf{z}$ *of the primitive compound network is a tensor on its own right and not a 2-matrix.*

CHAPTER XXI

MULTIPLE TENSORS

I. THE CREATION OF MORE COMPLEX ENTITIES

When a conglomeration of atoms organizes to form a new entity, a molecule, *by the very process of organization the new entity acquires new properties* that its component parts, the atoms, do not possess. The process of organization, however, continues *along several directions* toward the formation of more complex entities. A group of molecules may organize into a colloidal particle, a filterable virus, or a crystal, each having properties different from those of the component molecules. Colloidal particles may organize into cells, cells into organs, organs into plants or animals.

The organization of a conglomeration of mathematical expressions into "geometric objects" and "tensors" of various valence is only the first step in the organization of complicated mathematical entities. A second step was made in Chapter IX where a collection of tensors *of the same valence* has been organized into a "compound tensor." In this chapter a conglomeration of tensors *having different valences* are organized into entities with more complex structure, to be called "*multiple tensors.*"

Both "compound tensors" and "multiple tensors" are only links in several chains of increasingly complex mathematical entities. These mathematical entities represent more complex objective physical entities appearing in nature, whose existence can be inferred by measurements made along several types of reference frames.

II. MULTIPLE REFERENCE FRAMES

(*a*) In each problem so far considered only one set of reference frames occurred, each reference frame having n axes. The n axes were formed by n coils or by n hypothetical axes so that in $A_{\alpha\beta\gamma}$, for instance, each of the indices α or β or γ referred to all the n axes. In introducing compound tensors the n axes were subdivided into several groups so that in $A_{u\beta p}$, for instance, the index u referred to one of the groups of axes, p to another group, and β to all the axes.

(*b*) In electrical engineering numerous problems occur in which *several sets of reference frames are used*, where a reference frame belonging to one set may contain n axes while another reference frame belonging to a second set contains k axes, a third frame may contain h axes, and so on. *Each set of reference frames has its own transformation tensor*, so that in $A_{\alpha\beta pq}$, for instance, the transformation tensor of the set of axes α, β is $C_{\alpha'}^{\alpha}$ while those of pq is $C_{p'}^{p}$. The two tensors $C_{\alpha'}^{\alpha}$ and $C_{p'}^{p}$ represent different groups of transformation matrices.

As a special case no transformation tensor may be associated with one of the reference frames, so that the corresponding indices are closed indices as $A_{\alpha\beta(p)(q)}$.

Up till now, in each network, with each branch there was assumed to be associated only *one* component of e_α or of i^α, that is, in each branch only one known voltage or one known current was assumed to exist. However, *in each branch several sets of currents or voltages may exist*, each influencing the others in various manners. For instance, in each branch currents and voltages with several frequencies $\omega_1, \omega_2 \cdots \omega_k$ may exist so that *in each branch* the current is represented as i^p, where p assumes the value $\omega_1, \omega_2 \cdots \omega_k$ in succession. If there are n independent branches, the current vector is represented as $i^{\alpha p}$, where α represents the branch in which the current flows and p represents the various frequencies. The vector $i^{\alpha p}$ has $n \times k$ components.

When frequency conversion exists in a network represented by a transformation tensor $C_{p'}^{p}$, then the vector $i^{\alpha p}$ attracts two different types of transformation tensors:

1. $C_{\alpha'}^{\alpha}$ interconnects the coils, leaving the frequencies unchanged.
2. $C_{p'}^{p}$ changes the frequencies, leaving the coil interconnections unchanged.

That is, $i^{\alpha p}$ is a contravariant tensor of valence one in the α reference frame and also in the p reference frame. *In spite of its two indices $i^{\alpha p}$ is not a tensor of valence two.*

(*c*) Other examples of systems with multiple reference frames are plentiful in engineering. Such, for instance, are rotating electrical machines with d-c. and a-c. impressed voltages, or with small oscillations superimposed upon their steady rotation; transmission lines along which several waves with different velocities travel, and so on.

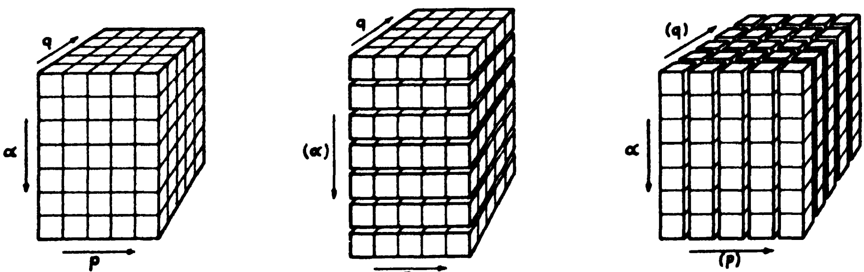
III. REPRESENTATION OF MULTIPLE TENSORS

(*a*) *A tensor containing two or more sets of indices* (each set referring to a different set of reference frames) *will be called here a "multiple tensor."* A base letter may have different number of indices in the

different frames. For instance, $z_{\alpha\beta}{}^{pqr}$ is a covariant tensor of valence two in the α frame, but it is a contravariant tensor of valence three in the p frame. The transformation matrices $C_{\alpha'}^{\alpha}$ belong to a different group from $C_{p'}^{p}$.

Also a base letter may be a tensor in one frame, a geometric object in another, and an n-matrix in a third set of frames. For instance, $A_{\alpha(p)(q)}$ is a vector in the α frame but it is a 2-matrix in the p frame. That is, $A_{\alpha(p)(q)}$ represents a set of k^2 vectors A_α. The α reference frame may be transformed to α' by $C_{\alpha'}^{\alpha}$, but *the closed indices are not transformed*.

(*b*) Multiple tensors are represented the same way as ordinary tensors shown in Chapter I, Section VII, with the difference that *along several directions the number of fixed indices is different*. For instance,



(a) THE MULTIPLE TENSOR $A^{\alpha pq}$ (b) A SET OF k 2-TENSORS $A^{(\alpha)pq}$ (c) A SET OF k^2 VECTORS $A^{\alpha(p)(q)}$

FIG. 21.1.—Various Representations of the Multiple Tensor $A^{\alpha pq}$

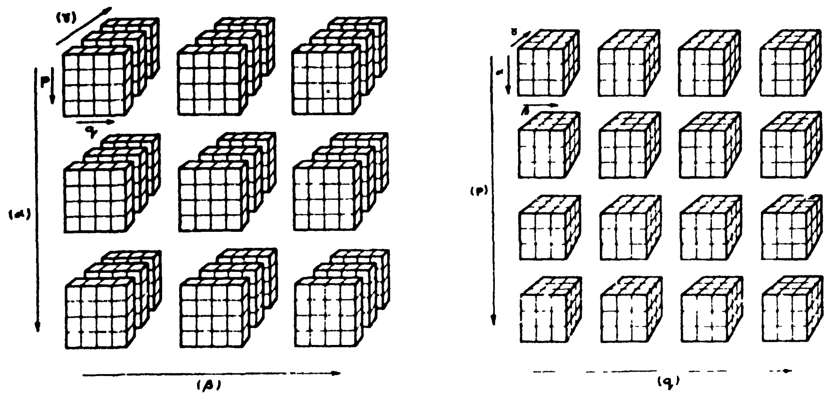
if in $A^{\alpha pq}$ there are seven axes in the α frame and five axes in the p frame, then the number of components in $A^{\alpha pq}$ is $7 \times 5 \times 5 = 175$, as shown in Fig. 21.1*a*.

When the α frame is assumed to be *temporarily* unchanged, that is when α is a closed index as $A^{(\alpha)pq}$, then the tensor is a set of seven 2-tensors A^{pq} arranged in a column (α) as shown in Fig. 21.1*b*. When the indices p and q are considered closed indices as $A^{\alpha(p)(q)}$, then the tensor is a set of $5^2 = 25$ vectors as shown in Fig. 21.1*c*.

Similarly the multiple tensor $A^{\alpha\beta\gamma pq}$ may be represented temporarily as a set of $3^3 = 27$ 2-tensors as $A^{(\alpha)(\beta)(\gamma)pq}$ shown in Fig. 21.2*a* or as a set of $4^2 = 16$ 3-tensors $A^{\alpha\beta\gamma(p)(q)}$ shown in Fig. 21.2*b*.

(*c*) *Multiple tensors can also be subdivided into component tensors so that the multiple tensor becomes a compound multiple tensor.* For instance, if in $A^{\alpha\beta\gamma(p)(q)}$ the indices α, β, γ represent both mesh and junction-pair axes, then say the index α may be subdivided into the

mesh axes m and the junction-pair axes u giving $A^{m\beta\gamma(p)(q)}$ and $A^{u\beta\gamma(p)(q)}$



(a) A SET OF k^3 2-TENSORS $A^{(\alpha)(\beta)(\gamma)pq}$

(b) A SET OF k^2 3-TENSORS $A^{\alpha\beta\gamma(p)(q)}$

FIG. 21.2.—Representation of the Multiple Tensor $A^{\alpha\beta\gamma pq}$

as shown in Fig. 21.3. Or m may represent the active and u the inactive junction-pairs, and so on.

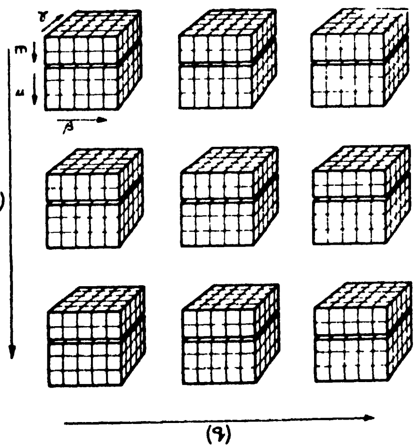


FIG. 21.3.—Compound Multiple Tensor $A^{\alpha\beta\gamma(p)(q)}$

IV. REPLACEMENT OF COMPLEX NUMBERS

A simple example of multiple tensors may be introduced in steady-state problems, where the components of the various tensors are complex numbers. Computation shows that *any complex number* $r + jx$

representing an impedance may be replaced by a matrix with two rows and columns containing only real numbers as

$$r + jX = \begin{bmatrix} r & -X \\ X & r \end{bmatrix}$$

so that, for instance, the impedance tensor $\mathbf{z}_{\alpha\beta}$ of an n-mesh network with n^2 complex components can be replaced by a multiple tensor $\mathbf{z}_{\alpha\beta(p)(q)}$ having twice as many rows and columns with all real components as

$$\mathbf{z} = \begin{array}{c|c} & \begin{array}{cc} \mathbf{a} & \mathbf{b} \end{array} \\ \hline \begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} & \begin{bmatrix} a+jb & jc \\ -d & -e-jf \end{bmatrix} \end{array} = \begin{array}{c|c} & \begin{array}{cccc} \mathbf{a}_r & \mathbf{a}_i & \mathbf{b}_r & \mathbf{b}_i \end{array} \\ \hline \begin{array}{c} \mathbf{a}_r \\ \mathbf{a}_i \\ \mathbf{b}_r \\ \mathbf{b}_i \end{array} & \begin{bmatrix} a & -b & & -c \\ b & a & c & \\ -d & & -e & f \\ & -d & -f & -e \end{bmatrix} \end{array} = \begin{array}{cc|cc} & & \begin{array}{c} a \\ (r)\;(i) \end{array} & \begin{array}{c} b \\ \end{array} \\ \hline a & \begin{array}{c} (r) \\ (i) \end{array} & \begin{bmatrix} a & -b \\ b & a \end{bmatrix} & \begin{bmatrix} & -c \\ c & \end{bmatrix} \\ b & & \begin{bmatrix} -d & \\ & -d \end{bmatrix} & \begin{bmatrix} -e & f \\ -f & -e \end{bmatrix} \end{array} \qquad (q) \rightarrow \beta, \quad (p) \downarrow \alpha \tag{21.1}$$

where α and β represent the n *circuit-axes* a and b, while p and q represent the two *time axes* r and i.

Along each reference frame (mesh or junction-pair) two new hypothetical time axes are now assumed, a real and an imaginary axis r and i, so that the voltage and current vectors are expressed in terms of their in-phase and out-of-phase components. For instance, if the current vector of the two-mesh network is

$$\mathbf{i} = \begin{array}{c} \begin{array}{cc} \mathbf{a} & \mathbf{b} \end{array} \\ \begin{bmatrix} 2+3j & 4-5j \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} \mathbf{a}_r & \mathbf{a}_i & \mathbf{b}_r & \mathbf{b}_i \end{array} \\ \begin{bmatrix} 2 & 3 & 4 & -5 \end{bmatrix} \end{array}$$

the voltage vector $\mathbf{e}$ is found by $\mathbf{z}\cdot\mathbf{i}$ as

$$\mathbf{e} = \begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \begin{bmatrix} 2a - 3b + 5c + j(3a + 2b + 4c) \\ -2d - 4e - 5f + j(-3d + 5e - 4f) \end{bmatrix} = \begin{array}{c} \mathbf{a}_r \\ \mathbf{a}_i \\ \mathbf{b}_r \\ \mathbf{b}_i \end{array} \begin{bmatrix} 2a - 3b + 5c \\ 3a + 2b + 4c \\ -2d - 4e - 5f \\ -3d + 5e - 4f \end{bmatrix}$$

V. MULTIELECTRODE TUBE NETWORKS

(*a*) A simple example of networks in which multiple tensors occur is a multielectrode-tube network with *small* impressed voltages $\Delta\mathbf{e}$ or

currents ΔI. *Because of the non-linearity of the tubes a voltage of one frequency* ω_1 *produces in each coil currents with several frequencies* ω_1, $\omega_2 \cdots \omega_n$, *and vice versa.* In the resultant multiple tensors one set of indices $m, n, k \cdots$ refers to the coils in which the currents flow, while the other set of indices $\alpha, \beta, \gamma \cdots$ refers to the various frequencies. Since in this chapter no frequency transformations are assumed, the indices $\alpha, \beta, \gamma \cdots$ are closed indices.

The equations of voltage and current for mesh and junction networks will be developed in the form of a Taylor's series that is analogous to the power series shown in Sections XIII and XIV, Chapter I. *The non-linear network may be looked upon also as a mesh, or a junction, or an orthogonal network, just as any linear network can.*

First the non-linear equations of a multielectrode tube *without the presence of the outside linear network* are developed. Afterward the interconnection of a tube or several tubes with a linear network is considered.

(*b*) Since the groups of transformation matrices $C_{\alpha'}^{\alpha}$ that will be used are not functions of the variables, *all geometric objects to be introduced in this chapter are tensors.*

VI. A MORE GENERAL FORMULATION OF THE "SECOND GENERALIZATION POSTULATE"

(*a*) In order to keep the physical picture clear the non-linear equations of a tube will be developed in four steps, assuming: (1) one electrode having one current, (2) one electrode having several frequencies of current, (3) several electrodes, each having several frequencies of currents, (4) several sets of electrodes, each set having a different function.

The step from 1 to 2 introduces a set of closed indices (α), (β) $\cdots$, the step from 2 to 3 adds an additional set of open indices $m, n \cdots$ (in mesh networks) or $u, v \cdots$ (in junction networks), and the step from 3 to 4 adds several sets of open indices.

Throughout the development, emphasis will be laid upon the important fact that, *as the complexity of the physical system increases, the number of tensors, the form of the equations, and the reasonings accompanying them do not change, only the number of the set of indices increases. Each increase in the complexity of the physical set-up adds an additional set of indices to each base letter.*

That is, *whatever theories, laws, equations, etc. are developed for ordinary networks are all valid for multiple networks by simply replacing single quantities by tensors and single tensors by appropriate multiple*

tensors. They are equally valid for compound multiple networks by replacing each multiple tensor by an appropriate compound multiple tensor.

This statement may be considered as a more general formulation of the " Second Generalization Postulate " in which the expression " tensor " may include compound tensors, multiple tensors and compound multiple tensors of any complexity.

(*b*) Corresponding to the equations of the physical system, Taylor's series is developed in the following steps:

1. Functions of real variables; in one variable and in several variables.

2. Functions of complex variables; in one set of variables and in several sets of variables.

Afterward the inverse series are developed for the same cases.

VII. TAYLOR'S SERIES

Let any equation $y = f(x)$ be given. Its graph is shown in Fig. 21.4. Suppose that for a given value of x, say x_0, the value of y is known to be y_0, but it is not known for any other value of x. Suppose also that for the given value of x not only y but also all its derivatives are known.

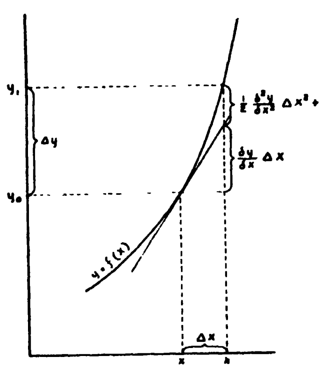


FIG. 21.4.—Graph of an equation $y = f(x)$

The problem is whether it is possible to find the value of y at *another point* x_1 in the neighborhood of x_0, even though the value of y is known only at point x_0.

Taylor's series gives the value of y at point x_1 in terms of the value of y at x_0, of all its derivatives at x_0, and of the distance $x_1 - x_0 = \Delta x$, all of which are known, provided certain conditions stated in textbooks are satisfied. The value of y at x_1 is

$$y_0 + \Delta y = y_0 + \frac{\partial y}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}(\Delta x)^2 + \frac{1}{6}\frac{\partial^3 y}{\partial x^3}(\Delta x)^3 + \cdots$$

where all derivatives are taken at point y_0. Since only Δy, the change in y, is needed, the equation becomes

$$\boxed{\Delta y = \frac{\partial y}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial^2 y}{\partial x^2}(\Delta x)^2 + \frac{1}{3!}\frac{\partial^3 y}{\partial x^3}(\Delta x)^3 + \cdots} \qquad 21.2$$

If the slope of the curve in the neighborhood of y_0 is constant, the curvature and $\partial^2 y/\partial x^2$ are zero, leaving $\Delta y = (\partial y/\partial x)\Delta x$.

VIII. TAYLOR'S SERIES IN AN INVARIANT FORM

(*a*) Let each of two variables be a function of two other independent variables; that is, let $y^a = f^a(x_a,x_b)$ and $y^b = f^b(x_a,x_b)$. (The position of the indices varies in various problems, depending upqn what the variables y and x represent.)

If each of the independent variables changes by an amount Δx_a and Δx_b, the change in the dependent variables is

$$\Delta y^a = \left[\frac{\partial y^a}{\partial x_a}\Delta x_a + \frac{\partial y^a}{\partial x_b}\Delta x_b\right]$$
$$+\frac{1}{2}\left[\frac{\partial^2 y^a}{\partial x_a \partial x_a}\Delta x_a \Delta x_a + \frac{\partial^2 y^a}{\Delta x_a \Delta x_b}\Delta x_a \Delta x_b + \frac{\partial^2 y^a}{\partial x_b \partial x_a}\Delta x_b \Delta x_a + \frac{\partial^2 v^a}{\partial x_b \partial x_b}\Delta x_b \Delta x_b\right]$$
$$+\frac{1}{6}\left[\frac{\partial^3 y^a}{\partial x_a \partial x_a \partial x_a}\Delta x_a \Delta x_a \Delta x_a + \cdots\right] + \cdots$$

$$\Delta y^b = \left[\frac{\partial y^b}{\partial x_a}\Delta x_a + \frac{\partial y^b}{\partial x_b}\Delta x^b\right]$$
$$+\frac{1}{2}\left[\frac{\partial^2 y^b}{\partial x_a \partial x_a}\Delta x_a \Delta x_a + \frac{\partial^2 y^b}{\partial x_a \partial x_b}\Delta x_a \Delta x_b + \frac{\partial^2 y^b}{\partial x_b \partial x_a}\Delta x_b \Delta x_a + \frac{\partial^2 y^b}{\partial x_b \partial x_b}\Delta x_b \Delta x_b\right]$$
$$+\frac{1}{6}\left[\frac{\partial^3 y^b}{\partial x_a \partial x_a \partial x_a}\Delta x_a \Delta x_a \Delta x_a + \cdots\right] + \cdots \qquad 21.3$$

(*b*) If, instead of two functions of two variables, there are n functions of n variables, then there are n such equations as the foregoing, each parenthesis containing n, n^2, or n^3 terms instead of 2, 2^2, or 2^3. In index notation the n equations are written as one invariant equation

$$\boxed{\Delta y^m = \frac{\partial y^m}{\partial x_n}\Delta x_n + \frac{1}{2!}\frac{\partial^2 y^m}{\partial x_n \partial x_k}\Delta x_n \Delta x_k + \frac{1}{3!}\frac{\partial^3 y^m}{\partial x_n \partial x_k \partial x_h}\Delta x_n \partial x_k \partial x_h + \cdots} \qquad 21.4$$

where the indices m, n, k, $\cdots$ may assume any one of the values a, b, c, $\cdots$.

This equation is similar to its scalar form, equation 21.2, except the nth power of Δx, that is $(\Delta x)^n$, is replaced by $\Delta x_a \Delta x_b \cdots \Delta x_n$.

IX. GEOMETRIC OBJECTS OF HIGHER VALENCE

(*a*) In the invariant equation 21.4, Δy^m is a contravariant vector and Δx_m is a covariant vector: $\partial y^m/\partial x_n$ is a doubly contravariant tensor, as has been shown in equation 15.16, and is represented, say, by Y^{mn}.

The expression $M^{mnk} = (\frac{1}{2})\partial^2 y^m/\partial x_n \partial x_k$ is a tensor of valence three where each term is a partial derivative of a component in Y^{mn}. It is

represented *in each particular reference frame* by n^3 numbers arranged in a cube as shown in Fig. 1.2. On paper for, say, $n = 4$, it can be represented by four matrices, each containing 4^2 components by assuming that in M^{mnk} the variable index m assumes the fixed indices, a, b, c, and d in succession. One of the four matrices (when m assumes the value a) is

$$2M^{ank} = \begin{array}{c|c|c|c|c|}
n \backslash k & a & b & c & d \\
\hline
a & \dfrac{\partial^2 y^a}{\partial x_a \partial x_a} & \dfrac{\partial^2 y^a}{\partial x_a \partial x_b} & \dfrac{\partial^2 y^a}{\partial x_a \partial x_c} & \dfrac{\partial^2 y^a}{\partial x_a \partial x_d} \\
\hline
b & \dfrac{\partial^2 y^a}{\partial x_b \partial x_a} & \dfrac{\partial^2 y^a}{\partial x_b \partial x_b} & \dfrac{\partial^2 y^a}{\partial x_b \partial x_c} & \dfrac{\partial^2 y^a}{\partial x_b \partial x_d} \\
\hline
c & \dfrac{\partial^2 y^a}{\partial x_c \partial x_a} & \dfrac{\partial^2 y^a}{\partial x_c \partial x_b} & \dfrac{\partial^2 y^a}{\partial x_c \partial x_c} & \dfrac{\partial^2 y^a}{\partial x_c \partial x_d} \\
\hline
d & \dfrac{\partial^2 y^a}{\partial x_d \partial x_a} & \dfrac{\partial^2 y^a}{\partial x_d \partial x_b} & \dfrac{\partial^2 y^a}{\partial x_d \partial x_c} & \dfrac{\partial^2 y^a}{\partial x_d \partial x_d} \\
\hline
\end{array} \qquad 21.5$$

In matrix $2M^{bnk}$ in each numerator y^b occurs instead of y^a; similarly, in the others, y^c and y^d occur respectively.

Instead of m, of course, n or k might have assumed the fixed indices a, b, c, d, resulting in matrices that represent different cuts of the original cube, or the same 4^3 components might have been arranged in two other ways.

(*b*) The expression $D^{mnkh} = (\frac{1}{6})\partial^3 y^m/\partial x_n \partial x_k \partial x_h$ is a tensor of valence four. It has *in each particular reference frame* n^4 components, each a partial derivative of a component in M^{mnk} with respect to x_h. On paper (Fig. 1.7) it may be represented by n^2 matrices, each with n^2 components, by replacing two of its variable indices by their fixed values in succession. For $n = 4$ one of the sixteen matrices (when $m = b$ and $n = c$) is

$$6D^{bckh} = \begin{array}{c|c|c|c|c|}
k \backslash h & a & b & c & d \\
\hline
a & \dfrac{\partial^3 y^b}{\partial x_c \partial x_a \partial x_a} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_a \partial x_b} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_a \partial x_c} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_a \partial x_d} \\
\hline
b & \dfrac{\partial^3 y^b}{\partial x_c \partial x_b \partial x_a} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_b \partial x_b} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_b \partial x_c} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_b \partial x_d} \\
\hline
c & \dfrac{\partial^3 y^b}{\partial x_c \partial x_c \partial x_a} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_c \partial x_b} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_c \partial x_c} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_c \partial x_d} \\
\hline
d & \dfrac{\partial^3 y^b}{\partial x_c \partial x_d \partial x_a} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_d \partial x_b} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_d \partial x_c} & \dfrac{\partial^3 y^b}{\partial x_c \partial x_d \partial x_d} \\
\hline
\end{array} \qquad 21.6$$

(*c*) In terms of these tensors Taylor's series can be written

$$\boxed{\Delta y^m = Y^{mn}\Delta x_n + M^{mnk}\Delta x_n \Delta x_k + D^{mnkh}\Delta x_n \Delta x_k \Delta x_h + \cdots} \qquad 21.7$$

It should be noted that the free index in each term is m.

(*d*) The indices for Δy and Δx may be upper or lower indices, depending on the problem. Accordingly, the position of the indices of the other tensors also varies in the various problems, as is shown later. For instance, when the series represents a transformation of the variables (when Δy^m represents the variables in the old reference system and Δx^m in the new reference system, a type of transformation that occurs frequently in tensor analysis), both y and x have upper indices and equation 21.7 becomes

$$\Delta x^m = Y^m_{.\alpha}\Delta x^\alpha + M^m_{.\alpha\beta}\Delta x^\alpha x^\beta + D^m_{.\alpha\beta\gamma}\Delta x^\alpha \Delta x^\beta \Delta x^\gamma + \cdots \qquad 21.8$$

X. THE MODULATION TENSOR

(*a*) If *small* voltages are applied at the various junction-pairs of a multielectrode tube in addition to the constant battery voltages, the *small* current changes are given by equation 21.7, where y is replaced by I and x by E, that is,

$$\boxed{\Delta I^u = Y^{uv}\Delta E_v + M^{uvw}\Delta E_v \Delta E_w + D^{uvws}\Delta E_v \Delta E_w \Delta E_s + \cdots} \qquad 21.9$$

The 4-tensor D^{uvws} introduces currents that cause distortion; hence it may be called the "*distortion tensor.*"

Usually it is sufficient to consider the first curvature of the I-E curve; hence the last term is neglected, leaving *the equation of current of multielectrode tubes* (considering them as junction networks)

$$\boxed{\mathbf{\Delta I} = \mathbf{Y}\cdot\mathbf{\Delta E} + \mathbf{\Delta E}\cdot\mathbf{M}\cdot\mathbf{\Delta E}} \quad \Bigg| \quad \boxed{\Delta I^u = Y^{uv}\Delta E_v + M^{uvw}\Delta E_v \Delta E_w} \qquad 21.10$$

Since the 3-tensor M^{uvw} determines the modulation characteristics of tubes it may be called the "*modulation tensor.*" If the curvature is neglected, it simplifies to $\Delta I^u = Y^{uv}\Delta E_v$, which has already been given in equation 15.15. The admittance tensor Y^{uv} of nonlinear networks may be called the "*amplification tensor.*"

For any tube

$$Y^{uv} = \frac{\partial I^u}{\partial E_v}$$

$$M^{uvw} = \frac{1}{2!}\frac{\partial Y^{uv}}{\partial E_w} = \frac{1}{2!}\frac{\partial^2 I^u}{\partial E_v \partial E_w} \qquad 21.11$$

$$D^{uvws} = \frac{1}{3!}\frac{\partial M^{uvw}}{\partial E_s} = \frac{1}{3!}\frac{\partial^2 Y^{uv}}{\partial E_w \partial E_s} = \frac{1}{3!}\frac{\partial^3 I^u}{\partial E_v \partial E_w \partial E_s}$$

(*b*) For a pentode the four matrices of the modulation tensor are given in equation 21.5 by replacing y by I and x by E. For a screen-grid tube the three matrices of the modulation tensor are

$$2M^{avw} = \begin{array}{c|c|c|c|} {}_{v}\diagdown^{w} & a & b & p \\ \hline a & -\dfrac{1}{r_a^2}\dfrac{\partial r_a}{\partial E_a} & -\dfrac{1}{r_a^2}\dfrac{\partial r_a}{\partial E_b} & -\dfrac{1}{r_a^2}\dfrac{\partial r_a}{\partial E_p} \\ \hline b & \dfrac{\partial G^{ab}}{\partial E_a} & \dfrac{\partial G^{ab}}{\partial E_b} & \dfrac{\partial G^{ab}}{\partial E_p} \\ \hline p & \dfrac{\partial G^{ap}}{\partial E_a} & \dfrac{\partial G^{ap}}{\partial E_b} & \dfrac{\partial G^{ap}}{\partial E_p} \\ \hline \end{array}$$

$$2M^{bvw} = \begin{array}{c|c|c|c|} {}_{v}\diagdown^{w} & a & b & p \\ \hline a & \dfrac{\partial G^{ba}}{\partial E_a} & \dfrac{\partial G^{ba}}{\partial E_b} & \dfrac{\partial G^{ba}}{\partial E_p} \\ \hline b & -\dfrac{1}{r_b^2}\dfrac{\partial r_b}{\partial E_a} & -\dfrac{1}{r_b^2}\dfrac{\partial r_b}{\partial E_b} & -\dfrac{1}{r_b^2}\dfrac{\partial r_b}{\partial E_p} \\ \hline p & \dfrac{\partial G^{bp}}{\partial E_a} & \dfrac{\partial G^{bp}}{\partial E_b} & \dfrac{\partial G^{bp}}{\partial E_p} \\ \hline \end{array} \qquad 21.12$$

$$2M^{pvw} = \begin{array}{c|c|c|c|} {}_{v}\diagdown^{w} & a & b & p \\ \hline a & \dfrac{\partial G^{pa}}{\partial E_a} & \dfrac{\partial G^{pa}}{\partial E_b} & \dfrac{\partial G^{pa}}{\partial E_p} \\ \hline b & \dfrac{\partial G^{pb}}{\partial E_a} & \dfrac{\partial G^{pb}}{\partial E_b} & \dfrac{\partial G^{pb}}{\partial E_p} \\ \hline p & -\dfrac{1}{r_p^2}\dfrac{\partial r_p}{\partial E_a} & -\dfrac{1}{r_p^2}\dfrac{\partial r_p}{\partial E_b} & -\dfrac{1}{r_p^2}\dfrac{\partial r_p}{\partial E_p} \\ \hline \end{array}$$

In the absence of grid currents the components of the first two matrices M^{avw} and M^{bvw} are all zero.

Since $M^{uvw} = \frac{1}{2}\partial Y^{uv}/\partial E_w$, the components of Y^{uv} are differentiated with respect to the third index, shown above the columns. For instance, M^{cpa} must be equal to half of the partial derivative of Y^{cp} with respect to E_a.

(*c*) For a triode, the two matrices of the modulation tensor are

$$2M^{gvw} = \begin{array}{c|cc} {}_{v}\diagdown^{w} & g & p \\ \hline g & -\frac{1}{r_g^2}\frac{\partial r_g}{\partial E_g} & -\frac{1}{r_g^2}\frac{\partial r_g}{\partial E_p} \\ p & \frac{\partial G^{gp}}{\partial E_g} & \frac{\partial G^{gp}}{\partial E_p} \end{array} \qquad 2M^{pvw} = \begin{array}{c|cc} {}_{v}\diagdown^{w} & g & p \\ \hline g & \frac{\partial G^{pg}}{\partial E_g} & \frac{\partial G^{pg}}{\partial E_p} \\ p & -\frac{1}{r_p^2}\frac{\partial r_p}{\partial E_g} & -\frac{1}{r_p^2}\frac{\partial r_p}{\partial E_p} \end{array} \qquad 21.13$$

In the absence of grid current the components of the first matrix M^{gvw} *are all zero.*

(*d*) *The equation of voltage of a tube considered as a mesh network is,* from equation 21.7,

$$\boxed{\Delta\mathbf{e} = \mathbf{z}\cdot\Delta\mathbf{i} + \Delta\mathbf{i}\cdot\mathbf{h}\cdot\Delta\mathbf{i}} \quad \Big| \quad \boxed{\Delta e_m = z_{mn}\Delta i^n + h_{mnk}\Delta i^n \Delta i^k} \qquad 21.14$$

where

$$z_{mn} = \frac{\partial e_m}{\partial i^n}; \qquad h_{mnk} = \frac{1}{2}\frac{\partial z_{mn}}{\partial i^k} = \frac{1}{2}\frac{\partial^2 e_m}{\partial i^n \partial i^k}$$

XI. COMPLEX TAYLOR'S SERIES

(*a*) In this section Taylor's series will be investigated where the variables and the coefficients are complex numbers instead of real numbers.

Let a set of n sinusoidal voltages be impressed across a non-linear junction circuit, say a crystal detector in series with an impedance, each voltage being of different frequency; that is, let

$$\Delta E_{(\alpha)} = \begin{array}{c} {}^{(\alpha)} \\ \end{array} \begin{array}{cccc} (1) & (2) & & (n) \\ \boxed{\Delta E_{(\omega_1)}} & \boxed{\Delta E_{(\omega_2)}} & \cdots & \boxed{\Delta E_{(\omega_n)}} \end{array} \qquad 21.15$$

each ΔE being a complex number

$$\Delta E_1 = \Delta E_1 + j\Delta E_1' = \sqrt{2}(E_1 \cos \omega_1 t - E_1' \sin \omega_1 t)$$

Since transformation of the frequencies to other frequencies is not intended in this chapter, the closed index α in $E_{(\alpha)}$ shows that the components of $E_{(\alpha)}$ are arranged in a row, but no formula of transformation is associated with the index.

Owing to the application of the n voltages with n different frequencies, the following sets of currents flow in the non-linear circuit:

1. A set of n currents $\Delta I^{(\alpha)}$, each current having the frequency of the corresponding voltage

$$\Delta I^{(\alpha)} = \begin{array}{cccc} {\scriptstyle (\alpha)} & & & \\ (1) & (2) & & (n) \\ \boxed{\Delta I^{(\omega_1)}} & \boxed{\Delta I^{(\omega_2)}} & \cdots & \boxed{\Delta I^{(\omega_n)}} \end{array} \qquad 21.16$$

2. A set of n^2 currents $\Delta I^{(\alpha)(\beta)}$, each having the frequency of the *sum* of two impressed voltage frequencies, including the case $\alpha = \beta$. These currents can be arranged in a square. They are caused by the curvature of the E-I curve.

3. A set of n^2 currents $\Delta I^{(\gamma)(\delta)}$, each having the frequency of the *difference* of two impressed voltage frequencies. The two preceding matrices are denoted as $\Delta I^{(\alpha)(\pm\beta)}$ so that

$$\Delta I^{(\alpha)(+\beta)} = \begin{array}{cccc} {\scriptstyle (\alpha)\backslash(+\beta)} & (1) & (2) & (n) \\ (1) & \boxed{\Delta I^{(\omega_1+\omega_1)}} & \boxed{\Delta I^{(\omega_1+\omega_2)}} & \boxed{\Delta I^{(\omega_1+\omega_n)}} \\ (2) & \boxed{\Delta I^{(\omega_2+\omega_1)}} & \boxed{\Delta I^{(\omega_2+\omega_2)}} & \boxed{\Delta I^{(\omega_2+\omega_n)}} \\ (n) & \boxed{\Delta I^{(\omega_n+\omega_1)}} & \boxed{\Delta I^{(\omega_n+\omega_2)}} & \boxed{\Delta I^{(\omega_n+\omega_n)}} \end{array} \qquad 21.17$$

A similar matrix exists for $\Delta I^{(\alpha)(-\beta)}$. The $2n^2$ currents are said to be product frequency currents, and $\Delta I^{(\alpha)(+\beta)}$ is a set of n^2 quantities arranged in a square.

4. A set of $4n^3$ currents $\Delta I^{(\alpha)(\pm\beta)(\pm\gamma)}$ arranged in four cubes, each having the frequency of the sums or differences of three of the impressed frequencies.

5. A set of $8n^4$ currents $\Delta I^{(\alpha)(\pm\beta)(\pm\gamma)(\pm\delta)}$, and so on.

Only the sets $\Delta I^{(\alpha)}$ and $\Delta I^{(\alpha)(\pm\beta)}$ are calculated here, giving $n + 2n^2$ components of current.

(*b*) The first set of n currents $\Delta I^{(\alpha)}$ is calculated by the formula

$$\Delta I^{(\alpha)} = Y^{(\alpha)(\beta)} \Delta E_{(\beta)} \qquad 21.18$$

where $Y^{(\alpha)(\beta)}$ usually is a matrix having only diagonal components, each component being calculated for the frequency of the applied voltage; hence, in equation 21.18, $\alpha = \beta$. In the most general case, such an equality is not necessarily true, however.

The second set of $2n^2$ currents is calculated by the formula

$$\Delta I^{(\alpha)(\pm\beta)} = M^{(\alpha)(\pm\beta)(\gamma)(\delta)} \Delta E_{(\gamma)} \Delta E_{(\delta)} \qquad 21.19$$

where usually $\gamma = \alpha$, and $\delta = \beta$ and M are two four-way matrices. Their calculation is shown later.

The matrix $\Delta E_{(\gamma)}\Delta E_{(\delta)}$ represents $2n^2$ complex numbers (two matrices) formed by all possible products of the components of $\Delta E_{(\gamma)}$, forming sum and difference frequency quantities.

(*c*) In multiplying two complex numbers $\Delta E_1 = A + jB$ and $\Delta E_2 = C + jD$, each representing a sinusoidal function of different frequencies ω_1 and ω_2, the following must be noted:

1. The product with $\omega_1 + \omega_2$ frequencies is found by $(A + jB) \times (C + jD)$.
2. The product with $\omega_1 - \omega_2$ frequencies is found by $(A + jB) \times (C - jD)$.
3. The component complex numbers represent square-root-of-mean-square values, and the products represent peak values.

(*d*) Equations 21.18 and 21.19 (a 1-matric and a 2-matric equation) may be combined as

$$\boxed{\Delta I^{(\epsilon)} + \Delta I^{(\alpha)(\pm\beta)} = Y^{(\epsilon)(\sigma)}\Delta E_{(\sigma)} + M^{(\alpha)(\pm\beta)(\gamma)(\delta)}\Delta E_{(\gamma)}\Delta E_{(\delta)}} \qquad 21.20$$

representing $n + 2n^2$ equations. Since the components of a 1-matrix $\Delta I^{(\epsilon)}$ and a 2-matrix $\Delta I^{(\alpha)(\beta)}$ cannot be added, the various sets of currents are kept separate.

XII. COMPOUND SERIES

(*a*) Assume now that additional sets of voltages with $(\alpha \pm \beta)$, $(\alpha \pm \beta \pm \gamma) \cdots$ frequencies are also impressed across the system. The set of $2n^2$ voltages with product frequencies then produces a set of $2n^2$ currents of the same frequency

$$\Delta I^{(\alpha)(\pm\beta)} = Y^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}\Delta E_{(\gamma)(\pm\delta)} \qquad 21.21$$

where usually $\gamma = \alpha$, $\delta = \beta$, and the components of $Y^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}$ are calculated at the various product frequencies. The additional currents due to the curvature of the *E-I* curves will not be calculated here.

The 2-matrix $\Delta E_{(\gamma)(\pm\delta)}$ contains $2n^2$ complex numbers representing product frequency impressed voltages, but they are not formed by products of fundamental frequency impressed voltages (as are the components of $\Delta E_{(\gamma)}\Delta E_{(\delta)}$), being independent of them.

Hence, if both fundamental frequency and product frequency voltages are impressed, the resultant fundamental and product frequency currents are

$$\begin{aligned}\Delta I^{(\epsilon)} + \Delta I^{(\alpha)(\pm\beta)} + \cdots &= Y^{(\epsilon)(\sigma)}\Delta E_{(\sigma)} + M^{(\alpha)(\pm\beta)(\gamma)(\delta)}\Delta E_{(\gamma)}\Delta E_{(\delta)} + \cdots \\ &\quad + Y^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}\Delta E_{(\gamma)(\pm\delta)} + \cdots + \cdots \end{aligned} \qquad 21.22$$

There is a matric form of Taylor's series for each set of impressed voltages, that is, one for $\Delta E_{(\alpha)}$, and another for $\Delta E_{(\alpha)(\pm\beta)}$, each set of voltages producing an infinite set of currents. Such a series is called a *compound series.* If only fundamental frequency voltages are impressed, the last term of equation 21.22 drops out.

(*b*) Conversely, if both fundamental frequency and product frequency currents flow, the resultant fundamental frequency and product frequency voltages are

$$\Delta E_{(\epsilon)} + \Delta E_{(\alpha)(\pm\beta)} + \cdots = Z_{(\epsilon)(\sigma)}\Delta I^{(\sigma)} + H_{(\alpha)(\pm\beta)(\gamma)(\delta)}\Delta I^{(\gamma)}\Delta I^{(\delta)} + \cdots$$
$$+ Z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}\Delta I^{(\gamma)(\pm\delta)} + \cdots + \cdots \qquad 21.23$$

where usually $\gamma = \alpha$, $\delta = \beta$, and $\epsilon = \sigma$.

(*c*) The important fact should be noted that the variables ΔI are not necessarily always arranged in a row to form a 1-matrix $\Delta I^{(\alpha)}$. Here some of the variables are arranged in squares, $\Delta I^{(\alpha)(\pm\beta)}$, and some in cubes. In modern matrix mechanics the variables are arranged in a square forming a matrix.

XIII. COMPLEX SERIES IN INVARIANT FORM

(*a*) Instead of a crystal detector let a multielectrode tube be connected to a network containing coils. If across several junction-pairs a set of fundamental voltages $\Delta E_{(\alpha)}$ is impressed, then in each circuit several sets of currents of various frequencies flow.

Here two sets of indices must be introduced: (1) open indices u, v, w representing the various circuits of the system a, b, c, g, p; and (2) closed indices (α), (β), (γ), representing the various impressed frequencies ω_1, ω_2, $\cdots$ ω_n.

In direct notation the open indices are not shown. With both fundamental and product frequency voltages applied along the various circuits, the currents (see equations 21.10 and 21.22) are

$$\Delta \mathbf{I}^{(\epsilon)} + \Delta \mathbf{I}^{(\alpha)(\pm\beta)} + \cdots = \mathbf{Y}^{(\epsilon)(\sigma)}\cdot\Delta \mathbf{E}_{(\sigma)} +$$
$$+ \Delta \mathbf{E}_{(\gamma)}\cdot\mathbf{M}^{(\alpha)(\pm\beta)(\gamma)(\delta)}\cdot\Delta \mathbf{E}_{(\delta)} + \cdots +$$
$$+ \mathbf{Y}^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}\cdot\Delta \mathbf{E}_{(\gamma)(\pm\delta)} + \cdots + \cdots \qquad 21.24$$

The order of the tensors is the same as in equation 21.10, and the order of the closed indices is the same as in equation 21.22.

In index notation

$$\Delta I^{u(\epsilon)} + \Delta I^{u(\alpha)(\pm\beta)} + \cdots = Y^{uv(\epsilon)(\sigma)}\Delta E_{v(\sigma)} +$$
$$+ M^{uvw(\alpha)(\pm\beta)(\gamma)(\delta)}\Delta E_{v(\gamma)}\Delta E_{w(\delta)} + \cdots +$$
$$+ Y^{uv(\alpha)(\pm\beta)(\gamma)(\pm\delta)}\Delta E_{v(\gamma)(\pm\delta)} + \cdots + \cdots \qquad 21.25$$

The order of the "open" indices is the same as in equation 21.10 and the order of the "closed" indices as in equation 21.22.

(*b*) The open indices have covariant and contravariant meaning, but the closed indices have no such significance; hence, for a more compact notation, *the position of the closed indices may be changed* and the previous equation may be written

$$\Delta I^{u}_{(\epsilon)} + \Delta I^{u}_{(\alpha)(\pm\beta)} + \cdots = Y^{uv}_{(\epsilon)(\sigma)}\Delta E^{(\sigma)}_{v} + M^{uvw}_{(\alpha)(\pm\beta)(\gamma)(\delta)}\Delta E^{(\gamma)}_{v}\Delta E^{(\delta)}_{w} + \cdots$$
$$+ Y^{uv}_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}\Delta E^{(\gamma)(\pm\delta)}_{v} + \cdots + \cdots \qquad 21.26$$

(*c*) Instead of writing n^2 different 2-tensors to represent $\mathbf{Y}^{(\alpha)(\beta)}$, the procedure in calculation is to write first *one* 2-tensor $\mathbf{Y}$, with each component still containing the closed indices, as $A^{(\alpha)(\beta)}$; thereby the operations represented by the open indices are disposed of, leaving the variable closed indices. The closed indices thereafter are assumed to vary through their own range, each variable closed index assuming the range of fixed indices in succession.

XIV. SPIN INDICES

(*a*) If the components of transformation tensors contain complex numbers $a + jb$, it is advantageous to use spin indices instead of tensor indices, that is, indices with bars over some of them. The bars over some of the indices also help to keep the correct order in multiplying several tensors of various valence. Spinor notation is valid, however, only if all complex numbers represent quantities of the same frequency.

Assuming the open indices as fixed indices and the closed ones as variable indices (if in one particular circuit all possible currents are considered), each complex number represents a quantity of different frequency. Hence, the closed indices cannot be spin indices in the present analysis.

Assuming the closed indices as fixed indices and the open ones as variable indices (that is, if the currents of one particular frequency in all circuits are considered), each complex number represents a quantity of the same frequency. Therefore, *the open indices may be considered as spin indices.*

In terms of spin indices the tensors hitherto introduced are

$$e_{\bar{m}},\ i^{m},\ \Delta e_{\bar{m}},\ \Delta i^{m},\ Y^{m\bar{n}},\ Z_{\bar{m}n},\ M^{m\bar{n}\bar{k}},\ Q_{\bar{m}nk},\ D^{m\bar{n}\bar{k}\bar{h}}$$

(*b*) Once the position of the bars in e_m and i^m is determined by the invariance of $e_m i^m$, their position in the other tensors automatically follows in the equations by the rule that the two dummy indices must be either both barred or both unbarred indices.

In terms of spin indices the previous equation becomes

$$\Delta I^{u}_{(\epsilon)} + \Delta I^{u}_{(\alpha)(\pm\beta)} + \cdots = Y^{u\bar{v}}_{(\epsilon)(\sigma)}E^{(\sigma)}_{\bar{v}} + M^{u\bar{v}\bar{w}}_{(\alpha)(\pm\beta)(\gamma)(\delta)}\Delta E^{(\gamma)}_{\bar{v}}\Delta E^{(\delta)}_{\bar{w}} + \cdots$$
$$+ Y^{u\bar{v}}_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}\Delta E^{(\gamma)(\pm\delta)}_{\bar{v}} + \cdots + \cdots \qquad 21.27$$

XV. THE INVERSE OF TAYLOR'S SERIES

In the foregoing sections the different forms of Taylor's series are established. In the following sections their inverse is found.

Since Taylor's series is a power series whose inverse has already been established in Section XII, Chapter II, for the case of no indices and for one set of indices, the calculation is not repeated here. For Taylor's series with one set of indices

$$\Delta\mathbf{e} = \mathbf{z}\cdot\Delta\mathbf{i} + \Delta\mathbf{i}\cdot\mathbf{h}\cdot\Delta\mathbf{i} \quad \Big| \quad \Delta e_{\bar{m}} = z_{\bar{m}n}\Delta i^{n} + h_{\bar{m}nk}\Delta i^{n}\Delta i^{k} \qquad 21.28$$

the inverse series is by equation 2.34

$$\Delta\mathbf{i} = \mathbf{y}\cdot\Delta\mathbf{e} + \Delta\mathbf{e}\cdot\mathbf{m}\cdot\Delta\mathbf{e} \quad \Big| \quad \Delta i^{m} = y^{m\bar{n}}\Delta e_{\bar{n}} + m^{m\bar{n}\bar{k}}\Delta e_{n}\Delta e_{\bar{k}} \qquad 21.29$$

where

$$\mathbf{y} = \mathbf{z}^{-1} \quad \Big| \quad y^{mn} = (z_{nm})^{-1} \qquad 21.30$$

$$\mathbf{m} = -\mathbf{y}_{t}\cdot(\mathbf{y}\cdot\mathbf{h})\cdot\mathbf{y} \quad \Big| \quad m^{m\bar{n}\bar{k}} = -\, h_{\bar{d}fg}y^{m\bar{d}}y^{f\bar{n}}y^{g\bar{k}} \qquad 21.31$$

Where a 3-tensor $\mathbf{h}$ and a two-tensor $\mathbf{y}$ are enclosed in parenthesis as $(\mathbf{y}\cdot\mathbf{h})$ it will be understood that *the free index of* $\mathbf{h}$ *is to be multiplied by the second index of* $\mathbf{y}$ *as* $(\mathbf{y}\cdot\mathbf{h}) = y^{md}h_{dfg}$ *so that the first index of* $\mathbf{y}$ *becomes the free index.* It should be noted that in equation 21.31 the free index d of $h^{\bar{d}fg}$ is multiplied by *second* index of $y^{m\bar{d}}$, while the other two indices of h_{dfg} are multiplied by the *first* indices of $y^{f\bar{n}}$ and $y^{g\bar{k}}$. Spin indices automatically take care of this order of multiplication.

XVI. INVERSE OF THE COMPLEX SERIES

(*a*) Let the voltage equation

$$\Delta e_{(\epsilon)} + \Delta e_{(\alpha)(\pm\beta)} = z_{(\epsilon)(\sigma)}\Delta i^{(\sigma)} + h_{(\alpha)(\pm\beta)(\gamma)(\sigma)}\Delta i^{(\gamma)}\Delta i^{(\delta)}$$
$$+ z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}\Delta i^{(\gamma)(\pm\delta)} \qquad 21.32$$

be given and let its inverse be calculated

$$\Delta i^{(\epsilon)} + \Delta i^{(\alpha)(\pm\beta)} = y^{(\epsilon)(\sigma)}\Delta e_{(\sigma)} + m^{(\alpha)(\pm\beta)(\gamma)(\delta)}\Delta e_{(\gamma)}\Delta e_{(\delta)}$$
$$+ y^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}\Delta e_{(\gamma)(\pm\delta)} \qquad 21.33$$

where y and m are unknown functions of z and h.

Following the previous steps, let the second equation be substituted into the first. Neglecting higher than second-order terms

$$\Delta e_{(\epsilon)} + \Delta e_{(\alpha)(\pm\beta)} = z_{(\epsilon)(\sigma)} y^{(\sigma)(\nu)} \Delta e_{(\nu)} + h_{(\alpha)(\pm\beta)(\gamma)(\delta)} y^{(\gamma)(\sigma)} y^{(\sigma)(\nu)} \Delta e_{(\sigma)} \Delta e_{(\nu)}$$

$$+ z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} m^{(\gamma)(\pm\delta)(\sigma)(\nu)} \Delta e_{(\sigma)} \Delta e_{(\nu)} + z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} y^{(\gamma)(\pm\delta)(\sigma)(\pm\nu)} \Delta e_{(\sigma)(\pm\nu)}$$

Equating corresponding coefficients of Δe on both sides of the equation

$$I_{(\epsilon)}^{(\nu)} = z_{(\epsilon)(\sigma)} y^{(\sigma)(\nu)}$$

$$I_{(\alpha)(\pm\beta)}^{(\sigma)(\pm\nu)} = z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} y^{(\gamma)(\pm\delta)(\sigma)(\pm\nu)}$$

$$0 = h_{(\alpha)(\pm\beta)(\gamma)(\delta)} y^{(\gamma)(\sigma)} Y^{(\delta)(\nu)} + z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} m^{(\gamma)(\pm\delta)(\sigma)(\nu)}$$

Solving for y and m

$$\boxed{y^{(\alpha)(\beta)} = \text{inverse of } z_{(\beta)(\alpha)}} \qquad 21.34$$

$$\boxed{y^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} = \text{inverse of } z_{(\gamma)(\pm\delta)(\alpha)(\pm\beta)}} \qquad 21.35$$

$$\boxed{m^{(\alpha)(\pm\beta)(\gamma)(\delta)} = -h_{(\sigma)(\pm\nu)(\epsilon)(\omega)} y^{(\alpha)(\pm\beta)(\sigma)(\pm\nu)} y^{(\epsilon)(\gamma)} y^{(\omega)(\delta)}} \qquad 21.36$$

where $\gamma = \alpha$ and $\delta = \beta$.

(*b*) Since in $\Delta e_{(\epsilon)} = z_{(\epsilon)(\sigma)} \Delta i^{(\sigma)}$ the matrix $z_{(\epsilon)(\sigma)}$ usually contains only diagonal components (one current producing only one voltage of its own frequency), its inverse $y^{(\epsilon)(\sigma)}$ is calculated in such cases by taking the inverse of each component separately.

Similarly, since in $\Delta e_{(\alpha)(\pm\beta)} = z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \Delta i^{(\gamma)(\pm\delta)}$ each current usually produces only one voltage of its own frequency (since $\gamma = \alpha$ and $\delta = \beta$), the inverse of $z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}$ is calculated by taking the inverse of each of its components.

XVII. INVERSE OF THE COMPLEX INVARIANT SERIES

(*a*) Let the set of invariant equations

$$\Delta \mathbf{e}_{(\epsilon)} + \mathbf{e}_{(\alpha)(\pm\beta)} = \mathbf{z}_{(\epsilon)(\sigma)} \cdot \Delta \mathbf{i}^{(\sigma)} + \Delta \mathbf{i}^{(\gamma)} \cdot \mathbf{h}_{(\alpha)(\pm\beta)(\gamma)(\delta)} \cdot \Delta \mathbf{i}^{(\delta)} +$$

$$+ \mathbf{z}_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \cdot \Delta \mathbf{i}_{(\gamma)(\pm\delta)} \qquad 21.37$$

be given and let its inverse be calculated

$$\Delta \mathbf{i}^{(\epsilon)} + \Delta \mathbf{i}^{(\alpha)(\pm\beta)} = \mathbf{y}^{(\epsilon)(\sigma)} \cdot \Delta \mathbf{e}_{(\sigma)} + \Delta \mathbf{e}_{(\gamma)} \cdot \mathbf{m}^{(\alpha)(\pm\beta)(\gamma)(\delta)} \cdot \Delta \mathbf{e}_{(\delta)\delta} +$$

$$+ \mathbf{y}^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \cdot \Delta \mathbf{e}_{(\gamma)(\pm\delta)} \qquad 21.38$$

in which $\mathbf{y}$ and $\mathbf{m}$ are unknown functions of $\mathbf{z}$ and $\mathbf{h}$.

Substituting the second equation into the first,

$$\begin{aligned}\Delta e_{(\epsilon)} + \Delta e_{(\alpha)(\pm\beta)} &= z_{(\epsilon)(\sigma)} \cdot y^{(\sigma)(\nu)} \cdot \Delta e_{(\nu)} \\ &+ \Delta e_{(\sigma)} \cdot y_t^{(\gamma)(\sigma)} \cdot h_{(\alpha)(\pm\beta)(\gamma)(\delta)} \cdot y^{(\delta)(\nu)} \cdot \Delta e_{(\nu)} \\ &+ \Delta e_{(\sigma)} \cdot (z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \cdot m^{(\gamma)(\pm\delta)(\sigma)(\nu)}) \cdot \Delta e_{(\nu)} \\ &+ z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \cdot y^{(\gamma)(\pm\delta)(\sigma)(\pm\nu)} \cdot \Delta e_{(\sigma)(\pm\nu)}\end{aligned}$$

Equating corresponding coefficients of Δe

$$I_{(\epsilon)}^{(\nu)} = z_{(\epsilon)(\sigma)} \cdot y^{(\sigma)(\nu)}$$

$$I_{(\alpha)(\pm\beta)}^{(\sigma)(\pm\nu)} = z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \cdot y^{(\gamma)(\pm\delta)(\sigma)(\pm\nu)}$$

$$0 = y_t^{(\gamma)(\sigma)} \cdot h_{(\alpha)(\pm\beta)(\gamma)(\delta)} \cdot y^{(\delta)(\nu)} + (z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \cdot m^{(\gamma)(\pm\delta)(\sigma)(\nu)})$$

Solving for the unknown y and m

$$\boxed{y^{(\alpha)(\beta)} = z^{-1}_{(\alpha)(\beta)}} \qquad 21.39$$

$$\boxed{y^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} = z^{-1}_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}} \qquad 21.40$$

$$\boxed{m^{(\alpha)(\pm\beta)(\gamma)(\delta)} = -y_t^{(\epsilon)(\gamma)} \cdot (y^{(\alpha)(\pm\beta)(\sigma)(\pm\nu)} \cdot h_{(q)(\pm\nu)(\epsilon)(\omega)}) \cdot y^{(\omega)(\delta)}} \qquad 21.41$$

(*b*) In spinor notation the last equation, by changing the position of the closed indices and using spin indices, is

$$\boxed{m^{m\bar{n}k}_{(\alpha)(\pm\beta)(\gamma)(\delta)} = -h^{(\sigma)(\pm\nu)(\epsilon)(\omega)}_{\bar{h}fg}\, y^{m\bar{h}}_{(\alpha)(\pm\beta)(\sigma)(\pm\nu)} y^{f\bar{n}}_{(\epsilon)(\gamma)} y^{g\bar{k}}_{(\omega)(\delta)}} \qquad 21.42$$

(*c*) It should be noted that in the last equation representing the most complete invariant form, the same number of base letters occur as in equation 2.43 representing the simplest possible scalar form.

XVIII. SIMPLIFICATION IN TUBE CIRCUITS

In thermionic tube circuits the previous multiple tensors assume special forms and the following simplifications may be introduced.

(*a*) For a tube *without any outside network* $\mathbf{Y}^{(\alpha)(\beta)}$ and $\mathbf{M}^{(\alpha)(\pm\beta)(\gamma)(\delta)}$ contain only real numbers; hence the closed indices may be removed from them, for their components are independent of the frequency of the terminal voltages ΔE. The equations of the tube (or tubes) are then

$$\Delta e_{(\alpha)} + \Delta e_{(\alpha)(\pm\beta)} = z \cdot \Delta i^{(\alpha)} + \Delta i^{(\alpha)} \cdot h \cdot \Delta i^{(\beta)} + z \cdot \Delta i^{(\alpha)(\pm\beta)} \qquad 21.43$$

$$\Delta I^{(\alpha)} + \Delta I^{(\alpha)(\pm\beta)} = Y \cdot \Delta E_{(\alpha)} + \Delta E_{(\alpha)} \cdot M \cdot \Delta E_{(\beta)} + y \cdot \Delta E_{(\alpha)(\pm\beta)} \qquad 21.44$$

considering them as a mesh or as a junction network respectively. As orthogonal networks their equations are analogous.

The inverse of $\mathbf{Y}$ is $\mathbf{Z}$ and the inverse of $\mathbf{M}$ (from equation 21.31) is $\mathbf{H} = -\mathbf{Z}_t \cdot (\mathbf{Z} \cdot \mathbf{M}) \cdot \mathbf{Z}$. Since $\mathbf{Y}$ does not have any closed indices, the inverse of $\mathbf{M}$ containing $\mathbf{Y}$ does not acquire any closed indices by multiplication with $\mathbf{Y}$.

(*b*) In the impedance and admittance tensors of the *outside network* all components are zero unless two of the closed indices are identical. For instance, $\mathbf{z}_{(\alpha)(\beta)}$ is

$$\mathbf{z}_{(\alpha)(\beta)} = \begin{array}{c|cccc} {}_{(\alpha)}\diagdown^{(\beta)} & (1) & (2) & & (n) \\ \hline (1) & \boxed{\mathbf{z}_{\omega 1}} & \boxed{0} & \vdots & \boxed{0} \\ (2) & \boxed{0} & \boxed{\mathbf{z}_{\omega 2}} & & \boxed{0} \\ (n) & \boxed{0} & \boxed{0} & & \boxed{\mathbf{z}_{\omega n}} \end{array} \qquad 21.45$$

Hence for the outside network $\mathbf{z}_{(\alpha)(\beta)}$ may also be considered as a set of n 2-tensors arranged in a row $\mathbf{z}_{(\alpha)}$

$$\mathbf{z}_{(\alpha)} = \begin{array}{cccc} \diagdown^{(\alpha)} (1) & (2) & & (n) \\ \boxed{\mathbf{z}_{\omega 1}} & \boxed{\mathbf{z}_{\omega 2}} & \cdots & \boxed{\mathbf{z}_{\omega n}} \end{array} \qquad 21.46$$

Similarly, $\mathbf{y}^{(\beta)(\alpha)}$, the inverse of $\mathbf{z}_{(\alpha)(\beta)}$, may be considered as a set of n 2-tensors arranged in a row

$$\mathbf{y}^{(\alpha)} = \begin{array}{cccc} \diagdown^{(\alpha)} (1) & (2) & & (n) \\ \boxed{\mathbf{z}^{-1}_{\omega 1}} & \boxed{\mathbf{z}^{-1}_{\omega 2}} & \cdots & \boxed{\mathbf{z}^{-1}_{\omega n}} \end{array} \qquad 21.47$$

For the outside network alone $\mathbf{M}$ is zero.

It should be noted that the inverse of $\mathbf{z}_{(\alpha)(\beta)}$ is calculated by finding the inverse of each of its component tensors. They, in turn, all have identical forms, except for frequencies.

Also $\mathbf{Y}^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}$, where $\gamma = \alpha$ and $\delta = \beta$, *may be replaced by* $Y^{(\alpha)(\pm\beta)}$, each component of the two matrices containing $\mathbf{Z}^{-1}_{\omega 1 + \omega 2}$, all having identical form, unless calculated for different product frequencies. This simpler notation will be used when tubes are connected to the network.

The equations of the *network without tubes* are, with fundamental and product frequency voltages or currents impressed

$$\Delta e_{(\alpha)} + \Delta e_{(\alpha)(\pm\beta)} = z_{(\alpha)(\beta)} \cdot \Delta i^{(\beta)} + z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \cdot \Delta i^{(\gamma)(\pm\delta)} \qquad 21.48$$

$$\Delta I^{(\alpha)} + \Delta I^{(\alpha)(\pm\beta)} = Y^{(\alpha)(\beta)} \cdot \Delta E_{(\beta)} + Y^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \cdot \Delta E_{(\gamma)(\pm\delta)} \qquad 21.49$$

XIX. INTERCONNECTION OF NON-LINEAR SYSTEMS

(*a*) In a tube circuit two physically different types of structures are interconnected, namely: the *non-linear* network containing one or more isolated tubes and the *linear* network connecting the tubes. Both of them may be considered as mesh, junction, or orthogonal networks.

The interconnection of two non-linear networks, or one linear and one non-linear network, follows analogously that of two linear networks given in Section XII, Chapter V, for mesh networks and in Section XII, Chapter XIV, for junction networks. The steps are:

1. The geometric objects **Y** and **M** (or **z** and **h**) of each component part are established. Some of them may be zero.
2. Their respective sums $\mathbf{Y}=\mathbf{Y}^1+\mathbf{Y}^2+\cdots$ and $\mathbf{M}=\mathbf{M}_1+\mathbf{M}_2+\cdots$ represents the geometric objects of the primitive system.
3. A transformation tensor **C** or $\mathbf{C}_t^{-1}$ is set up showing the manner of interconnection of the primitive system into the actual system. *The terminals of the tubes always form part of the new reference frame.*
4. The new components **Y′** and **M′** (or **z′** and **h′**) are found by their respective transformation formulae.

(*b*) The equations of voltage or current of the resultant system (leaving out the primes) are

$$\begin{aligned}\Delta e_{(\alpha)} + \Delta e_{(\alpha)(\pm\beta)} &= z_{(\alpha)(\beta)} \cdot \Delta i^{(\beta)} \\ &\quad + z_{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \cdot \Delta i^{(\gamma)(\pm\delta)} + \Delta i^{(\alpha)} \cdot \mathbf{h} \cdot i^{(\beta)} \end{aligned} \qquad 21.50$$

$$\begin{aligned}\Delta I^{(\alpha)} + \Delta I^{(\alpha)(\pm\beta)} &= Y^{(\alpha)(\beta)} \cdot \Delta E_{(\beta)}^{*} \\ &\quad + Y^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)} \cdot \Delta E_{(\gamma)(\pm\delta)} + \Delta E_{(\alpha)} \cdot \mathbf{M} \cdot E_{(\beta)} \end{aligned} \qquad 21.51$$

In these equations (where **h** and **M** still have no closed indices) it is assumed that *the terminals of the tube* (*or tubes*) *are considered as reference axes* in which Δi flows or across which ΔE appears.

(*c*) Instead of using the above method to find the equations 21.51 or 21.50, *the first term* $Y^{(\alpha)(\beta)}$ *may also be established in exactly the same manner as shown in Chapter X V, while* **M** *of the resultant system is found by multiplying* **M** *of the tube by* **A** *three times, changing, however, only its indices. The free indices of the modulation term refer only to the tube*

terminals $\mathbf{g}$ *and* $\mathbf{p}$, *while the free indices of the amplification term also include other junction-pairs.*

(*d*) *For purposes of manipulations they may be subdivided into several tensor equations:* For instance, in equation 21.51 three types of junction-pairs may be introduced:

1. $\Delta\mathbf{E}_1$ representing all tube junction-pairs.
2. $\Delta\mathbf{E}_2$ representing active junction-pairs with known impressed voltages.
3. $\Delta\mathbf{E}_3$ representing the remaining inactive junction-pairs.

In this case equation 21.51 is subdivided into three equations, the first one containing a term with $\mathbf{M}$. The elimination of $\mathbf{E}_3$ involves only the introduction of open-circuit admittances $\mathbf{Y}'^{(\alpha)(\beta)}$, while *the elimination of the inactive tube junction-pairs* $\mathbf{E}_1$ *involves the finding of the inverse of a series.*

(*e*) In setting up the equation of voltage as a mesh network it is necessary to calculate first the inverse of a series, since $\mathbf{z}_2$ and $\mathbf{h}_2$ of the tube are needed in the equation, while the inverse constants $\mathbf{y}_2$ and $\mathbf{m}_2$ of the tube are usually known.

The calculation of the inverse of equations 21.50 and 21.51 or of any subdivision of them containing a 3-tensor introduces closed indices for the 3-tensor also.

XX. SOLUTION OF THE EQUATIONS

(*a*) The equation of voltage and currents (21.50 and 21.51) may be solved for the unknowns (if some of the impressed quantities are known) *without subdivision.*

For *mesh* networks in finding the inverse of equation 21.50 the inverse of $\mathbf{h}$ would be (by equation 21.31) $\mathbf{m} = -\mathbf{y}_t\cdot(\mathbf{y}\cdot\mathbf{h})\cdot\mathbf{y}$ if $\mathbf{y} = \mathbf{z}^{-1}$ would contain no closed indices; but since $\mathbf{z}$ contains the closed indices $(\alpha)(\beta)$, *the inverse of* $\mathbf{h}$ *is* by equation 21.41.

$$\mathbf{m}^{(\alpha)(\pm\beta)(\gamma)(\delta)} = -\mathbf{y}_t^{(\gamma)}\cdot(\mathbf{y}^{(\alpha)(\pm\beta)}\cdot\mathbf{h})\cdot\mathbf{y}^{(\delta)} \qquad 21.52$$

where half the indices of $\mathbf{y}$ are dropped according to the simplifications of Section XVI.

Since $\mathbf{h}$ is found from the known $\mathbf{m}$ by $\mathbf{h} = -\mathbf{z}_t\cdot(\mathbf{z}\cdot\mathbf{m})\cdot\mathbf{z}$, equation 21.52 becomes in mesh networks

$$\boxed{\mathbf{m}^{(\alpha)(\pm\beta)(\gamma)(\delta)} = -\mathbf{y}_t^{(\gamma)}\cdot\mathbf{z}_t\cdot(\mathbf{y}^{(\alpha)(\pm\beta)}\cdot\mathbf{z}\cdot\mathbf{m})\cdot\mathbf{z}\cdot\mathbf{y}^{(\delta)}} \qquad 21.53$$

where $\mathbf{m}$ *is the modulation tensor of the tube before interconnecting it to the network and* $\mathbf{m}^{(\alpha)(\pm\beta)(\gamma)(\delta)}$ *is its modulation tensor after the interconnection. Also* $\mathbf{z}$ *is the impedance tensor of the tube alone and* $\mathbf{y}^{(\gamma)}$ *is the admittance tensor of the interconnected system.*

Hence, *for mesh networks the inverse of equation* 21.50 *is*

$$\boxed{\Delta \mathbf{i}^{(\alpha)} + \Delta \mathbf{i}^{(\alpha)(\pm\beta)} = \mathbf{y}^{(\alpha)(\beta)} \cdot \Delta \mathbf{e}_{(\beta)} + \Delta \mathbf{e}_{(\gamma)} \cdot \mathbf{m}^{(\alpha)(\pm\beta)(\gamma)(\delta)} \cdot \Delta \mathbf{e}_{(\delta)}} \qquad 21.54$$

giving the fundamental frequency and product frequency current changes due to fundamental frequency voltage changes $\Delta \mathbf{e}_{(\alpha)}$ *impressed around the meshes.* It should be noted that no impressed product frequency voltages $\Delta \mathbf{e}_{(\alpha)\pm(\beta)}$ are assumed.

(*b*) For *junction networks* the inverse of the known **M** is, analogously to equation 21.52,

$$\boxed{\mathbf{H}_{(\alpha)(\pm\beta)(\gamma)(\delta)} = -\mathbf{Z}_{t(\gamma)} \cdot (\mathbf{Z}_{(\alpha)(\pm\beta)} \cdot \mathbf{M}) \cdot \mathbf{Z}_{(\delta)}} \qquad 21.55$$

and the equation of voltage is

$$\boxed{\Delta \mathbf{E}_{(\alpha)} + \Delta \mathbf{E}_{(\alpha)(\pm\beta)} = \mathbf{Z}_{(\alpha)(\beta)} \Delta \mathbf{I}^{(\beta)} + \Delta \mathbf{I}^{(\gamma)} \cdot \mathbf{H}_{(\alpha)(\pm\beta)(\gamma)(\delta)} \cdot \Delta \mathbf{I}^{(\delta)}} \qquad 21.56$$

where $\Delta \mathbf{I}^{(\alpha)}$ are the known impressed fundamental frequency currents.

(*c*) The multiple tensor $\mathbf{m}^{(\alpha)(\pm\beta)(\gamma)(\delta)}$ (or rather $\mathbf{m}^{(\alpha)(\pm\beta)(\gamma)(\pm\delta)}$) contains $4n^4$ 3-tensors **m**. Since only those 3-tensors in which $\alpha = \gamma$ and $\beta = \delta$ are not zero, there are $2n^2$ 3-tensors **m** not equal to zero. They can be arranged in two squares each containing n^2 3-tensors, one giving $\alpha + \beta$, and the other $\alpha - \beta$ frequency currents. Each 3-tensor contains currents of the same frequency flowing in the various circuits.

If k of the circuits has an impressed voltage vector $\Delta e_{(\alpha)}$, each circuit containing n voltages of different frequencies, each of the possible k^2 products produces $2n^2$ product frequency currents in each circuit; that is, in each circuit flow $2n^2k^2$ product frequency currents and nk fundamental frequency currents.

XXI. EXAMPLE OF A TRIODE MESH NETWORK

(*a*) Let the grid and the plate of a triode be connected in series with two coils (Fig. 21.5), and let sets of voltages of various frequencies be impressed in the grid and the plate meshes.

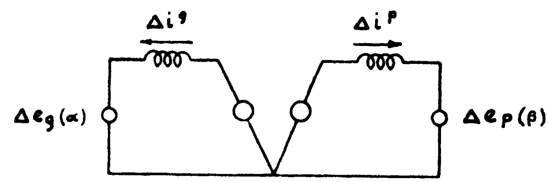


FIG. 21.5.—Triode Mesh Network

In order to find the modulation tensor of the interconnected system by equation 21.53 the following two 2-tensors and one 3-tensor have to be established. The admittance tensor **y** of the *interconnected system*

may be found by the method of Section XV, Chapter XV, and $\mathbf{z}_1$ of the *tube alone* is given in equation 15.55, so that

$$\mathbf{z}_1 = \begin{array}{c|c|c|} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & \dfrac{r_g}{1-\mu_p\mu_g} & \dfrac{-\mu_g r_p}{1-\mu_p\mu_g} \\ \hline \mathbf{p} & \dfrac{\mu_p r_g}{1-\mu_p\mu_g} & \dfrac{r_r}{1-\mu_p\mu_g} \\ \hline \end{array} = \begin{array}{c|c|c|} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & a & b \\ \hline \mathbf{p} & c & d \\ \hline \end{array} \qquad 21.57$$

$$\mathbf{y} = \begin{array}{c|c|c|} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & \left(\dfrac{r_p}{1-\mu_p\mu_g} + Z_p\right)\Big/ Det & \dfrac{\mu_g r_r}{1-\mu_p\mu_g}\Big/ Det \\ \hline \mathbf{p} & \dfrac{\mu_p r_g}{1-\mu_p\mu_g}\Big/ Det & \left(\dfrac{r_g}{1-\mu_p\mu_g} + Z_g\right)\Big/ Det \\ \hline \end{array} = \begin{array}{c|c|c|} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & e & f \\ \hline \mathbf{p} & g & h \\ \hline \end{array} \qquad 21.58$$

where $\quad Det = [(r_g + Z_g)(r_p + z_p) - Z_pZ_g\mu_p\mu_g]/(1 - \mu_p\mu_g).$

The modulation tensor of the tube $\mathbf{m}_1$ is given in equation 21.13 as

$$m^{gvw} = \mathbf{m}_{1g} = \begin{array}{c|c|c|} {}_v\diagdown^w & g & p \\ \hline g & k & l \\ \hline p & m & n \\ \hline \end{array} \qquad m^{pvw} = \mathbf{m}_{1p} = \begin{array}{c|c|c|} {}_v\diagdown^w & g & p \\ \hline g & p & q \\ \hline p & r & s \\ \hline \end{array} \qquad 21.59$$

(*b*) As a first step, the three product matrices of equation 21.53 are established. Replacing $(\alpha)(\pm\beta)$ by (ω) and leaving the closed indices attached to each component of $\mathbf{y}^{(\alpha)}$, $\mathbf{y}^{(\beta)}$, and $\mathbf{y}^{(\omega)}$

$$\mathbf{z}_1\cdot\mathbf{y}^\beta = \begin{array}{c|c|c|} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & ae_\beta + bg_\beta & af_\beta + bh_\beta \\ \hline \mathbf{p} & ce_\beta + dg_\beta & cf_\beta + dh_\beta \\ \hline \end{array} = \begin{array}{c|c|c|} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & A & B \\ \hline \mathbf{p} & C & D \\ \hline \end{array} = \mathbf{p}$$

$$(\mathbf{z}_1\cdot\mathbf{y}^\alpha)_t = \begin{array}{c|c|c|} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & ae_\alpha + bg_\alpha & ce_\alpha + dg_\alpha \\ \hline \mathbf{p} & af_\alpha + bh_\alpha & cf_\alpha + dh_\alpha \\ \hline \end{array} = \begin{array}{c|c|c|} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & E & F \\ \hline \mathbf{p} & G & H \\ \hline \end{array} = \mathbf{s}$$

$$\mathbf{y}^\omega\cdot\mathbf{z}_1 = \begin{array}{c|c|c|} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & e_\omega a + f_\omega c & e_\omega b + f_\omega d \\ \hline \mathbf{p} & g_\omega a + h_\omega c & g_\omega b + h_\omega d \\ \hline \end{array} = \begin{array}{c|c|c|} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & K & L \\ \hline \mathbf{p} & M & N \\ \hline \end{array} = \mathbf{r}$$

Now the value of $\mathbf{m}$ is to be found by equation 21.53; that is, by

$$\mathbf{m}^{\omega\alpha\beta} = (\mathbf{z}_1 \cdot \mathbf{y}^{\alpha})_t \cdot [(\mathbf{y}^{\omega} \cdot \mathbf{z}_1) \cdot \mathbf{m}_1] \cdot (\mathbf{z}_1 \cdot \mathbf{y}^{\beta}) = \mathbf{s} \cdot (\mathbf{r} \cdot \mathbf{m}_1) \cdot \mathbf{p} \qquad 21.60$$

(*c*) The center product $\mathbf{r} \cdot \mathbf{m}_1$ is $r_{zu}m^{uvw}$. Multiplying the cube of m^{uvw} and the square of r_{zu} *along the u direction* as shown in Fig. 21.6

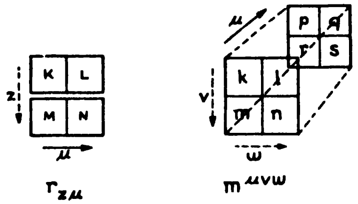


FIG. 21.6.—Calculation of the Product $r_{zu}m^{uvw}$

the product of the *first* row of $\mathbf{r}$ gives

$(\mathbf{r}\cdot\mathbf{m}_1)_g =$ g	g	p
g	$Kk + Lp$	$Kl + Lq$
p	$Km + Lr$	$Kn + Ls$

and the product of the *second* row of $\mathbf{r}$ gives

$(\mathbf{r}\cdot\mathbf{m}_1)_p =$ p	g	p
g	$Mk + Np$	$Ml + Nq$
p	$Mm + Nr$	$Mn + Ns$

These two matrices form part of the cube $\mathbf{r} \cdot \mathbf{m}_1$. (The multiplication may be performed in ways other than that shown.)

(*d*) Each of these matrices should be multiplied by $\mathbf{s}$ as

$\mathbf{s}\cdot(\mathbf{r}\cdot\mathbf{m}_1)_g =$ g	g	p
g	$E(Kk + Lp) + F(Km + Lr)$	$E(Kl + Lq) + F(Kn + Ls)$
p	$G(Kk + Lp) + H(Km + Lr)$	$G(Kl + Lq) + H(Kn + Ls)$

$\mathbf{s}\cdot(\mathbf{r}\cdot\mathbf{m}_1)_p =$ p	g	p
g	$E(Mk + Np) + F(Mm + Nr)$	$E(Ml + Nq) + F(Mn + Ns)$
p	$G(Mk + Np) + H(Mm + Nr)$	$G(Ml + Nq) + H(Mn + Ns)$

(*e*) The last two matrices should be multiplied by $\mathbf{p}$ as $\mathbf{s}\cdot(\mathbf{r}\cdot\mathbf{m}_1)_g\cdot\mathbf{p}$, *giving the component matrices of the final cube* as

$$\mathbf{m}^g = \begin{array}{c|c|c|} {}^{\diagdown g} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & \begin{matrix}[E(Kk+Lp)+F(Km+Lr)]A+\\ [E(Kl+Lq)+F(Kn+Ls)]C\end{matrix} & \begin{matrix}[E(Kk+Lp)+F(Km+Lr)]B+\\ [E(Kl+Lq)+F(Kn+Ls)]D\end{matrix} \\ \hline \mathbf{p} & \begin{matrix}[G(Kk+Lp)+H(Km+Lr)]A+\\ [G(Kl+Lq)+H(Kn+Ls)]C\end{matrix} & \begin{matrix}[G(Kk+Lp)H(Km+Lr)]B+\\ [G(Kl+Lq)+H(Kn+Ls)]D\end{matrix} \\ \hline \end{array} \qquad 21.61$$

With the aid of this matrix the product frequency components of the grid current are $\Delta\mathbf{i}^g = \Delta\mathbf{e}\cdot\mathbf{m}^g\cdot\Delta\mathbf{e}$.

The other matrix of the cube is $\mathbf{s}\cdot(\mathbf{r}\cdot\mathbf{m}_1)_p\cdot\mathbf{p}$ =

$$\mathbf{m}^p = \begin{array}{c|c|c|} {}^{\diagdown p} & \mathbf{g} & \mathbf{p} \\ \hline \mathbf{g} & \begin{matrix}[E(Mk+Np)+F(Mm+Nr)]A+\\ [E(Ml+Nq)+F(Mn+Ns)]C\end{matrix} & \begin{matrix}[E(Mk+Np)+F(Mm+Nr)]B+\\ [E(Ml+Nq)+F(Mn+Ns)]D\end{matrix} \\ \hline \mathbf{p} & \begin{matrix}[G(Mk+Np)+H(Mm+Nr)]A+\\ [G(Ml+Nq)+H(Mn+Ns)]C\end{matrix} & \begin{matrix}[G(Mk+Np)+H(Mm+Nr)]B+\\ [G(Ml+Nq)+H(Mn+Ns)]D\end{matrix} \\ \hline \end{array} \qquad 21.62$$

With the aid of this matrix the product frequency components of $\Delta\mathbf{i}^p$ are found as $\Delta\mathbf{i}^p = \Delta\mathbf{e}\cdot\mathbf{m}^p\cdot\Delta\mathbf{e}$.

The last two matrices $\mathbf{m}^g$ and $\mathbf{m}^p$ are components of the 3-tensor $\mathbf{m}$ *giving all the product frequency currents* (if $\omega = \alpha \pm \beta$)

$$\Delta\mathbf{i}^{(\omega)} = \Delta\mathbf{e}_{(\alpha)}\cdot\mathbf{m}^{(\omega)(\alpha)(\beta)}\cdot\Delta\mathbf{e}_{(\beta)} \qquad 21.63$$

due to the fundamental frequency voltages $\Delta\mathbf{e}_{(\alpha)}$.

(*f*) Evaluating one of the eight components of the 3-tensor $\mathbf{m}$, say M^{pgg}, the component in the upper left-hand corner of matrix 21.62

$$M' = M(AEk+AFm+CEl+CFn) + N(AEp+AFr+CEq+CFs)$$

But

$$M = -\frac{r_g\mu_p Z_{g\omega}}{D_\omega} \qquad A = \frac{r_g Z'_{p\beta}}{D_\beta} \qquad E = \frac{r_g Z'_{p\alpha}}{D_\alpha}$$

$$C = -\frac{r_g\mu_p Z_{p\beta}}{D_\beta} \qquad F = -\frac{r_g\mu_p Z_{p\alpha}}{D_\alpha D} \qquad N = \frac{r_p Z'_{g\omega}}{D_\omega}$$

where

$$D = (r_g + Z_g)(r_p + Z_p) - Z_p Z_g \mu_p \mu_g;\ Z' = r + Z;\ \omega = \alpha \pm \beta$$

Hence, one of the eight components is

$$
\begin{aligned}
M^{(\omega)(\alpha)(\beta)} = \frac{r_g\mu_p Z_{g\omega}}{2D_\alpha D_\beta D_\omega}\Bigg[& Z'_{p\alpha}Z'_{p\beta}\frac{\partial r_g}{\partial e_g} + r_g^2\mu_p Z_{p\alpha}Z'_{p\beta}\frac{\partial G^{gp}}{\partial e_g} \\
& - \mu_p Z_{p\beta}Z'_{p\alpha}\frac{\partial r_g}{\partial e_p} - r_g^2\mu_p^2 Z_{p\alpha}Z_{p\beta}\frac{\partial G^{gp}}{\partial e_p}\Bigg] \\
+ \frac{r_p Z'_{g\omega}}{2D_\alpha D_\beta D_\omega}\Bigg[& \frac{r_g^2}{r_p^2}\mu_p Z_{p\alpha}Z'_{p\beta}\frac{\partial r_p}{\partial e_g} + r_g^2 Z'_{p\alpha}Z'_{p\beta}\frac{\partial G^{pg}}{\partial e_g} \\
& - \frac{r_g^2}{r_p^2}\mu_p^2 Z_{p\alpha}Z_{p\beta}\frac{\partial r_p}{\partial e_p} - r_g^2\mu_p Z_{p\beta}Z'_{p\alpha}\frac{\partial G^{gp}}{\partial e_p}\Bigg]
\end{aligned}
\qquad 21.64
$$

Each Z is calculated for the frequency of its subscript. If the *grid voltage* is the sum of n voltages of different frequencies, there are $2n^2$ different components $M^{(\omega)(\alpha)(\beta)}$, giving $2n^2$ different *plate currents*

$$
\Delta i^{p(\alpha\pm\beta)} = m^{pgg(\alpha\pm\beta)(\alpha)(\beta)}\Delta e_{g(\alpha)}\Delta e_{g(\beta)} \qquad 21.65
$$

by allowing α and β assume the frequency range $1, 2, \cdots n$ (or $\omega_1, \omega_2 \cdots \omega_n$). In the equation, p and g are fixed open indices.

(*g*) When, similarly to equation 21.63, all eight components of $\mathbf{m}$ are calculated, for each frequency of current in $\Delta \mathrm{i}^{(\omega)} = \Delta \mathrm{i}^{(\alpha\pm\beta)}$ there is a cube. Altogether there are $2n^2$ cubes, each cube giving one particular product frequency current flowing in all the circuit. All the currents given by a cube have the same frequency.

CHAPTER XXII

THE ANALYSIS OF NETWORKS

I. TYPES OF NETWORK PROBLEMS

(*a*) The problems that occur in the study of networks may be classified under two main headings:

1. *Given a network, the performance of the network is to be found*, namely the various currents, voltages, powers, impedances, etc., of the given network. Such problems occur in "*network analysis*."

2. *Given the performance of a network, the network itself is to be found.* For instance, it is to be found what values the network impedances must have in order that the network should supply constant currents to certain loads. Such problems occur in "*network synthesis*."

(*b*) The *analysis* of networks may involve simple or complicated manipulations of the tensor equations depending on the problem at hand. Simpler types of manipulations are:

1. Given a network and certain voltages and currents, find the currents, voltages, power, etc., in other parts of the network.

2. A *change* is made in some of the currents or voltages or impedances of the network. Find the changes in various parts of the network.

More complicated manipulations are:

3. The changes are made so that the response of the network at some mesh or junction-pair is a *maximum* or *minimum*. For instance, some impedances may be varied so that the power output to certain loads should be a maximum.

4. The changes to be made depend on the knowledge of the unknown results to be attained.

Even though in many problems the manipulation of geometric objects is easily performed, the *solution* of the equations may become unwieldy or impossible, and approximate or step-by-step solutions have to be resorted to. Such cases occur for instance when the *absolute value* or the *phase angle* of the voltages, currents, or impedances plays a part in the analysis.

All networks will be assumed to be active and asymmetrical, hence they may include multielectrode tubes, rotating electrical machines, and other linear electrical and mechanical networks.

(*c*) *The great advantage of formulating and solving (if possible) network problems in terms of tensor equations is that it is possible to analyze each type of problem once and for all, irrespective of the number of meshes or junction-pairs and irrespective of the manner of interconnection of the coils or their mode of excitation. The analysis needs to be performed only once and then the final answer may be used for any particular case in a routine automatic manner.*

With the ordinary method of analysis both the setting up of the equations and the whole method of analysis have to be repeated for every single example that may come up in engineering practice. Since in ordinary analysis the large number of coils, the great variety of interconnections and hypothetical reference axes obscure the problem, quite often a different method of reasoning is required even for each particular example. In many instances the analysis simply breaks down after the first few steps because of mechanical difficulties in handling large numbers of equations.

II. A METHOD OF ATTACK

(*a*) The problems of network analysis may be formulated as: *Given a network, find its performance.* Since there is an innumerable variety of problems that may arise in network analysis, it is impossible to give general methods of attack. In many problems the following steps may be taken:

1. It is determined whether the network is to be considered a mesh, or a junction, or a complete network.

2. The equation of performance of the network is set up.

3. It is determined how many types of meshes and junction-pairs exist, those of the same type performing identical functions.

4. The equation of performance is divided into as many invariant equations as there are types of meshes and junction-pairs, as shown in equations 19.1 to 19.6.

5. *The set of tensor equations is manipulated analogously to ordinary equations* according to the requirement of the problem.

6. The unknowns, if there are any, are solved for by exact or approximate methods.

(*b*) In the analysis of compound orthogonal networks it should be remembered that:

1. In setting up the compound $\mathbf{z}$: (I) the presence of inactive compound *junction-pairs* is simply ignored; (II) the inactive compound *meshes* are eliminated by the reduction formulas of Chapter X.

2. In setting up the compound $\mathbf{Y}$: (I) the presence of inactive

compound *meshes* is simply ignored; (II) the inactive compound junction-pairs are eliminated by the reduction formulas of Chapter X.

Hence, *by ignoring or eliminating inactive meshes and junction-pairs, the analysis of networks may be performed in terms of active meshes and active junction-pairs only.*

(*c*) Two types of problems will be worked out as examples of this method of attack:

1. Given an *impressed* quantity, find the *response.*
2. Given a *change,* find the response.

During the analysis it will usually be found that at the most only as many inverse matrices have to be calculated as there are types of meshes and junction-pairs. Also each matrix will contain only as many rows and columns as the number of axes contained in the corresponding mesh or junction-pair group.

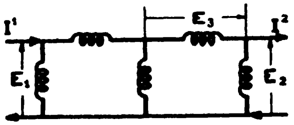
It is emphasized that each type of mesh or junction-pair may contain any number of individual meshes or junction-pairs.

In the manipulation of the equations it must constantly be remembered that even as an intermediate step the inverse of only a square 2-tensor can be used. They are usually only the diagonal components of a compound 2-tensor.

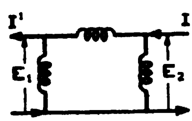
III. ANALYSIS OF π-NETWORKS

(*a*) Given the input $\mathbf{E}_1$ and the output $\mathbf{I}^2$, find the input current $\mathbf{I}^1$ and the difference of potential $\mathbf{E}_2$ across the load.

The junction-pairs of the network are of *three* types: (1) input, (2) output, (3) inactive, the last comprising all the remaining junction-pairs, as shown in Fig. 22.1*a*. If the inactive junction-pairs are elimin-



(*a*) Given junction network (3 junction-pairs)



(*b*) Reduced network (2 junction-pairs)

FIG. 22.1

ated from the equation of current by the method of Section IX, Chapter X, only the input and output junction-pairs remain, shown in Fig. 22.1*b*, whose equation is given in equation 19.2 as

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{E}_2 \\ \mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{Y}^{22}\cdot\mathbf{E}_2 \end{aligned} \qquad 22.1$$

The primes are left out.

(*b*) The load voltage $\mathbf{E}_2$ is found from the second equation as

$$\boxed{\mathbf{E}_2 = \mathbf{Y}^{22-1}\cdot(\mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{I}^2)} \tag{22.2}$$

by calculating the inverse of a tensor $\mathbf{Y}^{22}$ having as many rows and columns as there are load axes.

The input current $\mathbf{I}'$ is found by substituting $\mathbf{E}_2$ into the first equation.

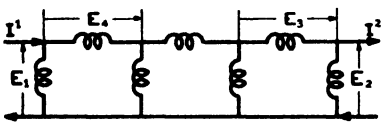
$$\mathbf{I}^1 = \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{Y}^{22-1}\cdot(\mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{I}^2)$$

$$\boxed{\mathbf{I}^1 = (\mathbf{Y}^{11} - \mathbf{Y}^{12}\cdot\mathbf{Y}^{22-1}\cdot\mathbf{Y}^{21})\cdot\mathbf{E}_1 + \mathbf{Y}^{12}\cdot\mathbf{Y}^{22-1}\cdot\mathbf{I}^2} \tag{22.3}$$

IV. OPEN-CIRCUIT VOLTAGES

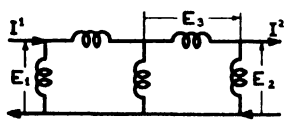
(*a*) Given again the input $\mathbf{E}_1$ and the output $\mathbf{I}^2$, find the difference of potentials $\mathbf{E}_3$ appearing across several (not all) of the remaining junction-pairs.

There are *four* types of junction-pairs as shown in Fig. 22.2*a*: (1) input, (2) output, (3) open-circuit terminals, (4) inactive. The



(*a*) Given junction network
(4 junction-pairs)

Given: E_1, I^2
Find: E_3



(*b*) Reduced network
(3 junction-pairs)

FIG. 22.2

compound network must have consequently four junction-pairs (or five junctions).

The equation of current $\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}$ of the junction network is to be divided into *four* equations. If the inactive equation is *not* eliminated, the four equations are:

$$\begin{aligned}
\mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{E}_2 - \mathbf{Y}^{13}\cdot\mathbf{E}_3 - \mathbf{Y}^{14}\cdot\mathbf{E}_4\\
\mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{Y}^{22}\cdot\mathbf{E}_2 - \mathbf{Y}^{23}\cdot\mathbf{E}_3 - \mathbf{Y}^{24}\cdot\mathbf{E}_4\\
0 &= \mathbf{Y}^{31}\cdot\mathbf{E}_1 - \mathbf{Y}^{32}\cdot\mathbf{E}_2 - \mathbf{Y}^{33}\cdot\mathbf{E}_3 - \mathbf{Y}^{34}\cdot\mathbf{E}_4\\
0 &= \mathbf{Y}^{41}\cdot\mathbf{E}_1 - \mathbf{Y}^{42}\cdot\mathbf{E}_2 - \mathbf{Y}^{43}\cdot\mathbf{E}_3 - \mathbf{Y}^{44}\cdot\mathbf{E}_4
\end{aligned} \tag{22.4}$$

All the *diagonal* tensors $\mathbf{Y}^{11}$, $\mathbf{Y}^{22}$, $\mathbf{Y}^{33}$, and $\mathbf{Y}^{44}$ have square matrices, and their inverse can be calculated.

The problem is to find $\mathbf{E}_3$ if $\mathbf{E}_1$ and $\mathbf{I}_2$ are known.

(*b*) The first step is to eliminate the inactive $\mathbf{E}_4$ from the last equation.

$$\mathbf{E}_4 = \mathbf{Y}^{44-1}\cdot(\mathbf{Y}^{41}\cdot\mathbf{E}_1 - \mathbf{Y}^{42}\cdot\mathbf{E}_2 - \mathbf{Y}^{43}\cdot\mathbf{E}_3)$$

Substituting into the remaining equations

$$\begin{aligned}
\mathbf{I}^1 &= (\mathbf{Y}^{11} - \mathbf{Y}^{14}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{41})\cdot\mathbf{E}_1 - (\mathbf{Y}^{12} - \mathbf{Y}^{14}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{42})\cdot\mathbf{E}_2 \\
&\qquad - (\mathbf{Y}^{13} - \mathbf{Y}^{14}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{43})\cdot\mathbf{E}_3 \\
\mathbf{I}^2 &= (\mathbf{Y}^{21} - \mathbf{Y}^{24}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{41})\cdot\mathbf{E}_1 - (\mathbf{Y}^{22} - \mathbf{Y}^{24}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{42})\cdot\mathbf{E}_2 \\
&\qquad - (\mathbf{Y}^{23} - \mathbf{Y}^{24}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{43})\cdot\mathbf{E}_3 \qquad 22.5 \\
0 &= (\mathbf{Y}^{31} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{41})\cdot\mathbf{E}_1 - (\mathbf{Y}^{32} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{42})\cdot\mathbf{E}_2 \\
&\qquad - (\mathbf{Y}^{33} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{43})\cdot\mathbf{E}_3
\end{aligned}$$

If so desired they may be replaced by

$$\begin{aligned}
\mathbf{I}^1 &= \mathbf{Y}^{11'}\cdot\mathbf{E}_1 - \mathbf{Y}^{12'}\cdot\mathbf{E}_2 - \mathbf{Y}^{13'}\cdot\mathbf{E}_3 \\
\mathbf{I}^2 &= \mathbf{Y}^{21'}\cdot\mathbf{E}_1 - \mathbf{Y}^{22'}\cdot\mathbf{E}_2 - \mathbf{Y}^{23'}\cdot\mathbf{E}_3 \qquad 22.6 \\
0 &= \mathbf{Y}^{31'}\cdot\mathbf{E}_1 - \mathbf{Y}^{32'}\cdot\mathbf{E}_2 - \mathbf{Y}^{33'}\cdot\mathbf{E}_3
\end{aligned}$$

represented by an ***equivalent double-π network*** with three junction-pairs as shown in Fig. 22.2*b*. The primed admittances represent the self- and mutual admittances of the first three junction-pairs measured with the fourth junction-pair open-circuited, analogously to the admittances of equation 22.1.

(*c*) In the last set of equations $\mathbf{E}_1$ and $\mathbf{I}^2$ are known and $\mathbf{E}_3$ is to be found. Finding $\mathbf{E}_2$ from the second equation as

$$\mathbf{E}_2 = \mathbf{Y}^{22'-1}\cdot(\mathbf{Y}^{21'}\cdot\mathbf{E}_1 - \mathbf{Y}^{23'}\cdot\mathbf{E}_3 - \mathbf{I}^2)$$

and substituting it into the third equation

$$0 = \mathbf{Y}^{31'}\cdot\mathbf{E}_1 - \mathbf{Y}^{32'}\cdot\mathbf{Y}^{22'-1}\cdot(\mathbf{Y}^{21'}\cdot\mathbf{E}_1 - \mathbf{Y}^{23'}\cdot\mathbf{E}_3 - \mathbf{I}^2) - \mathbf{Y}^{33'}\cdot\mathbf{E}_3$$

an equation is found containing only the known $\mathbf{E}_1$ and $\mathbf{I}^2$ and the unknown $\mathbf{E}_3$. Rearranging

$$\begin{aligned}
0 = (\mathbf{Y}^{31'} - \mathbf{Y}^{32'}\cdot\mathbf{Y}^{22'-1}\cdot\mathbf{Y}^{21'})\cdot\mathbf{E}_1 - (\mathbf{Y}^{33'} - \mathbf{Y}^{32'}\cdot\mathbf{Y}^{22'-1}\cdot\mathbf{Y}^{23'})\cdot\mathbf{E}_3 \\
+ \mathbf{Y}^{32'}\cdot\mathbf{Y}^{22'-1}\cdot\mathbf{I}^2
\end{aligned}$$

Solving for the unknown $\mathbf{E}_3$

$$\begin{aligned}
\mathbf{E}_3 = (\mathbf{Y}^{33'} - \mathbf{Y}^{32'}\cdot\mathbf{Y}^{22'-1}\cdot\mathbf{Y}^{23'})^{-1}\cdot[(\mathbf{Y}^{31'} - \mathbf{Y}^{32'}\cdot\mathbf{Y}^{22'-1}\cdot\mathbf{Y}^{21'})\cdot\mathbf{E}_1 \\
+ \mathbf{Y}^{32'}\cdot\mathbf{Y}^{22'-1}\cdot\mathbf{I}^2] \qquad 22.7
\end{aligned}$$

(*d*) In the calculation of $\mathbf{E}_3$ three inverse matrices have to be calculated:

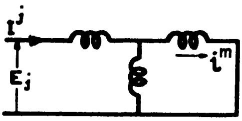
1. First the inverse of $\mathbf{Y}^{44}$, containing as many rows and columns as there are inactive junction-pairs ($\mathbf{E}_4$).

2. Then the inverse of $\mathbf{Y}^{22'} = \mathbf{Y}^{22} - \mathbf{Y}^{24}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{42}$, containing as many rows and columns as $\mathbf{Y}^{22}$ has, that is as many as there are load junction-pairs.

3. Finally the inverse of $\mathbf{Y}^{33'} - \mathbf{Y}^{32'}\cdot\mathbf{Y}^{22'-1}\cdot\mathbf{Y}^{23'}$, having as many rows and columns as $\mathbf{Y}^{33'} = \mathbf{Y}^{33} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{43}$ or $\mathbf{Y}^{33}$ has, that is, as many as the number of open-circuited terminals.

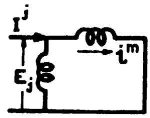
V. MESH CURRENTS IN AN ORTHOGONAL NETWORK

(*a*) As an example of a problem involving an orthogonal network, assume that the current $\mathbf{I}^j$ flowing through *some* (not all) of the junction-pairs of a network into outside loads are known as shown in Fig. 22.3*a*. Find *all* the mesh-currents $\mathbf{i}^m$.



(*a*) Given orthogonal network (1 mesh, 2 junction-pairs)

Given : I^j
Find : i^m



(*b*) Ignoring inactive junction-pair

FIG. 22.3

Since the voltages do not play any part, the equation of voltage $\mathbf{E} + \mathbf{e} = \mathbf{z}\cdot(\mathbf{i} + \mathbf{I})$ is set up, containing as many equations as the *total* number of meshes and the number of *needed* junction-pairs. In the present case the number of known currents $\mathbf{I}$ is the same as the number of needed junction-pairs.

The orthogonal equations of voltage may be expressed from equation 19.3 as

$$\begin{aligned} 0 &= \mathbf{z}_{mm}\cdot\mathbf{i}^m - \mathbf{z}_{mj}\cdot\mathbf{I}^j \\ \mathbf{E}_j &= \mathbf{z}_{jm}\cdot\mathbf{i}^m - \mathbf{z}_{jj}\cdot\mathbf{I}^j \end{aligned} \qquad 22.8$$

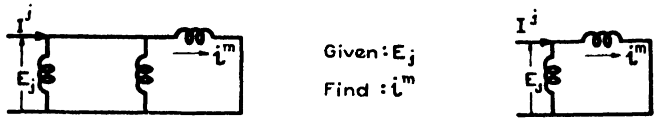
since no impressed coil voltages $\mathbf{e}_m$ are assumed. In these equations $\mathbf{I}^j$ are known and $\mathbf{i}^m$ (and $\mathbf{E}_j$) are unknown. *The unknown mesh currents* $\mathbf{i}^m$ *are found from* the first equation

$$\boxed{\mathbf{i}^m = -\mathbf{z}_{mm}^{-1}\cdot\mathbf{z}_{mj}\cdot\mathbf{I}^j} \qquad 22.9$$

The matrix of $\mathbf{z}_{mm}$ whose inverse has to be calculated has as many rows and columns as the number of meshes. A numerical example has

been worked out in Section XI, Chapter XVI, containing also impressed mesh voltages $\mathbf{e}_m$.

(*b*) Instead of $\mathbf{I}$ let $\mathbf{E}$ be assumed to be known as shown in Fig. 22.4, and let some (not all) of the mesh currents $\mathbf{i}^m$ be found. In that case the equation of current $\mathbf{i} + \mathbf{I} = \mathbf{Y}\cdot(\mathbf{E} + \mathbf{e})$ is set up, containing as many equations as the *total* number of junction-pairs and the number



(*a*) Given orthogonal network (2 meshes, 1 junction-pair)

(*b*) Ignoring active mesh in setting up Y

FIG. 22.4

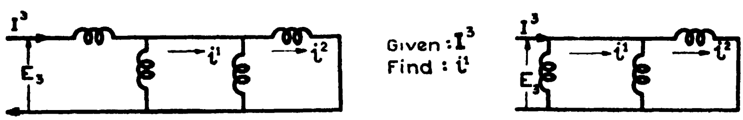
of *needed* meshes. (If only a few mesh currents $\mathbf{i}^m$ are to be calculated then the number of needed meshes is less than the number of total meshes.) In setting up $\mathbf{Y}$ the inactive meshes may be ignored; only the needed meshes are to be considered.

The orthogonal equations of currents are from equation 19.5 (where $\mathbf{e}_m = \mathbf{e}_j = \mathbf{I}^m = \mathbf{i}^j = \mathbf{E}_m = 0$)

$$\begin{aligned} \mathbf{i}^m &= \mathbf{Y}^{mj}\cdot\mathbf{E}_j \\ \mathbf{I}^j &= \mathbf{Y}^{jj}\cdot\mathbf{E}_j \end{aligned} \qquad 22.10$$

The currents $\mathbf{i}^m$ are found without calculating any inverse matrices.

(*c*) Again let $\mathbf{I}$ flowing through *some* of the junction-pairs be known and let only *some* of the mesh currents $\mathbf{i}^m$ be calculated. Now the meshes are to be divided into two types, while of the junction-pairs only the *needed* ones are used as shown in Fig. 22.5.



(*a*) Given orthogonal network (2 meshes, 2 junction-pairs)

(*b*) Ignoring inactive junction-pair

FIG. 22.5

Hence in the orthogonal equation of voltage the first equation (representing the mesh axes) is divided into two equations as

$$\begin{aligned} \mathbf{z}_{m_1m_1}\mathbf{i}^{m_1} + \mathbf{z}_{m_1m_2}\mathbf{i}^{m_2} - \mathbf{z}_{m_1j_1}\mathbf{I}^{j_1} &= 0 = \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{i}^2 - \mathbf{z}_{13}\cdot\mathbf{I} \\ \mathbf{z}_{m_2m_1}\mathbf{i}^{m_1} + \mathbf{z}_{m_2m_2}\mathbf{i}^{m_2} - \mathbf{z}_{m_2j_1}\mathbf{I}^{j_1} &= 0 = \mathbf{z}_{21}\cdot\mathbf{i}^1 + \mathbf{z}_{22}\cdot\mathbf{i}^2 - \mathbf{z}_{23}\cdot\mathbf{I} \\ \mathbf{z}_{j_1m_1}\mathbf{i}^{m_1} + \mathbf{z}_{j_1m_2}\mathbf{i}^{m_2} - \mathbf{z}_{j_1j_1}\mathbf{I}^{j_1} &= \mathbf{E} = \mathbf{z}_{31}\cdot\mathbf{i}^1 + \mathbf{z}_{32}\cdot\mathbf{i}^2 - \mathbf{z}_{33}\cdot\mathbf{I} \end{aligned} \qquad 22.11$$

where $\mathbf{i}^1$ represents those mesh currents about which knowledge is needed, $\mathbf{i}^2$ the remaining mesh currents, and $\mathbf{I}^3$ the junction-pair currents as shown in Fig. 22.5*b*. In these equations $\mathbf{I}^3$ is known and $\mathbf{i}^1$ is to be calculated ($\mathbf{E}_3$ is unknown).

Eliminating $\mathbf{i}^2$ from the second equation

$$\mathbf{i}^2 = \mathbf{z}_{22}^{-1}\cdot(\mathbf{z}_{23}\cdot\mathbf{I}^3 - \mathbf{z}_{21}\cdot\mathbf{i}^1) \qquad 22.12$$

Substituting into the first equation

$$0 = \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot(\mathbf{z}_{23}\cdot\mathbf{I}^3 - \mathbf{z}_{21}\cdot\mathbf{i}^1) - \mathbf{z}_{13}\cdot\mathbf{I}^3$$

Rearranging

$$(\mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{21})\cdot\mathbf{i}^1 = (\mathbf{z}_{13} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{23})\cdot\mathbf{I}^3$$

From this *the mesh-currents* $\mathbf{i}^1$ *are in terms of the juuction currents* $\mathbf{I}^3$

$$\boxed{\mathbf{i}^1 = (\mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{21})^{-1}\cdot(\mathbf{z}_{13} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{23})\cdot\mathbf{I}^3} \qquad 22.13$$

The two inverse matrices to be calculated have as many rows as the number of inactive and active meshes respectively.

The currents $\mathbf{i}^2$ in the inactive meshes may be calculated by equation 22.12, and the differences of potentials $\mathbf{E}$ across the junction-pairs may be calculated by substituting the values of $\mathbf{i}^1$ and $\mathbf{i}^2$ into the third equation of 22.11.

VI. TRIODE CIRCUITS

(*a*) It was shown in Section XVII, Chapter XV, that a triode may be replaced by a single plate coil with impedance r_p in series with an impressed voltage $\mu_p E_g$, where E_g is the difference of potential appearing between the grid and the filament. Hence when the grid coil is eliminated and the number of meshes is reduced by one, its place is taken by a junction-pair. With several tubes several meshes are eliminated, their place being taken by the same number of junction-pairs.

In replacing the grid coil by $\mu_p E_g$ the number of equations to be set up is not decreased since a mesh network is replaced by an orthogonal network. However, *the rank of the matrix whose inverse has to be calculated is decreased.* The rank of this matrix is the same as the number of meshes.

(*b*) Considering the orthogonal network of Fig. 22.6, its impedance tensor $\mathbf{z}$ is calculated by temporarily short-circuiting the grid junction-pairs. All the other junction-pairs are inactive, and in setting up $\mathbf{z}$ their presence may be ignored.

After $\mathbf{z}$ of the actual network as an orthogonal network has been established, its meshes may be divided into two types, those containing μE_g and the remaining ones. Hence its compound orthogonal network has two meshes and one junction-pair as shown in Fig. 22.6, and its equation of voltage is

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{i}^2 \\ \boldsymbol{\mu}\cdot\mathbf{E}_g &= \mathbf{z}_{21}\cdot\mathbf{i}^1 + \mathbf{z}_{22}\cdot\mathbf{i}^2 \\ -\mathbf{E}_g &= \mathbf{z}_{31}\cdot\mathbf{i}^1 + \mathbf{z}_{32}\cdot\mathbf{i}^2 \end{aligned} \qquad 22.14$$

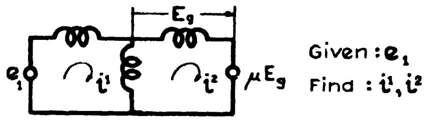


FIG. 22.6.—Tube Network as an Orthogonal Network

the junction-pair current $\mathbf{I}^3$ being zero.

It should be noted that there are two impressed mesh voltages, $\mathbf{e}_1$ and $\boldsymbol{\mu}\cdot\mathbf{E}_g$ (where $\boldsymbol{\mu}$ has a diagonal matrix), but only one of them is known, namely $\mathbf{e}_1$. The other is a function of the difference of potential $-\mathbf{E}_g$ appearing across the compound junction-pair.

(c) $\mathbf{E}_g$ may be eliminated by substituting the third equation into the second as

$$-\boldsymbol{\mu}\cdot(\mathbf{z}_{31}\cdot\mathbf{i}^1 + \mathbf{z}_{32}\cdot\mathbf{i}^2) = \mathbf{z}_{21}\cdot\mathbf{i}^1 + \mathbf{z}_{22}\cdot\mathbf{i}^2$$

$$(\boldsymbol{\mu}\cdot\mathbf{z}_{31} + \mathbf{z}_{21})\cdot\mathbf{i}^1 + (\boldsymbol{\mu}\cdot\mathbf{z}_{32} + \mathbf{z}_{22})\cdot\mathbf{i}^2 = 0$$

Hence the remaining two equations are

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{i}^2 \\ 0 &= \mathbf{z}'_{21}\cdot\mathbf{i}^1 + \mathbf{z}'_{22}\cdot\mathbf{i}^2 \end{aligned} \qquad 22.15$$

where

$$\mathbf{z}'_{21} = \boldsymbol{\mu}\cdot\mathbf{z}_{31} + \mathbf{z}_{21} \text{ and } \mathbf{z}'_{22} = \boldsymbol{\mu}\cdot\mathbf{z}_{32} + \mathbf{z}_{22} \qquad 22.16$$

In these two tensor equations the grid junction-pairs are not present. The second equation represents the plate meshes, the first equation the remaining meshes, some of them having impressed voltages.

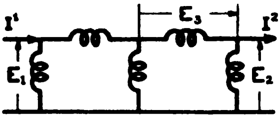
From this point the analysis is the same as that of any other compound mesh-network. The meshes represented by the first equation of 22.14 may be further subdivided into those containing impressed voltages and those that do not, or it may be subdivided into input, output, and inactive meshes, or in any other manner.

VII. GENERAL CIRCUIT PARAMETERS

(a) In transmission-line problems it often occurs that the input quantities $\mathbf{E}_1$ and $\mathbf{I}^1$ (or $\mathbf{e}_1$ and $\mathbf{i}^1$) are known while the output quan-

tities $\mathbf{E}_2$ and $\mathbf{I}^2$ (or $\mathbf{e}_2$ and $\mathbf{i}^2$) are unknown. The transmission system will be assumed as a junction network.

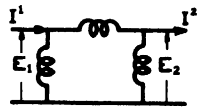
In the present problem there are three types of junction-pairs: (1) input, (2) output, (3) inactive (Fig. 22.7). Eliminating the inactive



(a) Given junction network (3 junction-pairs)

Given: E_1, I^1

Find : E_2, I^2



(b) Reduced network (2 junction-pairs)

FIG. 22.7

axes by the method of Section IX, Chapter X, the equations of the input and output junction-pairs (Fig. 22.7*b*) are

$$\begin{array}{l|l} \mathbf{I}^1 = \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{E}_2 & I^m = Y^{mn}E_n - Y^{mv}E_v \\ \mathbf{I}^2 = \mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{Y}^{22}\cdot\mathbf{E}_2 & I^u = Y^{un}E_n - Y^{uv}E_v \end{array} \qquad 22.17$$

where $\mathbf{E}_1$ and $\mathbf{I}^1$ are known, $\mathbf{E}_2$ and $\mathbf{I}^2$ are unknown.

(*b*) Eliminating $\mathbf{E}_2$ from the second equation

$$\mathbf{E}_2 = \mathbf{Y}^{22-1}\cdot(\mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{I}^2) \qquad 22.18$$

Substituting into the first equation

$$\mathbf{I}^1 = \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{Y}^{22-1}\cdot(\mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{I}^2)$$

Rearranging

$$\mathbf{I}^1 = (\mathbf{Y}^{11} - \mathbf{Y}^{12}\cdot\mathbf{Y}^{22-1}\cdot\mathbf{Y}^{21})\cdot\mathbf{E}_1 + \mathbf{Y}^{12}\cdot\mathbf{Y}^{22-1}\cdot\mathbf{I}^2 \qquad 22.19$$

In the last equation the unknown is $\mathbf{I}^2$. Solving for it

$$\mathbf{I}^2 = \mathbf{Y}^{22}\cdot\mathbf{Y}^{12-1}\cdot[\mathbf{I}^1 - (\mathbf{Y}^{11} - \mathbf{Y}^{12}\cdot\mathbf{Y}^{22-1}\cdot\mathbf{Y}^{21})\cdot\mathbf{E}_1]$$

$$\mathbf{I}^2 = \mathbf{Y}^{22}\cdot\mathbf{Y}^{12-1}\cdot\mathbf{I}^1 - \mathbf{Y}^{22}\cdot\mathbf{Y}^{12-1}\cdot\mathbf{Y}^{11}\cdot\mathbf{E}_1 + \mathbf{Y}^{21}\cdot\mathbf{E}_1$$

Hence the output current in terms of the input quantities is

$$\boxed{\mathbf{I}^2 = (\mathbf{Y}^{21} - \mathbf{Y}^{22}\cdot\mathbf{Y}^{12-1}\cdot\mathbf{Y}^{11})\cdot\mathbf{E}_1 + \mathbf{Y}^{22}\cdot\mathbf{Y}^{12-1}\cdot\mathbf{I}^1} \qquad 22.20$$

Since the inverse of $\mathbf{Y}^{12}$ has to be calculated, $\mathbf{Y}^{12}$ *is a square tensor only if the number of the assumed input and output junction-pairs is equal.*

When the compound input and output terminals have different number of junction-pairs, equation 22.20 represents *for any particular network* a set of ordinary linear equations in which the number of unknowns (the components of $\mathbf{I}^2$) is smaller or larger than the num-

ber of equations (the components of $\mathbf{I}^1$). Such linear equations may be solved with the method shown in Section XVI, Chapter X.

Substituting the above value of $\mathbf{I}^2$ into equation 22.18, the value of $\mathbf{E}_2$ is

$$\mathbf{E}_2 = \mathbf{Y}^{22-1}\cdot\mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{Y}^{22-1}\cdot(\mathbf{Y}^{21} - \mathbf{Y}^{22}\cdot\mathbf{Y}^{12-1}\cdot\mathbf{Y}^{11})\cdot\mathbf{E}_1 - \mathbf{Y}^{12-1}\cdot\mathbf{I}^1$$

Hence the output voltage in terms of the input quantities is

$$\boxed{\mathbf{E}_2 = \mathbf{Y}^{12-1}\cdot\mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12-1}\cdot\mathbf{I}^1} \qquad 22.21$$

The above two equations may be written as

$$\begin{array}{l|l} \mathbf{E}_2 = \mathbf{A}\cdot\mathbf{E}_1 + \mathbf{B}\cdot\mathbf{I}^1 & E_u = A_u^{\cdot m}E_m + B_{um}I^m \\ \mathbf{I}^2 = \mathbf{C}\cdot\mathbf{E}_1 + \mathbf{D}\cdot\mathbf{I}^1 & I^u = C^{um}E_m + D^u_{\cdot m}I^m \end{array} \qquad 22.22$$

where the *coefficients* of the output quantities are called "*general circuit parameters*," defined as

$$\begin{array}{l|l} \mathbf{A} = \mathbf{Y}^{12-1}\cdot\mathbf{Y}^{11} & A_u^{\cdot m} = Z_{um}Y^{mn} \\ \mathbf{B} = -\mathbf{Y}^{12-1} & B_{um} = -Z_{um} \\ \mathbf{C} = \mathbf{Y}^{21} - \mathbf{Y}^{22}\cdot\mathbf{Y}^{12-1}\cdot\mathbf{Y}^{11} & C^{um} = Y^{um} - Y^{uv}Z_{vn}Y^{nm} \\ \mathbf{D} = \mathbf{Y}^{22}\cdot\mathbf{Y}^{12-1} & D^u_{\cdot m} = Y^{uv}Z_{vm} \end{array} \qquad 22.23$$

It should be especially noted that each general circuit parameter has its indices in different positions.

(*c*) These parameters are not independent of one another if the network is symmetrical. Then $\mathbf{Y}^{12} = \mathbf{Y}_t^{21}$, also $\mathbf{Y}^{11} = \mathbf{Y}_t^{11}$ and

$$\mathbf{A}\cdot\mathbf{D}_t - \mathbf{B}\cdot\mathbf{C}_t = \mathbf{Y}^{12-1}\cdot\mathbf{Y}^{11}\cdot\mathbf{Y}_t^{12-1}\cdot\mathbf{Y}_t^{22} + \mathbf{Y}^{12-1}\cdot(\mathbf{Y}_t^{21} - \mathbf{Y}_t^{11}\cdot\mathbf{Y}_t^{12-1}\cdot\mathbf{Y}_t^{22})$$

$$= \mathbf{Y}^{12-1}\cdot\mathbf{Y}_t^{21} = \mathbf{I} = \text{unit tensor}$$

That is

$$\boxed{\mathbf{A}\cdot\mathbf{D}_t - \mathbf{B}\cdot\mathbf{C}_t = \mathbf{I}} \quad \Big| \quad \boxed{A_u^{\cdot m}D_m^{\cdot v} - B_{um}C^{mv} = \delta_u^v} \qquad 22.24$$

With the aid of these equations the output quantities may be found by measurements made at the input.

For a mesh network similar relations may be derived with $\mathbf{z}$ instead of $\mathbf{Y}$.

VIII. DISTRIBUTION NETWORKS

(*a*) In a distribution network a set of junction-pairs is always kept at a *constant difference of potential* $\mathbf{E}$ by means of voltage regulators. Or rather the absolute value $\sqrt{E_1^2 + E_2^2}$ of each component

$E_1 + jE_2$ of $\mathbf{E}$ remains constant while other quantities vary. The corresponding $\mathbf{I}$ is zero.

As a simple example of a distribution network problem, let the junction-pairs be divided into three sets as shown in Fig. 22.8.

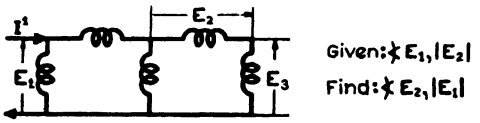


FIG. 22.8.—Distribution Network (3 junction-pairs)

1. One set contains the generators $\mathbf{E}_1$. The absolute value of each component of $\mathbf{E}_1$ is unknown, but their phase angles are known.

2. The other set contains the voltage regulators $\mathbf{E}_2$. The absolute value of each component of $\mathbf{E}_2$ is known and their phase angles are unknown.

3. The remaining junction-pairs with a difference of potential $\mathbf{E}_3$ are inactive.

The equation of current of the junction network $\mathbf{I} = \mathbf{Y} \cdot \mathbf{E}$ is divided into *three* tensor equations as

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{11} \cdot \mathbf{E}_1 - \mathbf{Y}^{12} \cdot \mathbf{E}_2 - \mathbf{Y}^{13} \cdot \mathbf{E}_3 \\ 0 &= \mathbf{Y}^{21} \cdot \mathbf{E}_1 - \mathbf{Y}^{22} \cdot \mathbf{E}_2 - \mathbf{Y}^{23} \cdot \mathbf{E}_3 \\ 0 &= \mathbf{Y}^{31} \cdot \mathbf{E}_1 - \mathbf{Y}^{32} \cdot \mathbf{E}_2 - \mathbf{Y}^{33} \cdot \mathbf{E}_3 \end{aligned} \qquad 22.25$$

(*b*) Eliminating the inactive $\mathbf{E}_3$ from the third equation

$$\mathbf{E}_3 = \mathbf{Y}^{33-1} \cdot (\mathbf{Y}^{31} \cdot \mathbf{E}_1 - \mathbf{Y}^{32} \cdot \mathbf{E}_2)$$

Substituting it into the second equation

$$0 = \mathbf{Y}^{21} \cdot \mathbf{E}_1 - \mathbf{Y}^{22} \cdot \mathbf{E}_2 - \mathbf{Y}^{23} \cdot \mathbf{Y}^{33-1} \cdot (\mathbf{Y}^{31} \cdot \mathbf{E}_1 - \mathbf{Y}^{32} \cdot \mathbf{E}_2)$$

Rearranging, *the following relation exists between the generator voltage* $\mathbf{E}_1$ *and the regulator voltage* $\mathbf{E}_2$

$$\boxed{(\mathbf{Y}^{21} - \mathbf{Y}^{23} \cdot \mathbf{Y}^{33-1} \cdot \mathbf{Y}^{31}) \cdot \mathbf{E}_1 = (\mathbf{Y}^{22} - \mathbf{Y}^{23} \cdot \mathbf{Y}^{33-1} \cdot \mathbf{Y}^{32}) \cdot \mathbf{E}_2} \qquad 22.26$$

The matrix of $\mathbf{Y}^{22}$ is a square matrix, similarly $\mathbf{Y}^{22} - \mathbf{Y}^{23} \cdot \mathbf{Y}^{33-1} \cdot \mathbf{Y}^{32}$. Solving for $\mathbf{E}_2$

$$\boxed{\mathbf{E}_2 = (\mathbf{Y}^{22} - \mathbf{Y}^{23} \cdot \mathbf{Y}^{33-1} \cdot \mathbf{Y}^{32})^{-1} \cdot (\mathbf{Y}^{21} - \mathbf{Y}^{23} \cdot \mathbf{Y}^{33-1} \cdot \mathbf{Y}^{31}) \cdot \mathbf{E}_1} \qquad 22.27$$

$$\boxed{\mathbf{E}_2 = \mathbf{Y}^{22'-1} \cdot \mathbf{Y}^{21'} \cdot \mathbf{E}_1 = \mathbf{Y}_2^{-1} \cdot \mathbf{E}_1} \qquad 22.28$$

representing a relation between the generator voltage $\mathbf{E}_1$ and the regulator voltages $\mathbf{E}_2$. *The unknowns are the absolute value of* $\mathbf{E}_1$ *and*

the phase angle of $\mathbf{E}_2$. The matrix of $\mathbf{Y}_2^{-1}$ has as many columns as there are voltage regulators and as many rows as there are generators. Exact method of solution of such equations is not undertaken in these pages.

(*c*) ***Approximate*** solutions may be found in special cases. For instance:

1. Let the number of generators be the same as the number of regulators.

2. Let all generators and all voltage regulators be in phase.

Then since the components of $\mathbf{E}_1$ and $\mathbf{E}_2$ are real numbers, equation 22.28 may be solved by *using only the real part of* $\mathbf{Y}_2^{-1}$, namely $\mathbf{Y}^r$, where $\mathbf{Y}_2^{-1} = \mathbf{Y}^r + j\mathbf{Y}^i$ so that the equation $\mathbf{E}_{2r} = \mathbf{Y}^r \cdot \mathbf{E}_{1r}$ may be solved as

$$\boxed{\mathbf{E}_{1r} = \mathbf{Y}^{r-1} \cdot \mathbf{E}_{2r}} \qquad 22.29$$

giving the unknown generated voltages $|\mathbf{E}_1|$.

(*d*) If it is assumed that the generated voltages only are in phase ($\mathbf{E}_1 = \mathbf{E}_{1r}$) then the real and imaginary components of $\mathbf{E}_2 = \mathbf{E}_{2r} + j\mathbf{E}_{2i}$ may be found by *successive approximations* if the *absolute* value of each of their components is known.

That is, if $\mathbf{E}_1$ has only real values, then equation 22.28 may be written as

$$\begin{aligned} \mathbf{E}_{2r} &= \mathbf{Y}^r \cdot \mathbf{E}_{1r} \\ \mathbf{E}_{2i} &= \mathbf{Y}^i \cdot \mathbf{E}_{1r} \end{aligned} \qquad 22.30$$

Now the following steps are made:

1. Assuming the absolute value of each component of $\mathbf{E}_2$ to be equal to its real component in $\mathbf{E}_{2r}$, the first equation gives $\mathbf{E}_{1r}$.

2. Substituting $\mathbf{E}_{1r}$ into the second equation gives the imaginary components of $\mathbf{E}_2$.

3. The *absolute* value of each component of $\mathbf{E}_2$ and their imaginary component being known, from the equations $(E)^2 = (E_r)^2 + (E_i)^2$, the ***real*** value of each component of $\mathbf{E}_2$ may be calculated.

The more correct values of $\mathbf{E}_{2r}$ being known, the above steps may be repeated several times, each time getting more correct results for $\mathbf{E}_{2r}$ and $\mathbf{E}_{2i}$.

The generator currents $\mathbf{I}^1$ are found by substituting the values of $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ (given in equation 22.29) into the first equation of equation 22.25.

IX. CHANGES IN ELECTRICAL QUANTITIES

(*a*) When the currents and voltages in a network are subjected to changes $\Delta\mathbf{E}$ and $\Delta\mathbf{I}$, because of the linearity of the system each impressed change produces its own response change irrespective of the presence of $\mathbf{E}$ and $\mathbf{I}$ at the instant of change. This may be seen from the reasoning in the case of a mesh network, whose equation of voltage *before* the change is

$$\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$$

Changing $\mathbf{e}$ to $\mathbf{e} + \Delta\mathbf{e}$ and $\mathbf{i}$ to $\mathbf{i} + \Delta\mathbf{i}$ (leaving $\mathbf{z}$ unchanged), the equation of voltage *after* the change is

$$\mathbf{e} + \Delta\mathbf{e} = \mathbf{z} \cdot (\mathbf{i} + \Delta\mathbf{i})$$

Subtracting the original equation $\mathbf{e} = \mathbf{z} \cdot \mathbf{i}$, the *equation of voltage change* is

$$\Delta\mathbf{e} = \mathbf{z} \cdot \Delta\mathbf{i}$$

That is, similar equations apply in the presence of $\Delta\mathbf{e}$, $\Delta\mathbf{E}$, $\Delta\mathbf{i}$, and $\Delta\mathbf{I}$ as in the presence of $\mathbf{e}$, $\mathbf{E}$, $\mathbf{i}$, and $\mathbf{I}$. Hence, *the following equations are used for the study of changes if the network itself remains unchanged:*

$$\boxed{\Delta\mathbf{e} = \mathbf{z} \cdot \Delta\mathbf{i}} \qquad \boxed{(\Delta\mathbf{E} + \Delta\mathbf{e}) = \mathbf{z} \cdot (\Delta\mathbf{i} + \Delta\mathbf{I})}$$
$$\boxed{\Delta\mathbf{I} = \mathbf{Y} \cdot \Delta\mathbf{E}} \qquad \boxed{(\Delta\mathbf{i} + \Delta\mathbf{I}) = \mathbf{Y} \cdot (\Delta\mathbf{E} + \Delta\mathbf{e})} \qquad 22.31$$

(*b*) In the presence of several types of meshes and junction-pairs *the resultant equations are divided into several tensor equations*, analogously to equations 19.1–19.6. For instance, $\Delta\mathbf{e} = \mathbf{z} \cdot \Delta\mathbf{i}$ is divided into

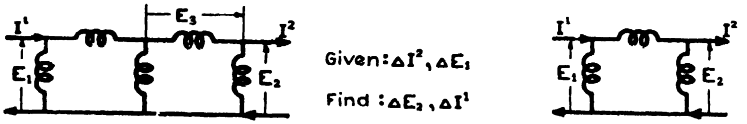
$$\begin{aligned}
\Delta\mathbf{e}_1 &= \mathbf{z}_{11} \cdot \Delta\mathbf{i}^1 + \mathbf{z}_{12} \cdot \Delta\mathbf{i}^2 + \mathbf{z}_{13} \cdot \Delta\mathbf{i}^3 + \cdots \\
\Delta\mathbf{e}_2 &= \mathbf{z}_{21} \cdot \Delta\mathbf{i}^1 + \mathbf{z}_{22} \cdot \Delta\mathbf{i}^2 + \mathbf{z}_{23} \cdot \Delta\mathbf{i}^3 + \cdots \qquad 22.32 \\
\Delta\mathbf{e}_3 &= \mathbf{z}_{31} \cdot \Delta\mathbf{i}^1 + \mathbf{z}_{32} \cdot \Delta\mathbf{i}^2 + \mathbf{z}_{33} \cdot \Delta\mathbf{i}^3 + \cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

(*c*) *The changes may be any amount, large or small.* A change, say $\Delta\mathbf{E}$, may consist of reducing some of the components of $\mathbf{E}$ to zero.

In non-linear networks, as in multielectrode tube circuits, the changes $\Delta\mathbf{E}$ and $\Delta\mathbf{I}$ may assume only small values.

X. CHANGE IN A π-NETWORK

(*a*) Let a network with three types of junction-pairs be given: (1) input, (2) output, (3) inactive, as shown in Fig. 22.9*a*, and let two types of changes be analyzed:



(*a*) Given junction network (3 junction-pairs)

FIG. 22.9

(*b*) Reduced network (2 junction-pairs)

1. Let the load vary. As a result the load current $\mathbf{I}^2$ changes by a value $\Delta\mathbf{I}^2$, while the impressed voltage $\mathbf{E}_1$ remains constant. The problem is to find how much the voltage across the load $\mathbf{E}_2$ and the input current $\mathbf{I}^1$ change.

Eliminating the inactive axes $\Delta\mathbf{E}_3$ the equation of the equivalent π of Fig. 22.9*b* is

$$\begin{aligned} \Delta\mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\Delta\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\Delta\mathbf{E}_2 \\ \Delta\mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\Delta\mathbf{E}_1 - \mathbf{Y}^{22}\cdot\Delta\mathbf{E}_2 \end{aligned} \qquad 22.33$$

Since the impressed voltage $\mathbf{E}_1$ does not change $\Delta\mathbf{E}_1 = 0$ and the above equations become

$$\begin{aligned} \Delta\mathbf{I}^1 &= -\mathbf{Y}^{12}\cdot\Delta\mathbf{E}_2 \\ \Delta\mathbf{I}^2 &= -\mathbf{Y}^{22}\cdot\Delta\mathbf{E}_2 \end{aligned}$$

From the second equation the change in the load terminal voltage is

$$\boxed{\Delta\mathbf{E}_2 = -\mathbf{Y}^{22^{-1}}\cdot\Delta\mathbf{I}^2} \qquad 22.34$$

Substituting $\Delta\mathbf{E}_2$ into the first equation of equation 22.33 *the change in the input current is*

$$\boxed{\Delta\mathbf{I}^1 = \mathbf{Y}^{12}\cdot\mathbf{Y}^{22^{-1}}\cdot\Delta\mathbf{I}^2} \qquad 22.35$$

2. Again let the load vary, and now let the input voltage $\mathbf{E}_1$ vary by $\Delta\mathbf{E}_1$ and let the load voltage $\mathbf{E}_2$ be maintained constant ($\Delta\mathbf{E}_2 = 0$). Then equation 22.33 may be written as

$$\boxed{\Delta\mathbf{I}^1 = \mathbf{Y}^{11}\cdot\Delta\mathbf{E}_1} \qquad 22.36$$

$$\boxed{\Delta\mathbf{I}^2 = \mathbf{Y}^{21}\cdot\Delta\mathbf{E}_1} \qquad 22.37$$

giving the current changes immediately.

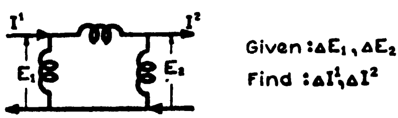


FIG. 22.10.—Given junction network (2 junction-pairs)

This last problem may be formulated by asking how much value ΔE_1 should have in order to maintain E_2 constant for a given ΔI^2. That is, in the last equation let ΔE_1 be unknown and ΔI^2 be known (Fig. 22.11).

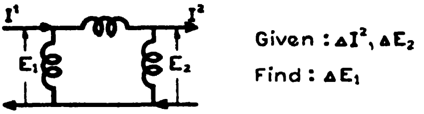


FIG. 22.11.—Given junction network (2 junction-pairs)

If the number of input and output terminals is the same, Y^{21} is square, and the necessary change in the impressed voltage is

$$\boxed{\Delta E_1 = Y^{21-1} \cdot \Delta I^2} \qquad 22.38$$

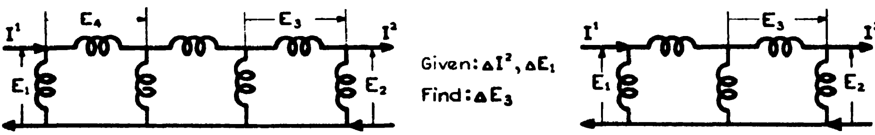
for a given change in load current in order to maintain the voltages across the load (E_2) constant.

If the number of input and output terminals is different, the method of Section XVI, Chapter X, should be followed.

XI. CHANGE IN THE OPEN-CIRCUIT VOLTAGES

(*a*) Let the load current vary by ΔI^2 while the input voltage E_1 remains constant. What is the change ΔE_3 across *part* of the inactive junction-pairs?

There are *four* types of junction-pairs as shown in Fig. 22.12*a*. Eliminating the inactive E_4, three types of junction-pairs remain,



(*a*) Given junction network (4 junction-pairs)

(*b*) Reduced network (3 junction -pairs)

FIG. 22.12

Fig. 22.12*b*, whose equation is given in equation 22.6. Leaving out the primes, the same equations for changes are

$$\begin{aligned} \Delta I^1 &= Y^{11} \cdot \Delta E_1 - Y^{12} \cdot \Delta E_2 - Y^{13} \cdot \Delta E_3 \\ \Delta I^2 &= Y^{21} \cdot \Delta E_1 - Y^{22} \cdot \Delta E_2 - Y^{23} \cdot \Delta E_3 \\ 0 &= Y^{31} \cdot \Delta E_1 - Y^{32} \cdot \Delta E_2 - Y^{33} \cdot \Delta E_3 \end{aligned} \qquad 22.39$$

(*b*) In these equations ΔI^2 is given and $\Delta E_1 = 0$. The problem is to find ΔE_3 in terms of ΔI^2.

Putting $\Delta E_1 = 0$

$$\begin{aligned} \Delta I^1 &= -Y^{12}\cdot\Delta E_2 - Y^{13}\cdot\Delta E_3 \\ \Delta I^2 &= -Y^{22}\cdot\Delta E_2 - Y^{23}\cdot\Delta E_3 \\ 0 &= -Y^{32}\cdot\Delta E_2 - Y^{33}\cdot\Delta E_3 \end{aligned} \qquad 22.40$$

Finding ΔE_2 from the second equation

$$\Delta E_2 = -Y^{22^{-1}}\cdot(Y^{23}\cdot\Delta E_3 + \Delta I^2)$$

Substituting into the third equation

$$0 = Y^{32}\cdot Y^{22^{-1}}\cdot(Y^{23}\cdot\Delta E_3 + \Delta I^2) - Y^{33}\cdot\Delta E_3$$

Rearranging

$$(Y^{33} - Y^{32}\cdot Y^{22^{-1}}\cdot Y^{23})\cdot\Delta E_3 = Y^{32}\cdot Y^{22^{-1}}\cdot\Delta I^2$$

Hence the change in the open-circuit voltage due to a change of load current ΔI^2 is

$$\boxed{\Delta E_3 = (Y^{33} - Y^{32}\cdot Y^{22^{-1}}\cdot Y^{23})^{-1}\cdot Y^{32}\cdot Y^{22^{-1}}\cdot\Delta I^2} \qquad 22.41$$

XII. GENERALIZATIONS OF THE "COMPENSATION THEOREM"

(*a*) In the previous sections the electrical quantities were changed while the network itself remained unchanged. (Of course the *load* supplied by the network may have changed.) *Now let the impedances* **z** *or admittances* **Y** *of some or all of the coils also change to* **z** + Δ**z** *or* **Y** + Δ**Y**, *without, however, changing the number or manner of interconnection of the coils.* For instance, the self-impedances of some of the coils may change or the mutual inductances between coils, etc.

Considering the equation of voltage of a complete network *before* the change

$$E + e = z\cdot(i + I) \qquad 22.42$$

let **z** change to **z** + Δ**z**, and also let all electrical quantities change. *After* the change their equation is

$$(E + \Delta E) + (e + \Delta e) = (z + \Delta z)\cdot[(i + \Delta i) + (I + \Delta I)] \qquad 22.43$$

Subtracting the original equation 22.42

$$\Delta E + \Delta e = (z + \Delta z)\cdot(\Delta i + \Delta I) + \Delta z(i + I) \qquad 22.44$$

so that *the equation of voltage change is*

$$\boxed{(\Delta E + \Delta e) - \Delta z\cdot(i + I) = (z + \Delta z)\cdot(\Delta i + \Delta I)} \qquad 22.45$$

That is, *if the impedance* $\mathbf{z}$ *of a network is changed by* $\Delta\mathbf{z}$, *the change of impedance is equivalent to impressing on the new system a voltage equal to* $-\Delta\mathbf{z}\cdot(\mathbf{i}+\mathbf{I})$, *where* $(\mathbf{i}+\mathbf{I})$ *are the total currents flowing in the system before the change.*

If the impressed voltages $\mathbf{E}+\mathbf{e}$ do not change, the last equation becomes

$$-\Delta\mathbf{z}\cdot(\mathbf{i}+\mathbf{I}) = (\mathbf{z}+\Delta\mathbf{z})\cdot(\Delta\mathbf{i}+\Delta\mathbf{I}) \qquad 22.46$$

showing that all current changes are due to this apparent additional impressed voltage. In a mesh network with no impressed current $\mathbf{I}$ equation 22.45 becomes

$$\boxed{\Delta\mathbf{e} - \Delta\mathbf{z}\cdot\mathbf{i} = (\mathbf{z}+\Delta\mathbf{z})\cdot\Delta\mathbf{i}} \qquad 22.47$$

(*b*) Similar reasoning applies to the equation of current of a complete network in which also the admittances change by $\Delta\mathbf{Y}$. *The equation of current change is*

$$\boxed{(\Delta\mathbf{i}+\Delta\mathbf{I}) - \Delta\mathbf{Y}\cdot(\mathbf{E}+\mathbf{e}) = (\mathbf{Y}+\Delta\mathbf{Y})\cdot(\Delta\mathbf{E}+\Delta\mathbf{e})} \qquad 22.48$$

In a junction network with no impressed $\mathbf{e}$ the above equation becomes

$$\boxed{\Delta\mathbf{I} - \Delta\mathbf{Y}\cdot\mathbf{E} = (\mathbf{Y}+\Delta\mathbf{Y})\cdot\Delta\mathbf{E}} \qquad 22.49$$

giving the apparent additional impressed current as $-\Delta\mathbf{Y}\cdot\mathbf{E}$.

Equations 22.45 and 22.48 are generalizations of the so-called "*compensation theorem*" which states that a single impedance change Δz is equivalent to impressing a voltage $-\Delta z i$, where i is the current flowing through z before the change.

(*c*) *Each of the above tensor equations of change may be subdivided into as many tensor equations as there are types of meshes and junction-pairs.* The network change $\Delta\mathbf{z}$ or $\Delta\mathbf{Y}$ usually occurs only in one type of mesh or junction-pair, hence usually *only one of the equations contains* $\Delta\mathbf{z}$ *or* $\Delta\mathbf{Y}$, while the others contain only $\mathbf{z}$ or $\mathbf{Y}$ and, of course, $\Delta\mathbf{i}$ $\Delta\mathbf{E}$, etc.

For instance, in a mesh network with *three* types of meshes, in which only $\mathbf{z}_{33}$ changes to $\mathbf{z}_{33}+\Delta\mathbf{z}_{33}$, equation 22.47 becomes

$$\begin{aligned}
\Delta\mathbf{e}_1 &= \mathbf{z}_{11}\cdot\Delta\mathbf{i}^1 + \mathbf{z}_{12}\cdot\Delta\mathbf{i}^2 + \mathbf{z}_{13}\cdot\Delta\mathbf{i}^3 \\
\Delta\mathbf{e}_2 &= \mathbf{z}_{21}\cdot\Delta\mathbf{i}^1 + \mathbf{z}_{22}\cdot\Delta\mathbf{i}^2 + \mathbf{z}_{23}\cdot\Delta\mathbf{i}^3 \qquad 22.50 \\
-\Delta\mathbf{z}_{33}\cdot\mathbf{i}^3 + \Delta\mathbf{e}_3 &= \mathbf{z}_{31}\cdot\Delta\mathbf{i}^1 + \mathbf{z}_{32}\cdot\Delta\mathbf{i}^2 + (\mathbf{z}_{33}+\Delta\mathbf{z}_{33})\cdot\Delta\mathbf{i}^3
\end{aligned}$$

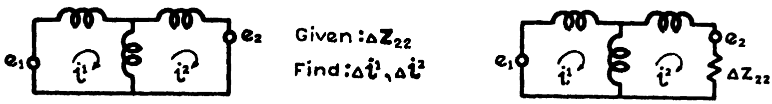
It should be noted that the amount of change $\Delta\mathbf{z}$ *and* $\Delta\mathbf{i}$, *etc., may be any amount, large or small.*

XIII. CHANGE OF IMPEDANCE

(*a*) As a simple example let a compound two-mesh network of Fig. 22.13*a* be given, whose equation of voltage is

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{i}^2 \\ \mathbf{e}_2 &= \mathbf{z}_{21}\cdot\mathbf{i}^1 + \mathbf{z}_{22}\cdot\mathbf{i}^2 \end{aligned} \qquad 22.51$$

Let the impedance $\mathbf{z}_{22}$ change by $\Delta\mathbf{z}_{22}$, as shown in Fig. 22.13*b*, while the impressed voltages $\mathbf{e}_1$ and $\mathbf{e}_2$ remain constant. The changes in currents $\Delta\mathbf{i}^1$ and $\Delta\mathbf{i}^2$ are to be found.



(*a*) Given mesh network (2 meshes)   (*b*) Addition of $\Delta\mathbf{Z}_{22}$

FIG. 22.13

The equation of voltage change of a mesh network 22.45 without **I** and $\Delta\mathbf{e}$,

$$-\Delta\mathbf{z}\cdot\mathbf{i} = (\mathbf{z} + \Delta\mathbf{z})\cdot\Delta\mathbf{i} \qquad 22.52$$

is divided into two component equations as

$$\begin{aligned} 0 &= \mathbf{z}_{11}\cdot\Delta\mathbf{i}^1 + \mathbf{z}_{12}\cdot\Delta\mathbf{i}^2 \\ -\Delta\mathbf{z}_{22}\cdot\mathbf{i}^2 &= \mathbf{z}_{21}\cdot\mathbf{i}^1 + (\mathbf{z}_{22} + \Delta\mathbf{z}_{22})\cdot\Delta\mathbf{i}^2 \end{aligned} \qquad 22.53$$

Eliminating $\Delta\mathbf{i}^1$ from the first equation

$$\Delta\mathbf{i}^1 = -\mathbf{z}_{11}^{-1}\cdot\mathbf{z}_{12}\cdot\Delta\mathbf{i}^2 \qquad 22.54$$

and substituting into the second equation

$$-\Delta\mathbf{z}_{22}\cdot\mathbf{i}^2 = -\mathbf{z}_{21}\cdot\mathbf{z}_{11}^{-1}\cdot\mathbf{z}_{12}\cdot\Delta\mathbf{i}^2 + (\mathbf{z}_{22} + \Delta\mathbf{z}_{22})\cdot\Delta\mathbf{i}^2$$

the change of currents in the changed impedances is

$$\boxed{\Delta\mathbf{i}^2 = -(\mathbf{z}_{22} + \Delta\mathbf{z}_{22} - \mathbf{z}_{21}\cdot\mathbf{z}_{11}^{-1}\cdot\mathbf{z}_{12})^{-1}\cdot\Delta\mathbf{z}_{22}\cdot\mathbf{i}^2} \qquad 22.55$$

Substituting $\Delta\mathbf{i}^2$ into equation 22.54, the remaining current changes are

$$\Delta\mathbf{i}^1 = \mathbf{z}_{11}^{-1}\cdot\mathbf{z}_{12}\cdot(\mathbf{z}_{22} + \Delta\mathbf{z}_{22} - \mathbf{z}_{21}\cdot\mathbf{z}_{11}^{-1}\cdot\mathbf{z}_{12})^{-1}\cdot\Delta\mathbf{z}_{22}\cdot\mathbf{i}^2 \qquad 22.56$$

(*b*) The value of $\mathbf{i}^2$ needed in the calculation of $\Delta\mathbf{i}_2$ is found from equation 22.51

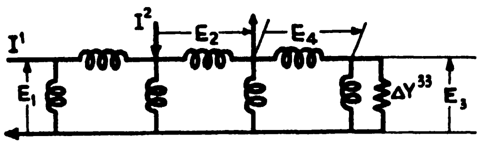
$$\begin{aligned} \mathbf{i}^1 &= \mathbf{z}_{11}^{-1}\cdot(\mathbf{e}_1 - \mathbf{z}_{12}\cdot\mathbf{i}^2) \\ \mathbf{e}_2 &= \mathbf{z}_{21}\cdot\mathbf{z}_{11}^{-1}\cdot(\mathbf{e}_1 - \mathbf{z}_{12}\cdot\mathbf{i}^2) + \mathbf{z}_{22}\cdot\mathbf{i}^2 \\ \mathbf{i}^2 &= (\mathbf{z}_{22} - \mathbf{z}_{21}\cdot\mathbf{z}_{11}^{-1}\cdot\mathbf{z}_{12})^{-1}\cdot(\mathbf{e}_2 - \mathbf{z}_{21}\cdot\mathbf{z}_{11}^{-1}\cdot\mathbf{e}_1) \end{aligned} \qquad 22.57$$

If this value of $\mathbf{i}^2$ is substituted into equation 22.55, then $\Delta\mathbf{i}^2$ *is found in terms of the impressed voltages* $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *and the impedance change* $\Delta\mathbf{z}_{22}$ *as*

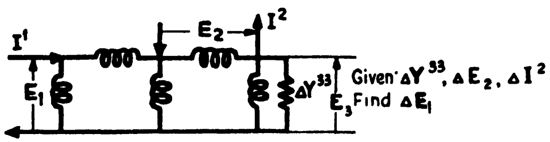
$$\begin{aligned}\Delta\mathbf{i}^2 &= -(\Delta\mathbf{z}_{22} + \mathbf{z}_{22} - \mathbf{z}_{21}\cdot\mathbf{z}_{11}^{-1}\cdot\mathbf{z}_{12})^{-1}\cdot\Delta\mathbf{z}_{22} \\ &\quad\cdot(\mathbf{z}_{22} - \mathbf{z}_{21}\cdot\mathbf{z}_{11}^{-1}\cdot\mathbf{z}_{12})^{-1}\cdot(\mathbf{e}_2 - \mathbf{z}_{21}\cdot\mathbf{z}_{11}^{-1}\cdot\mathbf{e}_1) \qquad 22.58 \\ \Delta\mathbf{i}^2 &= -(\Delta\mathbf{z}_{22} + \mathbf{z}'_{22})^{-1}\cdot\Delta\mathbf{z}_{22}\cdot\mathbf{z}'^{-1}_{22}\cdot\mathbf{e}'_2\end{aligned}$$

XIV. LOAD VOLTAGES AND CURRENTS UNCHANGED

(*a*) Let a junction network have *four* types of junction-pairs: (1) input, (2) output, (3) load, where the admittances $\mathbf{Y}$ change by $\Delta\mathbf{Y}$, (4) the remaining inactive junction-pairs, as shown in Fig. 22.14.



(*a*) Given junction network (4 junction-pairs)



(*b*) Reduced network (3 junction-pairs)

FIG. 22.14

If the inactive junction-pairs are eliminated, the equations of current of the *three* junction-pairs are

$$\begin{aligned}\mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{E}_2 - \mathbf{Y}^{13}\cdot\mathbf{E}_3 \\ \mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{Y}^{22}\cdot\mathbf{E}_2 - \mathbf{Y}^{23}\cdot\mathbf{E}_3 \qquad 22.59 \\ 0 &= \mathbf{Y}^{31}\cdot\mathbf{E}_1 - \mathbf{Y}^{32}\cdot\mathbf{E}_2 - \mathbf{Y}^{33}\cdot\mathbf{E}_3\end{aligned}$$

In these equations $\mathbf{E}_1$ and $\mathbf{E}_2$ (or $\mathbf{I}^2$) may be assumed to be known.

(*b*) Let it be assumed now that:

1. $\mathbf{Y}^{33}$ is changed to $\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33}$.
2. The input terminal voltage $\mathbf{E}_1$ is changed to $\mathbf{E}_1 + \Delta\mathbf{E}_1$, so that
3. The load voltage $\mathbf{E}_2$ and the load current $\mathbf{I}^2$ should not change, that is, $\Delta\mathbf{E}_2 = 0$ and $\Delta\mathbf{I}^2 = 0$.

The problem is to find *how much should* $\Delta\mathbf{E}_1$ *be changed for a given* $\Delta\mathbf{Y}^{33}$ *in order to keep* $\Delta\mathbf{E}_2 = 0$ *and* $\Delta\mathbf{I}^2 = 0$.

(*c*) The equation of current change

$$\Delta \mathbf{I} - \Delta \mathbf{Y} \cdot \mathbf{E} = (\mathbf{Y} + \Delta \mathbf{Y}) \cdot \Delta \mathbf{E} \qquad 22.60$$

can be divided into *three* component equations

$$\begin{aligned} \Delta \mathbf{I}^1 &= \mathbf{Y}^{11} \cdot \Delta \mathbf{E}_1 - \mathbf{Y}^{13} \cdot \Delta \mathbf{E}_3 \\ 0 &= \mathbf{Y}^{21} \cdot \Delta \mathbf{E}_1 - \mathbf{Y}^{23} \cdot \Delta \mathbf{E}_3 \qquad 22.61 \\ \Delta \mathbf{Y}^{33} \cdot \mathbf{E}_3 &= \mathbf{Y}^{31} \cdot \Delta \mathbf{E}_1 - (\mathbf{Y}^{33} + \Delta \mathbf{Y}^{33}) \cdot \Delta \mathbf{E}_3 \end{aligned}$$

where $\Delta \mathbf{E}_1$ has to be found in terms of $\mathbf{E}_3$, whose value is assumed to be known from equation 22.59.

Eliminating $\Delta \mathbf{E}_3$ from the third equation

$$\Delta \mathbf{E}_3 = (\mathbf{Y}^{33} + \Delta \mathbf{Y}^{33})^{-1} \cdot (\mathbf{Y}^{31} \cdot \Delta \mathbf{E}_1 - \Delta \mathbf{Y}^{33} \cdot \mathbf{E}_3)$$

Substituting it into the second equation

$$0 = \mathbf{Y}^{21} \cdot \Delta \mathbf{E}_1 - \mathbf{Y}^{23} \cdot (\mathbf{Y}^{33} + \Delta \mathbf{Y}^{33})^{-1} \cdot (\mathbf{Y}^{31} \cdot \Delta \mathbf{E}_1 - \Delta \mathbf{Y}^{33} \cdot \mathbf{E}_3)$$

$$\begin{aligned} 0 = [\mathbf{Y}^{21} - \mathbf{Y}^{23} \cdot (\mathbf{Y}^{33} + \Delta \mathbf{Y}^{33})^{-1} \cdot \mathbf{Y}^{31}] \cdot \Delta \mathbf{E}_1 \\ + \mathbf{Y}^{23} \cdot (\mathbf{Y}^{33} + \Delta \mathbf{Y}^{33})^{-1} \cdot \Delta \mathbf{Y}^{33} \cdot \mathbf{E}_3 \qquad 22.62 \end{aligned}$$

Hence for a given $\Delta \mathbf{Y}^{33}$ *the change in the applied terminal voltage* $\mathbf{E}_1$ *is*

$$\boxed{\Delta \mathbf{E}_1 = -[\mathbf{Y}^{21} - \mathbf{Y}^{23} \cdot (\mathbf{Y}^{33} + \Delta \mathbf{Y}^{33})^{-1} \cdot \mathbf{Y}^{31}]^{-1} \cdot \mathbf{Y}^{23} \cdot (\mathbf{Y}^{33} + \Delta \mathbf{Y}^{33})^{-1} \cdot \Delta \mathbf{Y}^{33} \cdot \mathbf{E}_3} \qquad 22.63$$

where $\mathbf{E}_3$ is calculated from equation 22.59 in terms of the known $\mathbf{E}_1$ and $\mathbf{E}_2$ (or $\mathbf{I}^2$).

While the matrix of $\mathbf{Y}^{33}$ is square, *the matrix of* $\mathbf{Y}^{21}$ *in general is not square*, hence the matrix in the bracket is not square either. They are square only if the same number of input and output junction-pairs are assumed.

When the matrix of $\mathbf{Y}^{21}$ is not square, equation 22.62 represents in any reference frame a set of ordinary linear equations, in which the number of unknowns $\Delta \mathbf{E}_1$ is not the same as the number of equations. The method of solution of such equations is given in Section XVI, Chapter X.

XV. THE FLOW OF POWER INTO LOADS

(*a*) Let the previous compound network be considered again and let it be assumed now that, while the admittance $\mathbf{Y}^{33}$ changes by $\Delta \mathbf{Y}^{33}$, *the power input into the load* $\mathbf{E}_2^* \cdot \mathbf{I}^2$ *remains unchanged*, although both $\mathbf{E}_2$ and $\mathbf{I}^2$ vary. (The load may consist, say, of synchronous motors.) The problem is to find the change in the load voltage, namely, $\Delta \mathbf{E}_2$

In the present problem the impressed terminal voltage $\mathbf{E}_1$ remains constant instead of $\mathbf{E}_2$ and $\mathbf{I}^2$, hence *the equation of current change is*

$$\begin{aligned}\Delta\mathbf{I}^1 &= -\mathbf{Y}^{12}\cdot\Delta\mathbf{E}_2 - \mathbf{Y}^{13}\cdot\Delta\mathbf{E}_3\\ \Delta\mathbf{I}^2 &= -\mathbf{Y}^{22}\cdot\Delta\mathbf{E}_2 - \mathbf{Y}^{23}\cdot\Delta\mathbf{E}_3\\ \Delta\mathbf{Y}^{33}\cdot\mathbf{E}_3 &= -\mathbf{Y}^{32}\cdot\Delta\mathbf{E}_2 - (\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})\cdot\Delta\mathbf{E}_3\end{aligned} \qquad 22.64$$

In addition to these equations it is also true that the power input into the load *before* the change $\mathbf{E}_2^*\cdot\mathbf{I}^2$ is the same as that *after* the change $(\mathbf{E}_2^* + \Delta\mathbf{E}_2^*)\cdot(\mathbf{I}^2 + \Delta\mathbf{I}^2)$. That is, the following equation also exists

$$\mathbf{E}_2^*\cdot\mathbf{I}^2 = (\mathbf{E}_2^* + \Delta\mathbf{E}_2^*)\cdot(\mathbf{I}^2 + \Delta\mathbf{I}^2)$$

Simplifying, *the fourth equation* representing power is

$$(\mathbf{E}_2^* + \Delta\mathbf{E}_2^*)\cdot\Delta\mathbf{I}^2 = -\Delta\mathbf{E}_2^*\cdot\mathbf{I}^2 \qquad 22.65$$

The last two equations of 22.64 and the power equation 22.65 contain three unknowns $\Delta\mathbf{E}_3$, $\Delta\mathbf{E}_2$, and $\Delta\mathbf{I}^2$.

(*b*) Eliminating $\Delta\mathbf{E}_3$ from the last equation of 22.64

$$\Delta\mathbf{E}_3 = -(\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})^{-1}\cdot(\mathbf{Y}^{32}\cdot\Delta\mathbf{E}_2 + \Delta\mathbf{Y}^{33}\cdot\mathbf{E}_3)$$

and substituting into the second equation

$$\Delta\mathbf{I}^2 = -\mathbf{Y}^{22}\cdot\Delta\mathbf{E}_2 + \mathbf{Y}^{23}\cdot(\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})^{-1}\cdot(\mathbf{Y}^{32}\cdot\Delta\mathbf{E}_2 + \Delta\mathbf{Y}^{33}\cdot\mathbf{E}_3)$$

$$\begin{aligned}\Delta\mathbf{I}^2 = &-[\mathbf{Y}^{22} - \mathbf{Y}^{23}\cdot(\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})^{-1}\cdot\mathbf{Y}^{32}]\cdot\Delta\mathbf{E}_2\\ &+ \mathbf{Y}^{23}\cdot(\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})^{-1}\cdot\Delta\mathbf{Y}^{33}\cdot\mathbf{E}_3\end{aligned} \qquad 22.66$$

This last equation and the power equation 22.65 contain *two unknowns* $\Delta\mathbf{E}$ and $\Delta\mathbf{I}$.

Substituting $\Delta\mathbf{I}^2$ from the last equation into equation 22.65

$$\begin{aligned}-(\mathbf{E}_2^* + \Delta\mathbf{E}_2^*)\cdot\{&[\mathbf{Y}^{22} - \mathbf{Y}^{23}\cdot(\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})^{-1}\cdot\mathbf{Y}^{32}]\cdot\Delta\mathbf{E}_2\\ &+ \mathbf{Y}^{23}\cdot(\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})^{-1}\cdot\Delta\mathbf{Y}^{33}\cdot\mathbf{E}_3\} = -\Delta\mathbf{E}_2^*\cdot\mathbf{I}^2\end{aligned}$$

or rearranged, the equation containing $\Delta\mathbf{E}_2$ as unknown is

$$\begin{aligned}&\Delta\mathbf{E}_2^*\cdot[\mathbf{Y}^{22} - \mathbf{Y}^{23}\cdot(\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})^{-1}\cdot\mathbf{Y}^{32}]\cdot\Delta\mathbf{E}_2\\ &-\Delta\mathbf{E}_2^*\cdot[\mathbf{I}^2 + \mathbf{Y}^{23}\cdot(\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})^{-1}\cdot\Delta\mathbf{Y}^{33}\cdot\mathbf{E}_3]\\ &+\mathbf{E}_2^*\cdot[\mathbf{Y}^{22} - \mathbf{Y}^{23}\cdot(\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})^{-1}\cdot\mathbf{Y}^{32}]\cdot\Delta\mathbf{E}_2\\ &-\mathbf{E}_2^*\cdot\mathbf{Y}^{23}\cdot(\mathbf{Y}^{33} + \Delta\mathbf{Y}^{33})^{-1}\cdot\Delta\mathbf{Y}^{33}\cdot\mathbf{E}_3 = 0\end{aligned} \qquad 22.67$$

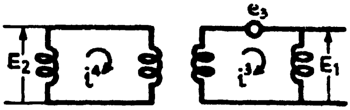
This is an invariant quadratic equation in the unknown vector $\Delta\mathbf{E}_2$. Its method of solution is not undertaken here. If only *one load terminal*

exists, $\Delta\mathbf{E}_2$ is a scalar and the above equation is an ordinary quadratic equation in $\Delta\mathbf{E}_2$ easily solvable.

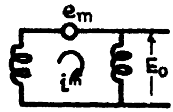
XVI. GENERALIZATION OF THÉVENIN'S THEOREM

(*a*) Let a complete compound network with two meshes and two junction-pairs be given as shown in Fig. 22.15*a*.

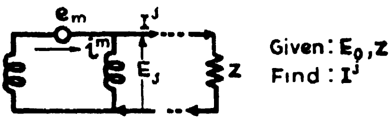
The problem is the following: If a load $\mathbf{z}$ (Fig. 22.15*c*) is connected across the open-circuited junction-pairs, with a terminal voltage $\mathbf{E}_0$,



(*a*) Given orthogonal network
(2 meshes, 2 junction-pairs)



(*b*) Reduced network
(1 mesh, 1 junction-pair)



(*c*) Addition of z

FIG. 22.15

what is the current* I *flowing into the load in terms of* $\mathbf{E}_0$ *existing before the introduction of the load?

Assuming the mesh of $\mathbf{i}^4$ and the junction-pair of $\mathbf{E}_2$ as inactive, the network may be reduced to one having one mesh and one junction-pair, Fig. 22.15*b*.

The equation of voltage of the reduced complete network of Fig. 22.15*b* is by equation 22.8 (leaving out the double primes)

$$\begin{aligned} \mathbf{e}_m &= \mathbf{z}_{mm}\cdot\mathbf{i}^m - \mathbf{z}_{mj}\cdot\mathbf{I}^j \\ \mathbf{E}_j &= \mathbf{z}_{jm}\cdot\mathbf{i}^m - \mathbf{z}_{jj}\cdot\mathbf{I}^j \end{aligned} \qquad 22.68$$

The first invariant equation stands for as many ordinary equations as there are active meshes, and the second one for as many as there are active junction-pairs.

(*b*) ***Before*** the introduction of the load (Fig. 22.15*b*) the load current $\mathbf{I}^j$ is zero and equation 22.68 becomes

$$\begin{aligned} \mathbf{e}_m &= \mathbf{z}_{mm}\cdot\mathbf{i}^m \\ -\mathbf{E}_0 &= \mathbf{z}_{jm}\cdot\mathbf{i}^m \end{aligned}$$

Solving the first equation for $\mathbf{i}^m$ (the mesh current)

22.69

Substituting into the second equation *the open-circuit terminal voltage* $\mathbf{E}_0$ *across the load is*

$$\boxed{\mathbf{E}_0 = \mathbf{z}_{jm}\cdot\mathbf{z}_{mm}^{-1}\cdot\mathbf{e}_m} \tag{22.70}$$

• (*c*) Now let $\mathbf{z}$ be introduced as shown in Fig. 22.15*c*. (That is, let impedances with mutual inductances between them be connected across several junction-pairs.) The effect of the load $\mathbf{z}$ is to make the terminal voltage $\mathbf{E}$ across it equal to $\mathbf{z}\cdot\mathbf{I}$. Hence in the presence of $\mathbf{z}$ equation 22.68 becomes

$$\begin{aligned} \mathbf{e}_m &= \mathbf{z}_{mm}\cdot\mathbf{i}^m - \mathbf{z}_{mj}\cdot\mathbf{I}^j \\ \mathbf{z}\cdot\mathbf{I}^j &= \mathbf{z}_{jm}\cdot\mathbf{i}^m - \mathbf{z}_{jj}\cdot\mathbf{I}^j \end{aligned} \tag{22.71}$$

Again solving the first equation for $\mathbf{i}^m$

$$\mathbf{i}^m = \mathbf{z}_{mm}^{-1}\cdot(\mathbf{e}_m + \mathbf{z}_{mj}\cdot\mathbf{I}^j) \tag{22.72}$$

and substituting into the second equation

$$\mathbf{z}\cdot\mathbf{I}^j = \mathbf{z}_{jm}\cdot\mathbf{z}_{mm}^{-1}\cdot(\mathbf{e}_m + \mathbf{z}_{mj}\cdot\mathbf{I}^j) - \mathbf{z}_{jj}\cdot\mathbf{I}^j$$

the load current $\mathbf{I}^j$ becomes

$$(\mathbf{z} + \mathbf{z}_{jj} - \mathbf{z}_{jm}\cdot\mathbf{z}_{mm}^{-1}\cdot\mathbf{z}_{mj})\cdot\mathbf{I}^j = \mathbf{z}_{jm}\cdot\mathbf{z}_{mm}^{-1}\cdot\mathbf{e}_m$$

However, from equation 10.7

$$\mathbf{z}_{jj} - \mathbf{z}_{jm}\cdot\mathbf{z}_{mm}^{-1}\cdot\mathbf{z}_{mj} = \mathbf{z}_{jj}' \tag{22.73}$$

represents the *short-circuit impedance of the network measured from the load.* Hence

$$(\mathbf{z} + \mathbf{z}_{jj}')\cdot\mathbf{I}^j = \mathbf{z}_{jm}\cdot\mathbf{z}_{mm}^{-1}\cdot\mathbf{e}_m$$

and the load current is

$$\boxed{\mathbf{I}^j = (\mathbf{z} + \mathbf{z}_{jj}')^{-1}\cdot\mathbf{z}_{jm}\cdot\mathbf{z}_{mm}^{-1}\cdot\mathbf{e}_m} \tag{22.74}$$

However, the expression beyond the parenthesis $\mathbf{z}_{jm}\cdot\mathbf{z}_{mm}^{-1}\cdot\mathbf{e}_m$ represents by equation 22.70 the open-circuit terminal voltage $\mathbf{E}_0$ across the load, since *the mesh impressed-voltage* $\mathbf{e}_m$ *is the same before and after the addition of the load* $\mathbf{z}$. Hence the current flowing into the load $\mathbf{z}$ is

$$\boxed{\mathbf{I} = (\mathbf{z} + \mathbf{z}_{jj}')^{-1}\cdot\mathbf{E}_0} \tag{22.75}$$

where $\mathbf{z}$ = impedance tensor of the load.

$\mathbf{z}_{jj}'$ = short-circuit impedance tensor of the network measured from the load terminals.

$\mathbf{E}_0$ = open-circuit terminal voltage across the load terminals before the introduction of $\mathbf{z}$.

The *mesh-current* $\mathbf{i}^m$ is found from the load current $\mathbf{I}^{\prime}$ by equation 22.72, or by

$$\boxed{\mathbf{i}^m = \mathbf{z}_{mm}^{-1}\cdot[\mathbf{e}_m + \mathbf{z}_{mj}\cdot(\mathbf{z} + \mathbf{z}_{jj}')^{-1}\cdot\mathbf{E}_0]} \qquad 22.76$$

where the first term $\mathbf{z}_{mm}^{-1}\cdot\mathbf{e}_m$ represents the mesh currents flowing *before* the introduction of the load.

(*d*) Equation 22.75 represents a *generalization of Thévenin's theorem* that gives a *single* load current I flowing into a single load Z as

$$\boxed{I^{\prime} = (Z + Z_{jj}')^{-1}E_0} \qquad 22.77$$

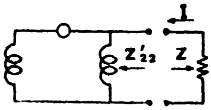


FIG. 22.16.—Thévenin's Theorem

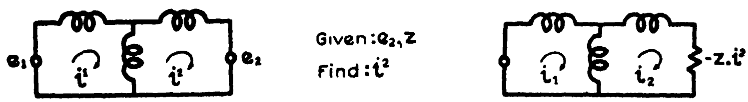
where Z_{jj}' is the network impedance viewed from the load and E_0 is the open-circuit terminal voltage. (Fig. 22.16)

It should be noted that the networks $\mathbf{z}_{jj}'$ and $\mathbf{z}$ may be asymmetrical networks, also they may contain impressed voltages $\mathbf{e}_m$ and currents $\mathbf{I}^j$ in the *eliminated* meshes and junction-pairs. That is, *the networks are asymmetrical, active networks with any number of meshes and terminals.*

(*e*) *The theorem of this section could have been developed also by assuming two compound meshes instead of one mesh and one junction-pair* and using

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{i}^2 \\ \mathbf{e}_2 &= \mathbf{z}_{21}\cdot\mathbf{i}^1 + \mathbf{z}_{22}\cdot\mathbf{i}^2 \end{aligned} \qquad 22.78$$

instead of equation 22.68 thereby replacing $\mathbf{e}_2$ by $-\mathbf{z}\cdot\mathbf{i}^2$, etc., as shown in Fig. 22.17. The results are the same. However, this last method



(*a*) Given mesh network (2 meshes)

(*b*) Replacing $\mathbf{e}_2$ by $-\mathbf{z}\cdot\mathbf{i}^2$

FIG. 22.17

of attack is not so clear-cut since each mesh of $\mathbf{e}_2$ must contain only *one* impressed voltage.

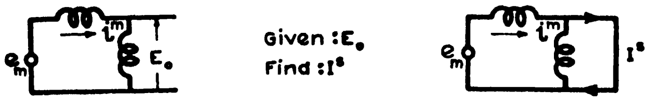
XVII. SHORT-CIRCUIT CURRENTS

(*a*) A special case of Thévenin's theorem, when $\mathbf{z} = 0$, represents the short-circuiting of the open-circuited junction-pair as shown in Fig. 22.18.

By equation 21.77 *the short-circuit current* is

$$\boxed{\mathbf{I}^j = \mathbf{z}_{jj}'^{-1}\cdot\mathbf{E}_0} \qquad 22.79$$

where $\mathbf{E}_0$ *is the difference of potential appearing across the junction-pairs before their short-circuit* and $\mathbf{z}_{jj}'$ is the impedance of the network viewed from the short-circuited terminals.



(*a*) Given orthogonal network (1 mesh, 1 junction-pair) (*b*) Short-circuit

FIG. 22.18

The *mesh currents* during the short circuits are by equation 22.76

$$\boxed{\mathbf{i}^m = \mathbf{z}_{mm}^{-1}\cdot(\mathbf{e}_m + \mathbf{z}_{mj}\cdot\mathbf{z}_{jj}'^{-1}\cdot\mathbf{E}_0)} \qquad 22.80$$

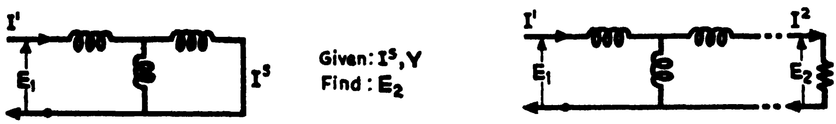
where the first term $\mathbf{z}_{mm}^{-1}\cdot\mathbf{e}_m$ represents the mesh currents existing *before* the short circuit. Hence the *additional* mesh currents due to the short circuits are

$$\mathbf{i}^m = \mathbf{z}_{mm}^{-1}\cdot\mathbf{z}_{mj}\cdot\mathbf{z}_{jj}'^{-1}\cdot\mathbf{E}_0 \qquad 22.81$$

XVIII. THE DUAL OF THÉVENIN'S THEOREM

(*a*) *Every method of reasoning, every theorem may be expressed in a dual form by replacing* **z** *with* **Y**, *etc., as shown in Section I, Chapter XIV.*

Instead of the dual of the complete network of Fig. 22.16, let the dual of the mesh network of Fig. 22.17 be taken, namely the junction



(*a*) Given short-circuit current $\mathbf{I}^s$ (*b*) Load **Y** inserted

FIG. 22.19

network of Fig. 22.19 having two junction-pairs (after reduction). Its equation of current is

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{E}_2 \\ \mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{Y}^{22}\cdot\mathbf{E}_2 \end{aligned} \qquad 22.82$$

Now let a load **Y** be inserted in parallel with the short-circuited junction-pair with a short-circuit current $\mathbf{I}^s$. The problem is to find the differences of potentials $\mathbf{E}_2$ appearing across the load in terms of the short-circuit current $\mathbf{I}^s$ existing before the introduction of the

load. It will have to be assumed that the input currents $\mathbf{I}^1$ are the same before and after the introduction of $\mathbf{Y}$ (just as $\mathbf{e}_1$ remained constant during the introduction of $\mathbf{z}$).

(*b*) *Before* the introduction of $\mathbf{Y}$ (Fig. 22.19*a*) the load voltage $\mathbf{E}_2$ is zero and equation 22.82 becomes

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 \\ \mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 \end{aligned} \qquad 22.83$$

Solving the first equation for $\mathbf{E}_1$ and substituting it into the second, the short-circuit current is

$$\boxed{\mathbf{I}^s = \mathbf{Y}^{21}\cdot\mathbf{Y}^{11-1}\cdot\mathbf{I}^1} \qquad 22.84$$

(*c*) The effect of the introduction of $\mathbf{Y}$ is to make the load current $\mathbf{I}^2$ equal to $\mathbf{Y}\cdot\mathbf{E}_2$. Hence in the presence of $\mathbf{Y}$ equation 22.82 becomes

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{E}_2 \\ \mathbf{Y}\cdot\mathbf{E}_2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{Y}^{22}\cdot\mathbf{E}_2 \end{aligned} \qquad 22.85$$

Solving the first equation for $\mathbf{E}_1$

$$\mathbf{E}_1 = \mathbf{Y}^{11-1}\cdot(\mathbf{I}^1 + \mathbf{Y}^{12}\cdot\mathbf{E}_2)$$

and substituting into the second equation

$$\mathbf{Y}\cdot\mathbf{E}_2 = \mathbf{Y}^{21}\cdot\mathbf{Y}^{11-1}\cdot(\mathbf{I}^1 + \mathbf{Y}^{12}\cdot\mathbf{E}) - \mathbf{Y}^{22}\cdot\mathbf{E}_2$$

the load voltage becomes

$$(\mathbf{Y} + \mathbf{Y}^{22} - \mathbf{Y}^{21}\cdot\mathbf{Y}^{11-1}\cdot\mathbf{Y}^{12})\cdot\mathbf{E}_2 = \mathbf{Y}^{21}\cdot\mathbf{Y}^{11-1}\cdot\mathbf{I}^1 \qquad 22.86$$

But

$$\mathbf{Y}^{22} - \mathbf{Y}^{21}\cdot\mathbf{Y}^{11-1}\cdot\mathbf{Y}^{12} = \mathbf{Y}^{22'}$$

represents the *open-circuit admittance* viewed from the load and the right-hand side of equation 22.86 is, by equation 22.84, the short-circuit current $\mathbf{I}^s$ existing before the load (since $\mathbf{I}^1$ is assumed to stay constant). Hence *the voltage across the load is*

$$\boxed{\mathbf{E}_2 = (\mathbf{Y} + \mathbf{Y}^{22'})^{-1}\cdot\mathbf{I}^s} \qquad 22.87$$

This equation is the *dual* of equation 22.75.

(*d*) The special case, when $\mathbf{Y} = 0$, represents the *open-circuiting* of the short-circuited junction-pair. *The open-circuit voltage is*

$$\boxed{\mathbf{E}_2 = \mathbf{Y}^{22'-1}\cdot\mathbf{I}^s} \qquad 22.88$$

expressed in terms of the short-circuit current existing before the open circuit.

CHAPTER XXIII

THE SYNTHESIS OF NETWORKS

I. THE TYPES OF PROBLEMS TO BE CONSIDERED

(*a*) In the problems considered in the previous chapter on the analysis of networks it was assumed that:

1. The design constants $\mathbf{z}$ or $\mathbf{Y}$ of the individual coils are all known.

2. The interconnections $\mathbf{C}$ or $\mathbf{C}_t^{*-1}$ of the coils are also known.

In problems of synthesis one or both of these two conditions are not satisfied. In their place some conditions of the performance are specified.

(*b*) In this chapter three general types of problems will be considered:

1. *Given certain desirable performance characteristics* (say the network has to supply constant currents $\mathbf{I}$ at certain terminals irrespective of the loads connected across those terminals), *the problem is to find the relation that must exist between the impedances or admittances of the meshes or junction-pairs of any network in order that the network should perform as it is desired. This relation between the components of the* $\mathbf{z}$*'s or* $\mathbf{Y}$*'s (also* $\mathbf{e}$*'s and* $\mathbf{i}$*'s) will be called the "criterion of performance." Each type of performance has its own "criterion."* For many types of performance several such criteria have to be established.

2. When a network is given in which the mesh impedances $\mathbf{z}'$ (or $\mathbf{Y}'$) are known to satisfy the desired "criterion of performance," the next problem is to find automatically the impedances (and impressed voltages) of a large variety of other arbitrary networks having different interconnections and coil impedances that still have the same performance. The impedance (or admittance) tensors $\mathbf{z}'$ or $\mathbf{Y}'$ (also $\mathbf{e}'$ or $\mathbf{I}'$) of these networks are found with the aid of a new type of transformation tensor, the so-called "synthesis tensor" $\mathbf{C}'_\sigma$. *For each criterion of performance a separate "synthesis tensor" has to be established that keeps the criterion (or sets of criteria) invariant.*

3. If the "synthesis tensor" $\mathbf{C}'_\sigma$ changing $\mathbf{z}'_1$ to $\mathbf{z}'_2$ (without examining the criterion) is known, the next problem is to establish a "*primitive synthesis tensor*" $\mathbf{C}_\sigma$ changing the $\mathbf{z}_1$ (or $\mathbf{Y}^1$) of the *individual* coils of one network to $\mathbf{z}_2$ (or $\mathbf{Y}^2$) of the *individual* coils of any other network

without going through the task of establishing $\mathbf{z}'_1$ and $\mathbf{z}'_2$ *of the resultant networks.*

That is, $\mathbf{C}'_\sigma$ eliminates the necessity of setting up the "criterion of performance" of each network, and $\mathbf{C}_\sigma$ eliminates even the necessity of setting up the impedance tensors $\mathbf{z}'$ (or $\mathbf{Y}'$) of each network. Hence *in problems of synthesis it is sufficient to consider:*

1. *The self and mutual impedances of isolated coils.*
2. *The connection tensors of the networks under consideration.*

The synthesis tensors $\mathbf{C}'_\sigma$ and $\mathbf{C}_\sigma$ represent a new type of "groups of transformations" that set up a correspondence between networks having in general *different number of coils.*

(*c*) The following examples will be worked out in detail, finding for each case the "*criterion of performance*" and the "*synthesis tensor*":

1. Keeping the *input* terminal voltages $\mathbf{E}_1$ constant and varying the outside *load* $\mathbf{Y}$, how to maintain in any network:

   (*a*) The load currents $\mathbf{I}^2$ constant.
   (*b*) The input currents $\mathbf{I}^1$ constant.
   (*c*) Both currents $\mathbf{I}^1$ and $\mathbf{I}^2$ constant.
   (*d*) The load voltages $\mathbf{E}_2$ constant.
   (*e*) The differences of potential $\mathbf{E}_3$ across a set of inactive terminals constant.

2. How to keep the input impedances $\mathbf{z}_i$ or admittances $\mathbf{Y}^i$ constant in any network if:

   (*a*) The network is passive.
   (*b*) The network is active with voltages (or currents) impressed around some or all of the closed (or open) meshes.
   (*c*) The network supplies a constant load.

Of course the variety of examples that occur in engineering practice is unlimited.

Where no misunderstanding may arise, the primes denoting *actual* networks will be omitted.

II. THE GENERALITY OF THE METHOD OF ATTACK

(*a*) By considering every network as a complete network having a non-singular $\mathbf{C}$, it is possible to establish new networks having desired performance characteristics, without establishing:

1. The impedance tensor $\mathbf{z}$ (or $\mathbf{Y}$) of any of the networks.
2. The criterion of performance of any of the networks.

That is, it is possible to establish a large variety of new networks by considering only:

1. The self- and mutual impedances (or admittances) of their *individual coils* (without ever establishing mesh impedances).

2. The *transformation tensors* of the networks showing their interconnections **C** and their synthesis tensor $\mathbf{C}_s$.

That is, *in problems of synthesis shown, the impedance tensor* **z** *of networks or any part of them does not need to be established.*

(*b*) *In all problems of synthesis it will be assumed that the networks are active asymmetrical networks, hence they may include vacuum tubes, rotating machines running at or oscillating around a constant speed, and other linear electrical and mechanical networks.*

(*c*) In problems of synthesis the question always comes up: *Is a coil with the necessary impedance or admittance physically realizable?* Or if it is physically realizable, is it the most economical or is it the most practical? These questions are not investigated in detail in this chapter.

Since the method of attack of this chapter is so organized that the final self- and mutual impedances defined may refer to *individual coils* if so desired, and also since these individual coil impedances are defined in terms of known impedances and of several *arbitrary complex numbers, or linear operators,* any eventual organized method of attack on the physical realizability of networks may consider only *isolated* coils that are functions of *arbitrary* parameters. The analysis need not be complicated by the interconnection of coils, and it need not be limited to real parameters.

(*d*) *It is emphasized that the method of attack shown is not the most general possible,* even for the cases considered. It represents just one out of many possibilities.

III. CRITERION OF CONSTANT LOAD CURRENTS

(*a*) Let a network with any number of meshes and junction-pairs be given which is supplied by several generators at a *constant-potential* **E**. Let the network supply currents **I** to several outside loads (say to a three-phase load) whose admittance **Y** does not appear on the network diagram. *The problem is: What relations must exist among the admittances of the network in order that the currents* **I** *supplied to the load should also be constant, no matter how the load admittances* **Y** *vary?*

(*b*) No matter how complicated the network is, let it be considered as a junction network with its junction-pairs divided into three functionally different groups (Fig. 23.1).

1. The first group contains all those junction-pairs that have constant voltages $\mathbf{E}_1$ impressed upon them.

2. The second group is assumed across the various loads, into which $\mathbf{I}^2$ flows and across which $\mathbf{E}_2$ appears.

3. The third group involves the remaining inactive junction-pairs with a difference of potential $\mathbf{E}_3$ appearing across them.

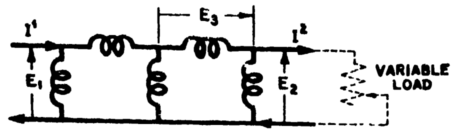


FIG. 23.1.—With $\mathbf{E}_1$ Constant, to Maintain $\mathbf{I}^2$ Constant

Hence if the network is considered as a compound junction-network with three junction-pairs its equation of current is

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{E}_2 - \mathbf{Y}^{13}\cdot\mathbf{E}_3 \\ \mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{Y}^{22}\cdot\mathbf{E}_2 - \mathbf{Y}^{23}\cdot\mathbf{E}_3 \\ 0 &= \mathbf{Y}^{31}\cdot\mathbf{E}_1 - \mathbf{Y}^{32}\cdot\mathbf{E}_2 - \mathbf{Y}^{33}\cdot\mathbf{E}_3 \end{aligned} \qquad 23.1$$

In these equations $\mathbf{E}_1$ is constant and $\mathbf{I}^2$ is to be maintained constant.

(*c*) Eliminating the inactive $\mathbf{E}_3$ from the third equation

$$\mathbf{E}_3 = \mathbf{Y}^{33-1}\cdot(\mathbf{Y}^{31}\cdot\mathbf{E}_1 - \mathbf{Y}^{32}\cdot\mathbf{E}_2)$$

Substituting into the other two equations, it gives

$$\begin{aligned} \mathbf{I}^1 &= (\mathbf{Y}^{11} - \mathbf{Y}^{13}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31})\cdot\mathbf{E}_1 - (\mathbf{Y}^{12} - \mathbf{Y}^{13}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{32})\cdot\mathbf{E}_2 \\ \mathbf{I}^2 &= (\mathbf{Y}^{21} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31})\cdot\mathbf{E}_1 - (\mathbf{Y}^{22} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{32})\cdot\mathbf{E}_2 \end{aligned} \qquad 23.2$$

The load current $\mathbf{I}^2$ is a function of the generator voltage $\mathbf{E}_1$ that is always constant and of $\mathbf{E}_2$, the voltage appearing across the load whose value varies with the load.

(*d*) Now *in order that* $\mathbf{I}^2$ *should be a function of the constant* $\mathbf{E}_1$ *and not of the variable* $\mathbf{E}_2$*, it is necessary and sufficient that the coefficent of* $\mathbf{E}_2$ *in the second equation should be zero. Hence if the admittances of the coils satisfy the "criterion of constant load currents"* (in which all "open-circuit admittances" as in equation 23.2 will in the succeeding pages be designated by a double prime instead of a single prime)

$$\boxed{\mathbf{Y}^{22''} = \mathbf{Y}^{22} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{32} = 0} \qquad 23.3$$

then the constant currents flowing into the loads are

$$\boxed{\mathbf{I}^2 = (\mathbf{Y}^{21} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31})\cdot\mathbf{E}_1} \qquad 23.4$$

Since the components of $\mathbf{E}_1$ *are constant, the components of* $\mathbf{I}^2$ *are also constant no matter what the loads are.* (If the components of $\mathbf{E}_1$ vary,

the components of $\mathbf{I}^2$ follow their variation by the proportionality factor $\mathbf{Y}^{21''}$.) That is, in any network the load currents are constant, if the admittance $\mathbf{Y}^{22''}$ looking from the load is zero. *The admittance may satisfy this condition either only at one particular frequency, or at all frequencies*, or along certain frequency band, depending on other conditions to be stated.

(*e*) The left-hand side of equation 23.3 is a matrix in each particular problem. A matrix is zero if each of its components is zero. Hence equating each component of the matrix of equation 23.3 to zero, *the criterion of performance represents a set of k^2 relations between the design constants of the network that must be satisfied, where k is the number of currents that are to be maintained constant.*

It may be mentioned that, if the network is considered a complete network instead of a junction network, then its admittance tensor contains sixteen component **Y**'s instead of nine, the other seven **Y**'s representing the meshes that are ignored. *These extra* **Y***'s may assume any value since they do not occur in the criterion anyway. Hence out of the sixteen compound tensors of the network fifteen have arbitrary values, while one of them,* $\mathbf{Y}^{22}$*, must be equal to* $\mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{32}$. Of course, in order to satisfy this last requirement by physically realizable coils, three of the other tensors $\mathbf{Y}^{23}$, $\mathbf{Y}^{32}$, and $\mathbf{Y}^{33}$ may not assume arbitrary values.

(*f*) If, instead of the load currents $\mathbf{I}^2$, the generator currents $\mathbf{I}^1$ are to be maintained constant, then the coefficient of $\mathbf{E}_2$ in the *first* equation of 23.2 is to be kept zero. That is, if

$$\mathbf{Y}^{12''} = \mathbf{Y}^{12} - \mathbf{Y}^{13}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{32} = 0 \qquad 23.5$$

this becomes the "criterion of constant generator currents" *and the generator currents* $\mathbf{I}^1$ are

$$\mathbf{I}^1 = (\mathbf{Y}^{11} - \mathbf{Y}^{13}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31})\cdot\mathbf{E}_1$$

(*g*) *If equations* 23.3 *and* 23.5 *are both satisfied, then at all loads both* $\mathbf{I}^1$ *and* $\mathbf{I}^2$ *are constant.*

(*h*) When only part of the load currents or generator currents are to be maintained constant, then the tensor equation of current of the network $\mathbf{I} = \mathbf{Y}\cdot\mathbf{E}$ is subdivided into more than three tensor equations that are to be analogously manipulated.

(*i*) *It should again be emphasized that all the above equations (and those to follow) apply to any asymmetrical network* that can be subdivided into the indicated number of groups of junction-pairs or meshes. Each group may contain any number of junction-pairs or meshes from one to infinity.

IV. CRITERION OF CONSTANT DIFFERENCES OF POTENTIAL

(*a*) Let the difference of potential $\mathbf{E}_3$ appearing across *some of the inactive junction-pairs* be maintained constant, while $\mathbf{E}_4$ of the remaining inactive terminals and $\mathbf{E}_2$ of the load terminals vary as the load varies (Fig. 23.2). The impressed voltages $\mathbf{E}_1$ are also constant.

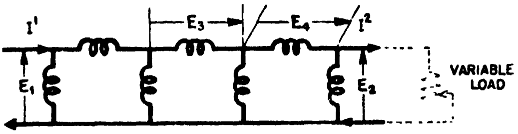


FIG. 23.2.– With $\mathbf{E}_1$ Constant, to Maintain $\mathbf{E}_3$ Constant

Dividing the junction-pairs into four types, their equation of current is

$$\begin{aligned}
\mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{E}_2 - \mathbf{Y}^{13}\cdot\mathbf{E}_3 - \mathbf{Y}^{14}\cdot\mathbf{E}_4\\
\mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{Y}^{22}\cdot\mathbf{E}_2 - \mathbf{Y}^{23}\cdot\mathbf{E}_3 - \mathbf{Y}^{24}\cdot\mathbf{E}_4\\
0 &= \mathbf{Y}^{31}\cdot\mathbf{E}_1 - \mathbf{Y}^{32}\cdot\mathbf{E}_2 - \mathbf{Y}^{33}\cdot\mathbf{E}_3 - \mathbf{Y}^{34}\cdot\mathbf{E}_4\\
0 &= \mathbf{Y}^{41}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{E}_2 - \mathbf{Y}^{43}\cdot\mathbf{E}_3 - \mathbf{Y}^{14}\cdot\mathbf{E}_4
\end{aligned} \qquad 23.6$$

(*b*) Eliminating $\mathbf{E}_4$ from the last equation

$$\mathbf{E}_4 = \mathbf{Y}^{44\,-1}\cdot(\mathbf{Y}^{41}\cdot\mathbf{E}_1 - \mathbf{Y}^{42}\cdot\mathbf{E}_2 - \mathbf{Y}^{43}\cdot\mathbf{E}_3)$$

Substituting it into the third equation

$$\begin{aligned}
0 = (\mathbf{Y}^{31} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{41})\cdot\mathbf{E}_1 &- (\mathbf{Y}^{32} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{42})\cdot\mathbf{E}_2\\
&- (\mathbf{Y}^{33} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{43})\cdot\mathbf{E}_3
\end{aligned}$$

The voltage $\mathbf{E}_3$ is constant if the coefficients of $\mathbf{E}_2$ are zero. That is, *the "criterion of constant voltage" is*

$$\boxed{\mathbf{Y}^{32\prime\prime} = \mathbf{Y}^{32} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{42} = 0} \qquad 23.7$$

The constant terminal potentials are

$$\begin{aligned}
\mathbf{E}_3 = (\mathbf{Y}^{33} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{43})^{-1}\cdot&(\mathbf{Y}^{31} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{41})\cdot\mathbf{E}_1\\
&= \mathbf{Y}^{33\prime\prime-1}\cdot\mathbf{Y}^{31\prime\prime}\cdot\mathbf{E}_1
\end{aligned} \qquad 23.8$$

V. CRITERION OF CONSTANT LOAD-VOLTAGES

(*a*) Now let the difference of potentials $\mathbf{E}_2$ across the *variable loads* be maintained constant, leaving some of the loads unchanged. That is,

let the network be divided into four types of junction-pairs (Fig. 23.3):

1. Constant potential ($\mathbf{E}_1$) input.
2. Constant potential ($\mathbf{E}_2$) across a variable load.
3. Variable potential ($\mathbf{E}_3$) across a constant load.
4. Inactive junction-pair ($\mathbf{E}_4$).

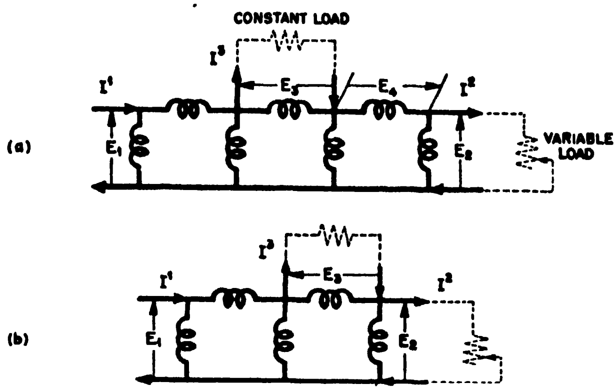


FIG. 23.3.—With $\mathbf{E}_1$ Constant to Maintain $\mathbf{E}_2$ Constant

(*b*) Eliminating the inactive junction-pairs by the reduction formula the equations of the active junction-pairs are

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{11''}\cdot\mathbf{E}_1 - \mathbf{Y}^{12''}\cdot\mathbf{E}_2 - \mathbf{Y}^{13''}\cdot\mathbf{E}_3 \\ \mathbf{I}^2 &= \mathbf{Y}^{21''}\cdot\mathbf{E}_1 - \mathbf{Y}^{22''}\cdot\mathbf{E}_2 - \mathbf{Y}^{23''}\cdot\mathbf{E}_3 \\ \mathbf{I}^3 &= \mathbf{Y}^{31''}\cdot\mathbf{E}_1 - \mathbf{Y}^{32''}\cdot\mathbf{E}_2 - \mathbf{Y}^{33''}\cdot\mathbf{E}_3 \end{aligned} \qquad 23.9$$

Let the value of constant loads be $\mathbf{Y}$ so that $\mathbf{I}^3 = \mathbf{Y}\cdot\mathbf{E}_3$. Substituting into the third equation

$$(\mathbf{Y} + \mathbf{Y}^{33''})\cdot\mathbf{E}_3 = \mathbf{Y}^{31''}\cdot\mathbf{E}_1 - \mathbf{Y}^{32''}\cdot\mathbf{E}_2$$

(*c*) The potentials $\mathbf{E}_2$ across the variable loads are constant if the coefficient of $\mathbf{E}_3$ is zero, that is, *the "criterion of constant load voltage" is*

$$\boxed{\mathbf{Y} + \mathbf{Y}^{33''} = 0} \qquad 23.10$$

The constant load potentials are

$$\mathbf{E}_2 = \mathbf{Y}^{32''-1}\cdot\mathbf{Y}^{31''}\cdot\mathbf{E}_1 \qquad 23.11$$

In order that the inverse of $\mathbf{Y}^{32''}$ should exist it is necessary that *as many constant load terminals should be assumed as there are variable*

load terminals. Any coil of the network may be considered as a constant load Y by simply considering its two junctions as a junction-pair.

(*d*) It is interesting that, in order to maintain constant load currents $\mathbf{I}^2$, the admittance of the network has to be examined from the variable load only. However, in order to maintain constant load voltages $\mathbf{E}_2$, the admittance of the network has to be examined both from the variable loads and from an equal number of other terminals.

VI. CRITERION OF EQUAL INPUT IMPEDANCE

(*a*) Let a network be given (Fig. 23.4) in which the input impedances $\mathbf{z}_i$ measured from several branches are known. The network may, for instance, have some desired impedance characteristics as the input frequency varies. It is desired to construct other networks that have the same input impedance $\mathbf{z}_i$, that is, that have the same impedance characteristics through the whole frequency range.

The meshes of the network are divided into two types:

1. The input meshes with impressed voltage $\mathbf{e}$.
2. The inactive meshes.

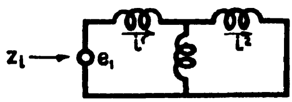


FIG. 23.4.—To Maintain the Input Impedance $\mathbf{z}_i$ Constant

The presence of the junction-pairs (a third group of opened meshes) is ignored.

(*b*) *The first step is to find the "criterion of performance,"* that is, to find the relation that must exist among the impedances of the network so that $\mathbf{z}_1$ should have the original value.

The equation of voltage is

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{i}^2 \\ 0 &= \mathbf{z}_{21}\cdot\mathbf{i}^1 + \mathbf{z}_{22}\cdot\mathbf{i}^2 \end{aligned} \qquad 23.12$$

To find the input impedance eliminate $\mathbf{i}^2$ from the second equation as $\mathbf{i}^2 = -\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{21}\cdot\mathbf{i}^1$. Substituting it into the first equation

$$\mathbf{e}_1 = (\mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{21})\cdot\mathbf{i}^1$$

The expression in parentheses is the input impedance of the network that has to be equal to the given $\mathbf{z}_i$. Hence, *the "criterion of constant input impedance"* is

$$\boxed{\mathbf{z}_i = \mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{21}} \qquad 23.13$$

VII. CRITERION OF EQUAL INPUT IMPEDANCE OF *ACTIVE* NETWORKS

(*a*) Let it be assumed that the network has the desired input characteristics when *impressed voltages* $\mathbf{e}_2$ *exist in some of the previously inactive meshes* (Fig. 23.5). In this case the given network also has a definite *equivalent input voltage* $\mathbf{e}$ that also must remain the same in all the other networks. Hence an *additional* criterion has to be constructed, which also has to be satisfied by all equivalent networks.

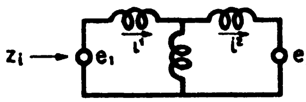


FIG. 23.5.—To Maintain z_i Constant in Active Network

The equation of the network is

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{i}^2 \\ \mathbf{e}_2 &= \mathbf{z}_{21}\cdot\mathbf{i}^1 + \mathbf{z}_{22}\cdot\mathbf{i}^2 \end{aligned} \qquad 23.14$$

(*b*) Eliminating $\mathbf{i}^2$ from the second equation and substituting into the first equation

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot(\mathbf{e}_2 - \mathbf{z}_{21}\cdot\mathbf{i}^1) \\ (\mathbf{e}_1 - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{e}_2) &= (\mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{21})\cdot\mathbf{i}^1 \end{aligned} \qquad 23.15$$

(This equation is the same as equation 10.17.)

The left-hand side represents the equivalent input voltage of the network, which has to be the same in all networks. Hence the additional "*criterion of constant input voltage*" is

$$\boxed{\mathbf{e}_i = \mathbf{e}_1 - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{e}_2} \qquad 23.16$$

(*c*) When only part of the previously inactive meshes contain impressed voltages, Fig. 23.6, then three groups of meshes are assumed

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{z}_{11}\cdot\mathbf{i}^1 + \mathbf{z}_{12}\cdot\mathbf{i}^2 + \mathbf{z}_{13}\cdot\mathbf{i}^3 \\ \mathbf{e}_2 &= \mathbf{z}_{21}\cdot\mathbf{i}^1 + \mathbf{z}_{22}\cdot\mathbf{i}^2 + \mathbf{z}_{23}\cdot\mathbf{i}^3 \\ 0 &= \mathbf{z}_{31}\cdot\mathbf{i}^1 + \mathbf{z}_{32}\cdot\mathbf{i}^2 + \mathbf{z}_{33}\cdot\mathbf{i}^3 \end{aligned} \qquad 23.17$$

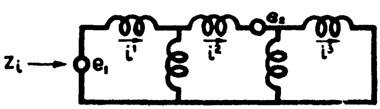


FIG. 23.6.—To Maintain z_i Constant in Active Network

Eliminating $\mathbf{i}^3$ and $\mathbf{i}^2$ by equations 10.33 and 10.34, the two criteria are

$$\boxed{\mathbf{z}_i = \mathbf{z}''_{11} - \mathbf{z}''_{12}\cdot\mathbf{z}''^{-1}_{22}\cdot\mathbf{z}''_{21}} \qquad 23.18$$

$$\boxed{\mathbf{e}_i = \mathbf{e}_1 - \mathbf{z}''_{12}\cdot\mathbf{z}''^{-1}_{22}\cdot\mathbf{e}_2} \qquad 23.19$$

where the primed quantities are short-circuit impedances.

VIII. CRITERION OF EQUAL INPUT OR OUTPUT ADMITTANCES

(*a*) Let a network be given (Fig. 23.7) in which the input admittances $\mathbf{Y}^i$ and the output admittances $\mathbf{Y}^0$ are known. *The problem is to construct other networks that have the same input and output admittances* $\mathbf{Y}^i$ and $\mathbf{Y}^0$.

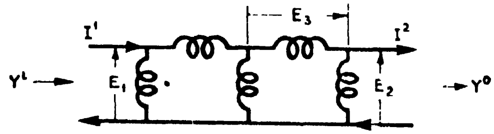


FIG 23.7 – To Maintain $\mathbf{Y}^i$ and $\mathbf{Y}^0$ Constant

The junction-pairs of the network are divided into three types.

1. The input junction-pairs.
2. The output junction-pairs.
3. The inactive junction-pairs.

The presence of the meshes (an additional group of closed junction-pairs) is ignored.

(*b*) The first step is to establish the conditions that must exist between the admittances of the network in order that $\mathbf{Y}^i$ and $\mathbf{Y}^0$ should be constant. That is, the first step is to find the "criterion of performance."

The equation of current of the network is

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{11}\cdot\mathbf{E}_1 - \mathbf{Y}^{12}\cdot\mathbf{E}_2 - \mathbf{Y}^{13}\cdot\mathbf{E}_3 \\ \mathbf{I}^2 &= \mathbf{Y}^{21}\cdot\mathbf{E}_1 - \mathbf{Y}^{22}\cdot\mathbf{E}_2 - \mathbf{Y}^{23}\cdot\mathbf{E}_3 \\ 0 &= \mathbf{Y}^{31}\cdot\mathbf{E}_1 - \mathbf{Y}^{32}\cdot\mathbf{E}_2 - \mathbf{Y}^{33}\cdot\mathbf{E}_3 \end{aligned} \qquad 23.20$$

Since the load $\mathbf{Y}^0$ is constant $\mathbf{I}^2 = \mathbf{Y}^0\cdot\mathbf{E}_2$. To find the input impedance of the network, let $\mathbf{E}_3$ be eliminated from the third equation and substituted into the other two, giving equation 23.2 again as

$$\begin{aligned} \mathbf{I}^1 &= (\mathbf{Y}^{11} - \mathbf{Y}^{13}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31})\cdot\mathbf{E}_1 - (\mathbf{Y}^{12} - \mathbf{Y}^{13}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{32})\cdot\mathbf{E}_2 \\ \mathbf{Y}^0\cdot\mathbf{E}_2 &= (\mathbf{Y}^{21} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31})\cdot\mathbf{E}_1 - (\mathbf{Y}^{22} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{32})\cdot\mathbf{E}_2 \end{aligned} \qquad 23.21$$

These equations may be written as

$$\begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{11''}\cdot\mathbf{E}_1 - \mathbf{Y}^{12''}\cdot\mathbf{E}_2 \\ \mathbf{Y}^0\cdot\mathbf{E}_2 &= \mathbf{Y}^{21''}\cdot\mathbf{E}_1 - \mathbf{Y}^{22''}\cdot\mathbf{E}_2 \end{aligned} \qquad 23.22$$

Eliminating $\mathbf{E}_2$ from the second equation

$$\mathbf{E}_2 = (\mathbf{Y}^0 + \mathbf{Y}^{22''})^{-1} \cdot \mathbf{Y}^{21''} \cdot \mathbf{E}_1$$

and substituting into the first

$$\mathbf{I}^1 = [\mathbf{Y}^{11''} - \mathbf{Y}^{12''} \cdot (\mathbf{Y}^0 + \mathbf{Y}^{22''})^{-1} \cdot \mathbf{Y}^{21''}] \cdot \mathbf{E}_1 \qquad 23.23$$

The expression in brackets is the input admittance of the network, which has to be equal to the given $\mathbf{Y}^i$. Hence "*the criterion of equal admittance*" is

$$\boxed{\mathbf{Y}^i = \mathbf{Y}^{11''} - \mathbf{Y}^{12''} \cdot (\mathbf{Y}^0 + \mathbf{Y}^{22''})^{-1} \cdot \mathbf{Y}^{21''}} \qquad 23.24$$

IX. DEFINITION OF THE "SYNTHESIS TENSOR" $\mathbf{C}_\sigma$

(*a*) Let it be assumed that for a given network (having a non-singular $\mathbf{C}_1$) the self- and mutual impedances or admittances of the individual coils have already been so selected that the network satisfies the desired "*criterion of performance*" and consequently it has the desired performance. Let it also be assumed that the impedance tensor $\mathbf{z}'_1$ (or $\mathbf{Y}^{1'}$) of the network is also known.

Now let *any other arbitrary network* $\mathbf{C}_2$ be assumed having different number of coils and interconnected in an arbitrary manner. The problem is to find $\mathbf{z}'_2$ (or $\mathbf{Y}^{2'}$) of the second network from $\mathbf{z}'_1$ (or $\mathbf{Y}^{1'}$) *by a transformation* so that the second network should behave in exactly the same manner as the first network. That is, $\mathbf{z}'_2$ or $\mathbf{Y}^{2'}$ of the second network is to be found without going through the labor of trial and error needed to establish a $\mathbf{z}'_2$ that will satisfy the "criterion of performance."

(*b*) *The transformation tensor that changes* $\mathbf{z}'_1$ *of a given network to* $\mathbf{z}'_2$ *of any other network but automatically keeps the "criterion of performance" invariant will be called the "synthesis tensor"* $\mathbf{C}'_\sigma$. *Instead of keeping the scalar* $\mathbf{e}^* \cdot \mathbf{i}$ *invariant,* $\mathbf{C}'_\sigma$ *keeps certain other combinations of tensors, namely, the "criteria of performance" invariant.*

The transformation tensors used up to now did several types of chores. They interconnected coils, changed number of turns, introduced hypothetical currents, and so on, doing several of these jobs in one step. *The transformation tensor* $\mathbf{C}'_\sigma$ *to be introduced now will do one other thing*; *it will change the impedances of the individual coils also, in addition to performing other functions, chiefly interconnecting coils.*

Since in the next few sections no primitive networks are considered, the primes will be left out.

X. ESTABLISHMENT OF THE SYNTHESIS TENSOR

(*a*) *A synthesis tensor* $\mathbf{C}_\sigma$, or $\mathbf{C}^{*-1}_{\sigma t} = \mathbf{A}_\sigma$, *will be established for each "criterion of performance" separately* as a compound tensor by the following steps. These steps are determined by the requirement that the "criterion of performance" should have the same form before and after transforming it by $\mathbf{C}_\sigma$. (Instead of transforming the criterion itself, it was found to be a quicker procedure first to transform $\mathbf{z}$ and then to reestablish the transformed form of the criterion.)

1. The impedance tensor $\mathbf{z}_1$ (or $\mathbf{Y}^1$) of the first network is set up as a *compound tensor* containing as many rows and columns as there are types of meshes and junction-pairs.
2. A non-singular compound transformation tensor $\mathbf{C}_\sigma$ (or $\mathbf{C}^{*-1}_{\sigma t}$) is set up, each component being an as yet undetermined 2-tensor.
3. The impedance tensor $\mathbf{z}_2$ (or $\mathbf{Y}^2$) of the new network is established by performing the multiplications $\mathbf{C}^*_{\sigma t}\cdot\mathbf{z}_1\cdot\mathbf{C}_\sigma$ (or $\mathbf{C}^{-1}_\sigma\cdot\mathbf{Y}^1\cdot\mathbf{C}^{*-1}_{\sigma t}$).
4. The "criterion of performance" is set up using the components of the new impedance tensor $\mathbf{z}_2$.

This criterion contains the same impedances as the original criterion; in addition, however, it contains all the component tensors of the compound synthesis tensor $\mathbf{C}_\sigma$.

5. *Some (not all) of the component tensors of* $\mathbf{C}_\sigma$ *are equated to zero or to the unit tensor* $\mathbf{I}$ *(or are changed to some other form) so that all components of* $\mathbf{C}_\sigma$ *drop out of the "criterion of performance."*

(*b*) That is, it will be found that, if the synthesis tensor $\mathbf{C}_\sigma$ assumes a simpler form by making some of its components zero or unity, etc., while leaving other components quite arbitrary, then after multiplying $\mathbf{z}_1$ of the network by this special $\mathbf{C}_\sigma$ the "criterion of performance" remains unchanged since all components of $\mathbf{C}_\sigma$ drop out of sight. Hence *the impedances of the coils, the interconnection of the coils and the number of coils all may be changed by* $\mathbf{C}_\sigma$ *without changing the "criterion of performance." The new network will perform in exactly the same manner as the given network.*

Since several of the components of $\mathbf{C}_\sigma$ contain arbitrary quantities (real or complex numbers, or linear operators) *there is a large number of networks, each with a large variety of coil impedances, that have the same performance characteristics.* The selection of a particular network with a particular coil impedance is determined by physical and economic considerations.

XI. THE SYNTHESIS TENSOR FOR CONSTANT OUTPUT CURRENT

(*a*) The admittance tensor of the junction network whose output current is to be maintained constant is, from equation 23.1,

$$\mathbf{Y} = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & \mathbf{Y}^{11} & \mathbf{Y}^{12} & \mathbf{Y}^{13} \\ \hline 2 & \mathbf{Y}^{21} & \mathbf{Y}^{22} & \mathbf{Y}^{23} \\ \hline 3 & \mathbf{Y}^{31} & \mathbf{Y}^{32} & \mathbf{Y}^{33} \\ \hline \end{array} \qquad 23.25$$

It is emphasized that $\mathbf{Y}$ of the complete orthogonal network contains one additional row and column representing the ignored meshes.

The "criterion of performance," that is to be maintained invariant, is, from equation 23.3,

$$\mathbf{Y}^{22} - \mathbf{Y}^{23} \cdot \mathbf{Y}^{33-1} \cdot \mathbf{Y}^{32} = 0 \qquad 23.26$$

It should be noted that:

1. The components of the *first* row and column of $\mathbf{Y}$ (the input axes) do not occur in the criterion.

2. The components of the *fourth* row and column of $\mathbf{Y}$, representing the ignored meshes and not shown in equation 23.1, also do not occur in the criterion.

(*b*) *The problem is to establish a transformation tensor* $\mathbf{A}_\sigma$ ("conjugate transpose inverse transformation tensor" $\mathbf{C}_{\sigma t}^{*-1}$ that is denoted by $\mathbf{A}_\sigma$) changing $\mathbf{Y}$ of the given network to $\mathbf{Y}'$ of any other network as

$$\mathbf{Y}' = \mathbf{A}_{\sigma t}^{*} \cdot \mathbf{Y} \cdot \mathbf{A}_\sigma = \begin{array}{c|c|c|c|} & 1' & 2' & 3' \\ \hline 1' & \mathbf{Y}^{11'} & \mathbf{Y}^{12'} & \mathbf{Y}^{13'} \\ \hline 2' & \mathbf{Y}^{21'} & \mathbf{Y}^{22'} & \mathbf{Y}^{23'} \\ \hline 3' & \mathbf{Y}^{31'} & \mathbf{Y}^{32'} & \mathbf{Y}^{33'} \\ \hline \end{array} \qquad 23.27$$

so that the "criterion of performance" remains unchanged, invariant. That is, after the transformation

$$\boxed{\mathbf{Y}^{22'} - \mathbf{Y}^{23'} \cdot \mathbf{Y}^{33'-1} \cdot \mathbf{Y}^{32'} = \mathbf{Y}^{22} - \mathbf{Y}^{23} \cdot \mathbf{Y}^{33-1} \cdot \mathbf{Y}^{32}} \qquad 23.28$$

from which the components of $\mathbf{A}_\sigma$ have dropped out. If this condition is satisfied that even *after* the transformation

$$\mathbf{Y}^{22'} - \mathbf{Y}^{23'} \cdot \mathbf{Y}^{33'-1} \cdot \mathbf{Y}^{32'} = 0 \qquad 23.29$$

and the output currents of the new network are constant.

The synthesis tensor has the form

$$
\mathbf{A}_\sigma = \begin{array}{c|c|c|c|} & 1' & 2' & 3' \\ \hline 1 & A^1_1 & A^2_1 & A^3_1 \\ \hline 2 & A^1_2 & A^2_2 & A^3_2 \\ \hline 3 & A^1_3 & A^2_3 & A^3_3 \\ \hline \end{array}
\qquad
\mathbf{A}^*_{\sigma t} = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1' & A^{1*}_{1t} & A^{1*}_{2t} & A^{1*}_{3t} \\ \hline 2' & A^{2*}_{1t} & A^{2*}_{2t} & A^{2*}_{3t} \\ \hline 3' & A^{3*}_{1t} & A^{3*}_{2t} & A^{3*}_{3t} \\ \hline \end{array}
\qquad 23.30
$$

in which all the components are as yet arbitrary.

(*c*) After transforming **Y** by the formula $\mathbf{A}^*_{\sigma t}\cdot\mathbf{Y}\cdot\mathbf{A}_\sigma$, the new $\mathbf{Y}'$ of equation 23.27 is $\mathbf{Y}'$ =

$$
\begin{array}{|l|l|l|}
\hline
\begin{array}{l} A^{1*}_{1t}\cdot(Y^{11}\cdot A^1_1 \\ \quad +Y^{12}\cdot A^1_2 + Y^{13}\cdot A^1_3) \\ \quad +A^{1*}_{2t}\cdot(Y^{21}\cdot A^1_1 \\ \quad +Y^{22}\cdot A^1_2 + Y^{23}\cdot A^1_3) \\ \quad +A^{1*}_{3t}\cdot(Y^{31}\cdot A^1_1 \\ \quad +Y^{32}\cdot A^1_2 + Y^{33}\cdot A^1_3) \end{array} &
\begin{array}{l} A^{1*}_{1t}\cdot(Y^{11}\cdot A^2_1 \\ \quad +Y^{12}\cdot A^2_2 + Y^{13}\cdot A^2_3) \\ \quad +A^{1*}_{2t}\cdot(Y^{21}\cdot A^2_1 \\ \quad +Y^{22}\cdot A^2_2 + Y^{23}\cdot A^2_3) \\ \quad +A^{1*}_{3t}\cdot(Y^{31}\cdot A^2_1 \\ \quad +Y^{32}\cdot A^2_2 + Y^{33}\cdot A^2_3) \end{array} &
\begin{array}{l} A^{1*}_{1t}\cdot(Y^{11}\cdot A^3_1 \\ \quad +Y^{12}\cdot A^3_2 + Y^{13}\cdot A^3_3) \\ \quad +A^{1*}_{2t}\cdot(Y^{21}\cdot A^3_1 \\ \quad +Y^{22}\cdot A^3_2 + Y^{23}\cdot A^3_3) \\ \quad +A^{1*}_{3t}\cdot(Y^{31}\cdot A^3_1 \\ \quad +Y^{32}\cdot A^3_2 + Y^{33}\cdot A^3_3) \end{array} \\
\hline
\begin{array}{l} A^{2*}_{1t}\cdot(Y^{11}\cdot A^1_1 \\ \quad +Y^{12}\cdot A^1_2 + Y^{13}\cdot A^1_3) \\ \quad +A^{2*}_{2t}\cdot(Y^{21}\cdot A^1_1 \\ \quad +Y^{22}\cdot A^1_2 + Y^{23}\cdot A^1_3) \\ \quad +A^{2*}_{3t}\cdot(Y^{31}\cdot A^1_1 \\ \quad +Y^{32}\cdot A^1_2 + Y^{33}\cdot A^1_3) \end{array} &
\begin{array}{l} A^{2*}_{1t}\cdot(Y^{11}\cdot A^2_1 \\ \quad +Y^{12}\cdot A^2_2 + Y^{13}\cdot A^2_3) \\ \quad +A^{2*}_{2t}\cdot(Y^{21}\cdot A^2_1 \\ \quad +Y^{22}\cdot A^2_2 + Y^{23}\cdot A^2_3) \\ \quad +A^{2*}_{3t}\cdot(Y^{31}\cdot A^2_1 \\ \quad +Y^{32}\cdot A^2_2 + Y^{33}\cdot A^2_3) \end{array} &
\begin{array}{l} A^{2*}_{1t}\cdot(Y^{11}\cdot A^3_1 \\ \quad +Y^{12}\cdot A^3_2 + Y^{13}\cdot A^3_3) \\ \quad +A^{2*}_{2t}\cdot(Y^{21}\cdot A^3_1 \\ \quad +Y^{22}\cdot A^3_2 + Y^{23}\cdot A^3_3) \\ \quad +A^{2*}_{3t}\cdot(Y^{31}\cdot A^3_1 \\ \quad +Y^{32}\cdot A^3_2 + Y^{33}\cdot A^3_3) \end{array} \\
\hline
\begin{array}{l} A^{3*}_{1t}\cdot(Y^{11}\cdot A^1_1 \\ \quad +Y^{12}\cdot A^1_2 + Y^{13}\cdot A^1_3) \\ \quad +A^{3*}_{2t}\cdot(Y^{21}\cdot A^1_1 \\ \quad +Y^{22}\cdot A^1_2 + Y^{23}\cdot A^1_3) \\ \quad +A^{3*}_{3t}\cdot(Y^{31}\cdot A^1_1 \\ \quad +Y^{32}\cdot A^1_2 + Y^{33}\cdot A^1_3) \end{array} &
\begin{array}{l} A^{3*}_{1t}\cdot(Y^{11}\cdot A^2_1 \\ \quad +Y^{12}\cdot A^2_2 + Y^{13}\cdot A^2_3) \\ \quad +A^{3*}_{2t}\cdot(Y^{21}\cdot A^2_1 \\ \quad +Y^{22}\cdot A^2_2 + Y^{23}\cdot A^2_3) \\ \quad +A^{3*}_{3t}\cdot(Y^{31}\cdot A^2_1 \\ \quad +Y^{32}\cdot A^2_2 + Y^{33}\cdot A^2_3) \end{array} &
\begin{array}{l} A^{3*}_{1t}\cdot(Y^{11}\cdot A^3_1 \\ \quad +Y^{12}\cdot A^3_2 + Y^{13}\cdot A^3_3) \\ \quad +A^{3*}_{2t}\cdot(Y^{21}\cdot A^3_1 \\ \quad +Y^{22}\cdot A^3_2 + Y^{23}\cdot A^3_3) \\ \quad +A^{3*}_{3t}\cdot(Y^{31}\cdot A^3_1 \\ \quad +Y^{32}\cdot A^3_2 + Y^{33}\cdot A^3_3) \end{array} \\
\hline
\end{array}
\qquad 23.31
$$

(*d*) To set up the criterion, it is necessary to find the inverse of the expression in the right lower 2-tensor $\mathbf{Y}^{33\prime}$ containing nine different **Y**'s. Since in the criterion (equation 23.26) only the inverse of $\mathbf{Y}^{33}$ occurs, *all terms in the right lower 2-tensor* $\mathbf{Y}^{33\prime}$ *may be reduced to zero except* $\mathbf{Y}^{33}$, *if* $\mathbf{A}^3_1$ *and* $\mathbf{A}^3_2$ *are assumed to be zero.* (Of course, other, more complicated assumptions could also produce the same result.)

Also in order that in the center 2-tensor $\mathbf{Y}^{22\prime}$ the term of $\mathbf{Y}^{22}$ needed in the criterion should have the form of $\mathbf{Y}^{22}$ without being multiplied by anything, $\mathbf{A}^2_2$ *may be equated to the unit tensor* **I**. (Other assumptions are also possible. This present assumption is made only to simplify the results.)

Hence if $A_1^3 = 0$, $A_2^3 = 0$, and $A_2^2 = I$, then Y' becomes

$$
Y' = \begin{array}{c|r|r|l|}
 & 1' & 2' & 3' \\
\hline
1' & \begin{array}{r} A_{1t}^{1*}\cdot(Y^{11}\cdot A_1^1 + Y^{12}\cdot A_2^1 \\ +Y^{13}\cdot A_3^1) \\ +A_{2t}^{1*}\cdot(Y^{21}\cdot A_1^1 + Y^{22}\cdot A_2^1 \\ +Y^{23}\cdot A_3^1) \\ +A_{3t}^{1*}\cdot(Y^{31}\cdot A_1^1 + Y^{32}\cdot A_2^1 \\ +Y^{33}\cdot A_3^1) \end{array} & \begin{array}{r} A_{1t}^{1*}\cdot(Y^{11}\cdot A_1^2 + Y^{12} \\ +Y^{13}\cdot A_3^2) \\ +A_{2t}^{1*}\cdot(Y^{21}\cdot A_1^2 + Y^{22} \\ +Y^{23}\cdot A_3^2) \\ +A_{3t}^{1*}\cdot(Y^{31}\cdot A_1^2 + Y^{32} \\ +Y^{33}\cdot A_3^2) \end{array} & \begin{array}{l} A_{1t}^{1*}\cdot Y^{13} + \\ A_{2t}^{1*}\cdot Y^{23} + \\ A_{3t}^{1*}\cdot Y^{33})\cdot A_3^3 \end{array} \\
\hline
2' & \begin{array}{r} A_{1t}^{2*}\cdot(Y^{11}\cdot A_1^1 + Y^{12}\cdot A_2^1 \\ +Y^{13}\cdot A_3^1) \\ +(Y^{21}\cdot A_1^1 + Y^{22}\cdot A_2^1 \\ +Y^{23}\cdot A_3^1) \\ +A_{3t}^{2*}\cdot(Y^{31}\cdot A_1^1 + Y^{32}\cdot A_2^1 \\ +Y^{33}\cdot A_3^3) \end{array} & \begin{array}{r} A_{1t}^{2*}\cdot(Y^{11}\cdot A_1^2 + Y^{12} \\ +Y^{13}\cdot A_3^2) \\ +(Y^{21}\cdot A_1^2 + Y^{22} \\ +Y^{23}\cdot A_3^2) \\ +A_{3t}^{2*}\cdot(Y^{31}\cdot A_1^2 + Y^{32} \\ +Y^{33}\cdot A_3^2) \end{array} & \begin{array}{l} (A_{1t}^{2*}\cdot Y^{13} + \\ Y^{23} + \\ A_{3t}^{2*}\cdot Y^{33})\cdot A_3^3 \end{array} \\
\hline
3' & \begin{array}{r} A_{3t}^{3*}\cdot(Y^{31}\cdot A_1^1 + Y^{32}\cdot A_2^1 \\ +Y^{33}\cdot A_3^1) \end{array} & \begin{array}{r} A_{3t}^{3*}\cdot(Y^{31}\cdot A_1^2 + Y^{32} \\ +Y^{33}\cdot A_3^2) \end{array} & A_{3t}^{3*}\cdot Y^{33}\cdot A_3^3 \\
\hline
\end{array} \qquad 23.32
$$

(*e*) The "criterion of performance" $Y^{22'} - Y^{23'}\cdot Y^{33'-1}\cdot Y^{32'}$ as calculated from this tensor assumes the form

$$
\begin{aligned}
A_{1t}^{2*}\cdot(Y^{11}\cdot A_1^2+Y^{12}+Y^{13}\cdot A_3^2)+(Y^{21}\cdot A_1^2+Y^{22}+Y^{23}\cdot A_3^2)+A_{3t}^{2*}\cdot(Y^{31}\cdot A_1^2 \\
+Y^{32}+Y^{33}\cdot A_3^2)-(A_{1t}^{2*}\cdot Y^{13}+Y^{23} \\
+A_{3t}^{2*}\cdot Y^{33})\cdot Y^{33^{-1}}\cdot(Y^{31}\cdot A_1^2+Y^{32}+Y^{33}\cdot A_3^2)
\end{aligned} \qquad 23.33
$$

Simplifying, the criterion assumes the form

$$
\begin{aligned}
A_{1t}^{2*}\cdot(Y^{11}\cdot A_1^2+Y^{12})+Y^{21}\cdot A_1^2+Y^{22}-(A_{1t}^{2*}\cdot Y^{13} \\
+Y^{23})\cdot Y^{33^{-1}}\cdot(Y^{31}\cdot A_1^2+Y^{32})
\end{aligned} \qquad 23.34
$$

If A_1^2 *is also made equal to zero, then the criterion becomes* $Y^{22} - Y^{23}\cdot Y^{33-1}\cdot Y^{32}$. This is the form sought for the criterion.

(*f*) Hence *if the synthesis tensor assumes the form*

$$
A_\sigma = \begin{array}{c|c|c|c|}
 & 1' & 2' & 3' \\
\hline
1 & A_1^1 & 0 & 0 \\
\hline
2 & A_2^1 & I & 0 \\
\hline
3 & A_3^1 & A_3^2 & A_3^3 \\
\hline
\end{array} \qquad 23.35
$$

in which the inverse of A_3^3 *exists, then the "criterion of constant load current," equation* 23.26, *remains invariant.*

In other words, (1) if Y is multiplied by A_σ to find Y', and (2) if the criterion is established from the components of Y', then *all the components of* A_σ *of equation* 23.35 *disappear from the criterion, no matter what the various component* A*'s are.*

The component A's may contain real numbers, complex numbers, linear operators, etc. No matter what they are, they do not influence the constancy of the load currents. However, *the inverse of* $\mathbf{A}_3^3$ *must exist,* since it occurs in establishing the criterion, equation 23.33.

XII. THE SINGULAR SYNTHESIS TENSOR

(*a*) *The very important fact should be noted that since the inverse of* $\mathbf{A}_{1'}^1$ *is not needed, its matrix may have any rectangular form, so that the number of axes in group* 1′ *may vary from zero to any number. Hence the new network may have either more or fewer junction-pairs than the original network has. There is however an absolute minimum number of junction-pairs that the new network may have, namely the sum of the junction-pairs in groups* 2 *and* 3.

Since the matrix of most $\mathbf{A}_\sigma$ is rectangular, their inverse can not be calculated and the various matrices of $\mathbf{A}_\sigma$ form only a "*semi-group*" (Chapter XI, Section III) instead of a "group." While all previous transformation tensors have transformed into each other only such networks that had the *same* number of coils, the synthesis tensor transforms into each other networks that have *different* number of coils.

The synthesis tensor $\mathbf{A}_\sigma = A_\sigma^\alpha$ *may be considered as a "multiple tensor"* with one index σ belonging to the references axes possessed by all n-coil networks and with the other index α belonging to those possessed by all k-coil networks. In the previous studies the n-coil networks and their reference axes were independent of the k-coil networks and their axes.

(*b*) The transformation $\mathbf{A}_\sigma$ may involve (among others) the following three changes:

1. The interconnection of the network may be changed.
2. The impedances of the coils may be changed.
3. The number of coils may be changed.

(*c*) *It must be emphasized that the given* $\mathbf{A}_\sigma$ *is not the only transformation tensor that leaves the output current invariant.* By knowing the value of the Y's it is possible to give values other than zero to $\mathbf{A}_{1'}^2$, $\mathbf{A}_{1'}^3$, and $\mathbf{A}_2^3$ that keep the criterion invariant. These other possible forms of $\mathbf{A}_\sigma$ are not investigated here.

XIII. THE EFFECT OF INACTIVE JUNCTION-PAIRS ON $\mathbf{A}_\sigma$

(*a*) It was mentioned that the criterion $\mathbf{Y}^{22} - \mathbf{Y}^{23} \cdot \mathbf{Y}^{33^{-1}} \cdot \mathbf{Y}^{32}$ did not involve the *first* rows and columns of $\mathbf{Y}$ of equation 23.1 and con-

sequently it would have been sufficient to calculate the *second* and *third* rows and columns of $\mathbf{Y}'$ by a similarly decreased $\mathbf{A}_\sigma$ as

$$\mathbf{Y} = \begin{array}{c|cc|} & 2 & 3 \\ \hline 2 & Y^{22} & Y^{23} \\ 3 & Y^{32} & Y^{33} \\ \hline \end{array} \qquad \mathbf{A}_\sigma = \begin{array}{c|cc|} & 2' & 3' \\ \hline 2 & I & 0 \\ 3 & A^2_3 & A^3_3 \\ \hline \end{array} \qquad \mathbf{Y}' = \begin{array}{c|cc|} & 2' & 3' \\ \hline 2' & Y^{22'} & Y^{23'} \\ 3' & Y^{32'} & Y^{33'} \\ \hline \end{array} \tag{23.36}$$

The effect of an extra row and column in $\mathbf{Y}$ *(that is not needed in the criterion) is to add an extra column to* $\mathbf{A}_\sigma$ *(containing the arbitrary components* A^1_1, A^1_2, *and* A^1_3*) since multiplication with this extra column of* $\mathbf{A}_\sigma$ *does not change the needed components of* $\mathbf{Y}'$. That is

$$\mathbf{Y} = \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & Y^{11} & Y^{12} & Y^{13} \\ 2 & Y^{21} & Y^{22} & Y^{23} \\ 3 & Y^{31} & Y^{32} & Y^{33} \\ \hline \end{array} \qquad \mathbf{A}_\sigma = \begin{array}{c|ccc|} & 1' & 2' & 3' \\ \hline 1 & A^1_1 & 0 & 0 \\ 2 & A^1_2 & I & 0 \\ 3 & A^1_3 & A^2_3 & A^3_3 \\ \hline \end{array} \qquad \mathbf{Y}' = \begin{array}{c|ccc|} & 1' & 2' & 3' \\ \hline 1' & Y^{11'} & Y^{12'} & Y^{13'} \\ 2' & Y^{21'} & Y^{22'} & Y^{23'} \\ 3' & Y^{31'} & Y^{32'} & Y^{33'} \\ \hline \end{array} \tag{23.37}$$

It should be recalled (Section XV, Chapter XVI) that the addition to $\mathbf{C}$ of *columns* along the extra junction-pair axes (and the corresponding addition to $\mathbf{z}$) does not change the components of $\mathbf{z}'$ along the other mesh axes. Similarly here the addition of column $1'$ to $\mathbf{A}_\sigma$ (and the corresponding addition of row and column 1 to $\mathbf{Y}$) does not change the components of $\mathbf{Y}'$ along the other axes $2'$ and $3'$. Since $\mathbf{A}_\sigma$ can have additional non-zero components only along the added columns, the additional components along its other columns are zero.

Hence, *in calculating the synthesis tensor it is sufficient to transform those compound axes of* $\mathbf{Y}$ *(or* $\mathbf{z}$*) only that play a part in the criterion. The remaining compound axes add the same number of arbitrary columns to the synthesis tensor.*

(*b*) Since in the present example $\mathbf{Y}$ contains still another compound axis 4 representing the ignored mesh axes, therefore the synthesis tensor contains another arbitrary column as

$$\mathbf{Y} = \begin{array}{c|cccc|} & 4 & 1 & 2 & 3 \\ \hline 4 & Y^{44} & Y^{41} & Y^{42} & Y^{43} \\ 1 & Y^{14} & Y^{11} & Y^{12} & Y^{13} \\ 2 & Y^{24} & Y^{21} & Y^{22} & Y^{23} \\ 3 & Y^{34} & Y^{31} & Y^{32} & Y^{33} \\ \hline \end{array} \tag{23.38}$$

$$\mathbf{A}_\sigma = \begin{array}{c|cccc|} & 4' & 1' & 2' & 3' \\ \hline 4 & A^4_4 & A^1_4 & 0 & 0 \\ 1 & A^4_1 & A^1_1 & 0 & 0 \\ 2 & A^4_2 & A^1_2 & I & 0 \\ 3 & A^4_3 & A^1_3 & A^2_3 & A^3_3 \\ \hline \end{array} \tag{23.39}$$

In using this tensor the network should be considered as a complete network.

Since $\mathbf{A}_\sigma$ need not have an inverse, *the new network may have any arbitrary number of mesh axes* **4′**.

(*c*) In other words, if any network has equation 23.38 as its admittance tensor and it supplies constant load current $\mathbf{I}^2$, then it can be multiplied by $\mathbf{A}_\sigma$ of equation 23.39, changing its interconnection, its number of coils and the admittances of its coils, and the new network still supplies constant load current $\mathbf{I}^{2\prime}$.

However, this current $\mathbf{I}^{2\prime}$ is not the same as the current $\mathbf{I}^2$, since by equation 23.4 $\mathbf{I}^2 = (\mathbf{Y}^{21} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31})\cdot\mathbf{E}_1$ and the coefficient of $\mathbf{E}_1$ does not remain invariant after transforming $\mathbf{Y}$ by $\mathbf{A}_\sigma$ of equation 23.39.

XIV. SYNTHESIS TENSOR FOR UNVARYING OUTPUT CURRENT

(*a*) Let it be now required not only to keep $\mathbf{I}^{2\prime}$ constant while the load varies, but also *to keep the new value of* $\mathbf{I}^{2\prime}$ *the same after the transformation as it is before the transformation.* In other words, let $\mathbf{A}_\sigma$ keep not only the coefficient of $\mathbf{E}_2$ in the second equation of 23.2, namely, $\mathbf{Y}^{22} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{32}$, invariant, but also the coefficient of $\mathbf{E}_1$, namely, $\mathbf{Y}^{21} - \mathbf{Y}^{32}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31}$.

In this second criterion all three axes of $\mathbf{Y}$ contain a component, hence the only ignorable axes are the mesh axes **4**.

(*b*) From equation 23.32 (where now also $\mathbf{A}_1^2$ is zero), this last criterion is

$$\begin{aligned}\mathbf{Y}^{21}\cdot\mathbf{A}_1^1 + \mathbf{Y}^{22}\cdot\mathbf{A}_2^1 + \mathbf{Y}^{23}\cdot\mathbf{A}_3^1 + \mathbf{A}_{3t}^{2*}\cdot(\mathbf{Y}^{31}\cdot\mathbf{A}_1^1 + \mathbf{Y}^{32}\cdot\mathbf{A}_2^1 + \mathbf{Y}^{33}\cdot\mathbf{A}_3^1) - \\ - (\mathbf{Y}^{23}\cdot\mathbf{A}_3^3 + \mathbf{A}_{3t}^{2*}\cdot\mathbf{Y}^{33}\cdot\mathbf{A}_3^3)\cdot\mathbf{A}_3^{3-1}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{A}_{3t}^{3*-1}\cdot\mathbf{A}_{3t}^{3*}\cdot \\ \cdot(\mathbf{Y}^{31}\cdot\mathbf{A}_1^1 + \mathbf{Y}^{32}\cdot\mathbf{A}_2^1 + \mathbf{Y}^{33}\cdot\mathbf{A}_3^1)\end{aligned} \qquad 23.40$$

Simplifying, the criterion becomes

$$\mathbf{Y}^{21}\cdot\mathbf{A}_1^1 + \mathbf{Y}^{22}\cdot\mathbf{A}_2^1 - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot(\mathbf{Y}^{31}\cdot\mathbf{A}_1^1 + \mathbf{Y}^{32}\cdot\mathbf{A}_2^1)$$

If $\mathbf{A}_2^1$ *is made equal to zero and* $\mathbf{A}_1^1$ *is made equal to* $\mathbf{I}$, the criterion becomes $\mathbf{Y}^{21} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31}$.

(*c*) Hence, also the magnitude of $\mathbf{I}^2$ remains the same in all transformations if $\mathbf{A}_\sigma$ has the form

$$\mathbf{A}_\sigma = \begin{array}{c|c|c|c|} & 1' & 2' & 3' \\ \hline 1 & \mathbf{I} & & \\ \hline 2 & & \mathbf{I} & \\ \hline 3 & \mathbf{A}_3^1 & \mathbf{A}_3^2 & \mathbf{A}_3^3 \\ \hline \end{array} \qquad 23.41$$

The effect of the ignored mesh axes is to add an arbitrary fourth column to $\mathbf{A}_\sigma$ hence *the complete synthesis tensor is*

$$\mathbf{A}_\sigma = \begin{array}{c|c|c|c|c|} & 4' & 1' & 2' & 3' \\ \hline 4 & \mathbf{A}_4^4 & & & \\ \hline 1 & \mathbf{A}_1^4 & \mathbf{I} & & \\ \hline 2 & \mathbf{A}_2^4 & & \mathbf{I} & \\ \hline 3 & \mathbf{A}_3^4 & \mathbf{A}_3^1 & \mathbf{A}_3^2 & \mathbf{A}_3^3 \\ \hline \end{array} \qquad 23.42$$

where $\mathbf{A}_4^4$ may have any rectangular form.

(*d*) A simple computation shows that this last $\mathbf{A}_\sigma$ leaves also the coefficients of $\mathbf{E}_1$ and $\mathbf{E}_2$ in the first equation of 23.2 invariant. That is, $\mathbf{A}_\sigma$ *of equation* 23.42 *leaves all phenomena viewed from the input and output axes* 1 *and* 2 *unchanged. It should be noted that along these invariant axes* $\mathbf{A}_\sigma$ *contains only the unit tensor* $\mathbf{I}$.

(*e*) Computation also shows that the above synthesis tensor would have been found if the criterion had been $\mathbf{Y}^{21} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31}$ alone, or $\mathbf{Y}^{12} - \mathbf{Y}^{13}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{32}$ alone.

This latter criterion leaves the generator current $\mathbf{I}^1$ constant. It appears that in order to do that it is also necessary to leave the output current $\mathbf{I}^2$ also constant at the same time. This occurs only because some of the $\mathbf{A}$'s were made equal to zero, which steps were not at all necessary. They could have been defined as functions of some of the $\mathbf{Y}$'s. In order to simplify the results, that definition was not introduced here.

XV. SYNTHESIS TENSOR FOR CONSTANT DIFFERENCES OF POTENTIAL

(*a*) In order to keep the differences of potential $\mathbf{E}_3$ *across some of the inactive junction-pairs* constant the criterion is that in the compound admittance tensor $\mathbf{Y}$ having four rows and columns $\mathbf{Y}^{32} - \mathbf{Y}^{34}\cdot\mathbf{Y}^{44-1}\cdot\mathbf{Y}^{42} = 0$. (Equation 23.7.)

Since this criterion does not involve the components of the *first* row and column of $\mathbf{Y}$, it is sufficient to examine the changes in the remaining rows and columns of $\mathbf{Y}$, namely, in

$$\mathbf{Y} = \begin{array}{c|c|c|c|} & 2 & 3 & 4 \\ \hline 2 & \mathbf{Y}^{22} & \mathbf{Y}^{23} & \mathbf{Y}^{24} \\ \hline 3 & \mathbf{Y}^{32} & \mathbf{Y}^{33} & \mathbf{Y}^{34} \\ \hline 4 & \mathbf{Y}^{42} & \mathbf{Y}^{43} & \mathbf{Y}^{44} \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|c|} & 2 & 3 & 4 \\ \hline 2 & \mathbf{Y}^{11} & \mathbf{Y}^{12} & \mathbf{Y}^{13} \\ \hline 3 & \mathbf{Y}^{21} & \mathbf{Y}^{22} & \mathbf{Y}^{23} \\ \hline 4 & \mathbf{Y}^{31} & \mathbf{Y}^{32} & \mathbf{Y}^{33} \\ \hline \end{array} \qquad \mathbf{A}_\sigma = \begin{array}{c|c|c|c|} & 2' & 3' & 4' \\ \hline 2 & \mathbf{A}_1^1 & \mathbf{A}_1^2 & \mathbf{A}_1^3 \\ \hline 3 & \mathbf{A}_2^1 & \mathbf{A}_2^2 & \mathbf{A}_2^3 \\ \hline 4 & \mathbf{A}_3^1 & \mathbf{A}_3^2 & \mathbf{A}_3^3 \\ \hline \end{array}. \qquad 23.43$$

(*b*) In order to use some of the results of Section XI, let the indices of the component Y's be changed *temporarily* as shown. The criterion becomes $\mathbf{Y}^{21} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33-1}\cdot\mathbf{Y}^{31} = 0$.

The product $\mathbf{A}^{*}_{\sigma t}\cdot\mathbf{Y}\cdot\mathbf{A}_{\sigma}$ is shown in equation 23.31. Since here it is also necessary to keep the inverse of $\mathbf{Y}^{33}$ unchanged, here also $\mathbf{A}^3_1 = 0$ and $\mathbf{A}^3_2 = 0$.

In order to keep the tensor $\mathbf{Y}^{21'}$ as $\mathbf{Y}^{21}$, let $\mathbf{A}^2_2 = \mathbf{I}$ and $\mathbf{A}^1_1 = \mathbf{I}$. With these values equation 23.31 becomes equation 23.32, in which also $\mathbf{A}^1_1 = \mathbf{I}$.

(*c*) The criterion of performance $\mathbf{Y}^{21'} - \mathbf{Y}^{23'}\cdot\mathbf{Y}^{33'-1}\cdot\mathbf{Y}^{31'}$ is from equation 23.32

$$\begin{aligned}\mathbf{A}^{2*}_{1t}\cdot(\mathbf{Y}^{11} + \mathbf{Y}^{12}\cdot\mathbf{A}^1_2 + \mathbf{Y}^{13}\cdot\mathbf{A}^1_3) + \mathbf{Y}^{21} + \mathbf{Y}^{22}\cdot\mathbf{A}^1_2 + \mathbf{Y}^{23}\cdot\mathbf{A}^1_3 + \\ + \mathbf{A}^{2*}_{3t}\cdot(\mathbf{Y}^{31} + \mathbf{Y}^{32}\cdot\mathbf{A}^1_2 + \mathbf{Y}^{33}\cdot\mathbf{A}^1_3) - (\mathbf{A}^{2*}_{1t}\cdot\mathbf{Y}^{13} + \mathbf{Y}^{23} + \\ + \mathbf{A}^{2*}_{3t}\cdot\mathbf{Y}^{33})\cdot\mathbf{Y}^{33-1}\cdot(\mathbf{Y}^{31} + \mathbf{Y}^{32}\cdot\mathbf{A}^1_2 + \mathbf{Y}^3_3\cdot\mathbf{A}^1_3)\end{aligned} \tag{23.44}$$

Simplifying, the criterion becomes

$$\begin{aligned}\mathbf{A}^{2*}_{1t}\cdot(\mathbf{Y}^{11} + \mathbf{Y}^{12}\cdot\mathbf{A}^1_2) + \mathbf{Y}^{21} + \mathbf{Y}^{22}\cdot\mathbf{A}^1_2 - \\ - (\mathbf{A}^{2*}_{1t}\cdot\mathbf{Y}^{13} + \mathbf{Y}^{23})\cdot\mathbf{Y}^{33-1}\cdot(\mathbf{Y}^{31} + \mathbf{Y}^{32}\cdot\mathbf{A}^1_2)\end{aligned} \tag{23.44a}$$

If $\mathbf{A}^2_1 = 0$ and $\mathbf{A}^1_2 = 0$, the criterion becomes $\mathbf{Y}^{21} - \mathbf{Y}^{23}\cdot\mathbf{Y}^{33\,-1}\cdot\mathbf{Y}^{31}$. Hence the synthesis tensor has the same form as equation 23.41. Changing the subscripts of the components of $\mathbf{A}_\sigma$ and adding two arbitrary columns along the ignored junction-pair axis **1** and along the ignored mesh axis **5**, the synthesis tensor is

$$\mathbf{A}_\sigma = \begin{array}{c|ccccc|} & 5' & 1' & 2' & 3' & 4' \\ \hline 5 & \mathbf{A}^5_5 & \mathbf{A}^1_5 & & & \\ 1 & \mathbf{A}^5_1 & \mathbf{A}^1_1 & & & \\ 2 & \mathbf{A}^5_2 & \mathbf{A}^1_2 & \mathbf{I} & & \\ 3 & \mathbf{A}^5_3 & \mathbf{A}^1_3 & & \mathbf{I} & \\ 4 & \mathbf{A}^5_4 & \mathbf{A}^1_4 & \mathbf{A}^2_4 & \mathbf{A}^3_4 & \mathbf{A}^4_5 \\ \hline \end{array} \tag{23.45}$$

XVI. SYNTHESIS TENSOR FOR EQUAL INPUT IMPEDANCE

(*a*) In order to keep the impedance of the network constant when viewed from several input terminals, the criterion (equation 23.13) is that $\mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{21}$ should remain invariant while the coil inter-

connections or the coil impedances are changed by $\mathbf{C}_\sigma$. The problem is to find the form of $\mathbf{C}_\sigma$. Let

$$\mathbf{z} = \begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & \mathbf{z}_{11} & \mathbf{z}_{12} \\ \hline 2 & \mathbf{z}_{21} & \mathbf{z}_{22} \\ \hline \end{array} \qquad \mathbf{C}_\sigma = \begin{array}{c|c|c|} & 1' & 2' \\ \hline 1 & \mathbf{C}^1_1 & \mathbf{C}^1_2 \\ \hline 2 & \mathbf{C}^2_1 & \mathbf{C}^2_2 \\ \hline \end{array} \tag{23.46}$$

After transformation the network impedance is by $\mathbf{C}^*_{\sigma t}\cdot\mathbf{z}\cdot\mathbf{C}_\sigma$

$$\mathbf{z}' = \begin{array}{c|c|c|} & 1' & 2' \\ \hline 1' & \begin{matrix}(\mathbf{C}^{1*}_{1t}\cdot\mathbf{z}_{11} + \mathbf{C}^{2*}_{1t}\cdot\mathbf{z}_{21})\cdot\mathbf{C}^1_1 + \\ (\mathbf{C}^{1*}_{1t}\cdot\mathbf{z}_{12} + \mathbf{C}^{2*}_{1t}\cdot\mathbf{z}_{22})\cdot\mathbf{C}^2_1\end{matrix} & \begin{matrix}(\mathbf{C}^{1*}_{1t}\cdot\mathbf{z}_{11} + \mathbf{C}^{2*}_{1t}\cdot\mathbf{z}_{21})\cdot\mathbf{C}^1_2 + \\ (\mathbf{C}^{1*}_{1t}\cdot\mathbf{z}_{12} + \mathbf{C}^{2*}_{1t}\cdot\mathbf{z}_{22})\cdot\mathbf{C}^2_2\end{matrix} \\ \hline 2' & \begin{matrix}(\mathbf{C}^{1*}_{2t}\cdot\mathbf{z}_{11} + \mathbf{C}^{2*}_{2t}\cdot\mathbf{z}_{21})\cdot\mathbf{C}^1_1 + \\ (\mathbf{C}^{1*}_{2t}\cdot\mathbf{z}_{12} + \mathbf{C}^{2*}_{2t}\cdot\mathbf{z}_{22})\cdot\mathbf{C}^2_1\end{matrix} & \begin{matrix}(\mathbf{C}^{1*}_{2t}\cdot\mathbf{z}_{11} + \mathbf{C}^{2*}_{2t}\cdot\mathbf{z}_{21})\cdot\mathbf{C}^1_2 \\ (\mathbf{C}^{1*}_{2t}\cdot\mathbf{z}_{12} + \mathbf{C}^{2*}_{2t}\cdot\mathbf{z}_{22})\cdot\mathbf{C}^2_2\end{matrix} \\ \hline \end{array} \tag{23.47}$$

(*b*) In order to find the inverse of the right-lower corner matrix, assume $\mathbf{C}^1_2 = 0$. In order to leave $\mathbf{z}_{11}$ in the upper left-hand corner tensor unchanged, assume $\mathbf{C}^1_1 = \mathbf{I}$. Hence assuming

$$\mathbf{C}_\sigma = \begin{array}{c|c|c|} & 1' & 2' \\ \hline 1 & \mathbf{I} & 0 \\ \hline 2 & \mathbf{C}^2_1 & \mathbf{C}^2_2 \\ \hline \end{array} \tag{23.48}$$

as the synthesis tensor, the impedance tensor is after the transformation

$$\mathbf{z}' = \begin{array}{c|c|c|} & 1' & 2' \\ \hline 1' & \mathbf{z}_{11} + \mathbf{z}_{12}\cdot\mathbf{C}^2_1 + \mathbf{C}^{2*}_{1t}\cdot\mathbf{z}_{21} + \mathbf{C}^{2*}_{1t}\cdot\mathbf{z}_{22}\cdot\mathbf{C}^2_1 & \mathbf{z}_{12}\cdot\mathbf{C}^2_2 + \mathbf{C}^{2*}_{1t}\cdot\mathbf{z}_{22}\cdot\mathbf{C}^2_2 \\ \hline 2' & \mathbf{C}^{2*}_{2t}\cdot\mathbf{z}_{21} + \mathbf{C}^{2*}_{2t}\cdot\mathbf{z}_{22}\cdot\mathbf{C}^2_1 & \mathbf{C}^{2*}_{2t}\cdot\mathbf{z}_{22}\cdot\mathbf{C}^2_2 \\ \hline \end{array} \tag{23.49}$$

(*c*) The input impedance as viewed from axis $1'$ is by the reduction formula

$$\begin{aligned} \mathbf{z}'_{11} - \mathbf{z}'_{12}\cdot\mathbf{z}'^{-1}_{22}\cdot\mathbf{z}'_{21} &= (\mathbf{z}_{11} + \mathbf{z}_{12}\cdot\mathbf{C}^2_1 + \mathbf{C}^{2*}_{1t}\cdot\mathbf{z}_{21} + \\ &+ \mathbf{C}^{2*}_{1t}\cdot\mathbf{z}_{22}\cdot\mathbf{C}^2_1) - (\mathbf{z}_{12}\cdot\mathbf{C}^2_2 + \mathbf{C}^{2*}_{1t}\cdot\mathbf{z}_{22}\cdot\mathbf{C}^2_2)\cdot \\ &\cdot(\mathbf{C}^{2*}_{2t}\cdot\mathbf{z}_{22}\cdot\mathbf{C}^2_2)^{-1}\cdot(\mathbf{C}^{2*}_{2t}\cdot\mathbf{z}_{21} + \mathbf{C}^{2*}_{2t}\cdot\mathbf{z}_{22}\cdot\mathbf{C}^2_1) \end{aligned}$$

Simplifying

$$\mathbf{z}'_{11} - \mathbf{z}'_{12}\cdot\mathbf{z}'^{-1}_{22}\cdot\mathbf{z}'_{21} = \mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}^{-1}_{22}\cdot\mathbf{z}_{21} \tag{23.50}$$

Hence the input impedance (the criterion of performance) is the same *after* the transformation as it is *before* the transformation. *The components of* $\mathbf{C}_\sigma$ *dropped out of the picture.*

Assuming the ignored junction-pair axes **3′** also, *the synthesis tensor that keeps the input impedance of an asymmetrical network unchanged is*

$$\mathbf{C}_\sigma = \begin{array}{c|c|c|c|} & \mathbf{1'} & \mathbf{2'} & \mathbf{3'} \\ \hline \mathbf{1} & \mathbf{I} & & \mathbf{C}^1_3 \\ \hline \mathbf{2} & \mathbf{C}^2_1 & \mathbf{C}^2_2 & \mathbf{C}^2_3 \\ \hline \mathbf{3} & & & \mathbf{C}^3_3 \\ \hline \end{array} \qquad 23.51$$

where $\mathbf{C}^3_3$ may have any rectangular form.

(*d*) *Similarly* $\mathbf{C}_\sigma$ *applies if instead of the input impedance* $\mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{21}$ *the input admittance* $\mathbf{Y}^{11} - \mathbf{Y}^{12}\cdot\mathbf{Y}^{22-1}\cdot\mathbf{Y}^{21}$ *is kept invariant.* For instance in order to keep both input and output admittances, equation 23.24 (containing open-circuit admittances), constant along axes **1** and **2**, the synthesis tensor is

$$\mathbf{A}_\sigma = \begin{array}{c|c|c|c|c|} & \mathbf{1'} & \mathbf{2'} & \mathbf{3'} & \mathbf{4'} \\ \hline \mathbf{1} & \mathbf{I} & & & \mathbf{A}^4_1 \\ \hline \mathbf{2} & & \mathbf{I} & & \mathbf{A}^4_2 \\ \hline \mathbf{3} & \mathbf{A}^1_3 & \mathbf{A}^2_3 & \mathbf{A}^3_3 & \mathbf{A}^4_3 \\ \hline \mathbf{4} & & & & \mathbf{A}^4_4 \\ \hline \end{array} \qquad 23.52$$

(*e*) Or as another example, in order to maintain the voltages $\mathbf{E}_2$ across the loads constant, it is necessary to keep by equation 23.10 the admittance of axis **3**, namely $\mathbf{Y}^{33''}$, constant in the presence of the inactive junction-pair axes **4**. Hence

$$\mathbf{A}_\sigma = \begin{array}{c|c|c|c|c|} & \mathbf{1'} & \mathbf{2'} & \mathbf{3'} & \mathbf{4'} \\ \hline \mathbf{1} & \mathbf{A}^1_1 & \mathbf{A}^2_1 & & \\ \hline \mathbf{2} & \mathbf{A}^1_2 & \mathbf{A}^2_2 & & \\ \hline \mathbf{3} & \mathbf{A}^1_3 & \mathbf{A}^2_3 & \mathbf{I} & \\ \hline \mathbf{4} & \mathbf{A}^1_4 & \mathbf{A}^2_4 & \mathbf{A}^3_4 & \mathbf{A}^4_4 \\ \hline \end{array} \quad \text{or} \quad \mathbf{A}_\sigma = \begin{array}{c|c|c|c|c|c|} & \mathbf{1'} & \mathbf{2'} & \mathbf{3'} & \mathbf{4'} & \mathbf{5'} \\ \hline \mathbf{1} & \mathbf{A}^1_1 & \mathbf{A}^2_1 & & & \mathbf{A}^5_1 \\ \hline \mathbf{2} & \mathbf{A}^1_2 & \mathbf{A}^2_2 & & & \mathbf{A}^5_2 \\ \hline \mathbf{3} & \mathbf{A}^1_3 & \mathbf{A}^2_3 & \mathbf{I} & & \mathbf{A}^5_3 \\ \hline \mathbf{4} & \mathbf{A}^1_4 & \mathbf{A}^2_4 & \mathbf{A}^3_4 & \mathbf{A}^4_4 & \mathbf{A}^5_4 \\ \hline \mathbf{5} & \mathbf{A}^1_5 & \mathbf{A}^2_5 & & & \mathbf{A}^5_5 \\ \hline \end{array} \qquad 23.53$$

where **5** is the ignored compound mesh axis.

(*f*) Let it be assumed that *all* the inactive meshes of the previous network contain impressed voltages $\mathbf{e}_2$ so that the voltage vector $\mathbf{e}$ before and after the transformation $\mathbf{C}_\sigma$ (equation 23.48) is

$$\mathbf{e} = \begin{array}{c|c|} \hline \mathbf{1} & \mathbf{e}_1 \\ \hline \mathbf{2} & \mathbf{e}_2 \\ \hline \end{array} \qquad 23.54 \qquad \mathbf{e}' = \mathbf{C}^*_{\sigma t}\cdot\mathbf{e} = \begin{array}{c|c|} \hline \mathbf{1'} & \mathbf{e}_1 + \mathbf{C}^{2*}_{1t}\cdot\mathbf{e}_2 \\ \hline \mathbf{2'} & \mathbf{C}^{2*}_{2t}\cdot\mathbf{e}_2 \\ \hline \end{array} \qquad 23.55$$

(*g*) *The second criterion* (equation 23.16) is by equation 23.54 and 23.49

$$\mathbf{e}_i' = \mathbf{e}_1' - \mathbf{z}_{12}'\cdot\mathbf{z}_{22}'^{-1}\cdot\mathbf{e}_2' =$$

$$= \mathbf{e}_1 + \mathbf{C}_{1t}^{2*}\cdot\mathbf{e}_2 - (\mathbf{z}_{12}\cdot\mathbf{C}_2^2 + \mathbf{C}_{1t}^{2*}\cdot\mathbf{z}_{22}\cdot\mathbf{C}_2^2)\cdot\mathbf{C}_2^{2-1}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{C}_{2t}^{2*-1}\cdot(\mathbf{C}_{2t}^{2*}\cdot\mathbf{e}_2)$$

$$= \mathbf{e}_1 - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{e}_2 \qquad 23.56$$

Hence *the previous synthesis tensors are valid also if the inactive meshes or junction-pairs contain impressed voltages or currents, that is, they keep both the input impedances and input voltages of active networks invariant.*

(*h*) A special case of the simpler synthesis tensor of 23.48 (without including the junction-pair axes 3) has been established by Cauer. In the synthesis tensor of Cauer:

1. The components of $\mathbf{C}_\sigma$, namely $\mathbf{C}_1^2$ and $\mathbf{C}_2^2$, may contain only *real* numbers.

2. The network is *symmetrical*, that is, it contains only lumped resistances, inductances, and elastances.

3. The network is *passive*.

The limitations of Cauer's synthesis tensor are due to the fact that it was *not* established by keeping a "criterion of performance," namely, the input impedance $\mathbf{z}_{11} - \mathbf{z}_{12}\cdot\mathbf{z}_{22}^{-1}\cdot\mathbf{z}_{21}$, invariant, but it was established by *keeping three real quadratic forms simultaneously invariant.* These three real quadratic forms are $\mathbf{i}\cdot\mathbf{r}\cdot\mathbf{i}$, $\mathbf{i}\cdot\mathbf{l}\cdot\mathbf{i}/2$, and $\mathbf{i}\cdot\mathbf{s}\cdot\mathbf{i}/2$ of mesh networks.

It should be noted that in general no quadratic (or hermitian) forms are associated with the networks considered in this chapter, since they are not necessarily stationary symmetrical networks.

XVII. THE "PRIMITIVE" SYNTHESIS TENSOR

(*a*) The synthesis tensor $\mathbf{C}_\sigma'$ changes $\mathbf{z}_1'$ (or $\mathbf{Y}^{1'}$) of a network to $\mathbf{z}_2'$ (or $\mathbf{Y}^{2'}$) of another network having the same performance. Since the components of both $\mathbf{z}_1'$ and $\mathbf{z}_2'$ represent the self- and mutual impedances of several coils connected into closed or open *meshes*, it is quite awkward to determine from them the impedances of the *individual* coils.

A synthesis tensor $\mathbf{C}_\sigma$ *may be established between the individual coils of two networks instead of between their meshes or junction-pairs.* That is, it is possible to find immediately the self- and mutual impedances of the individual coils of the second network in terms of those of the first network with the aid of $\mathbf{C}_\sigma$. This additional step is quite important since it is much easier to examine the self and mutual impedances

of individual coils for, say, their physical realizability, than the self and mutual impedances of whole meshes, open or closed.

(*b*) $\mathbf{C}_\sigma$ of the individual coils may be found from $\mathbf{C}'_\sigma$ of the meshes (established in the previous sections) by the following steps. Let:

1. $\mathbf{C}_1$ = transformation tensor of the original network changing $\mathbf{z}_1$ to $\mathbf{z}'_1$ by $\mathbf{C}^*_{1t}\cdot\mathbf{z}_1\cdot\mathbf{C}_1 = \mathbf{z}'_1$.
2. $\mathbf{C}_2$ = transformation tensor of the new network changing $\mathbf{z}_2$ to $\mathbf{z}'_2$ by $\mathbf{C}^*_{2t}\cdot\mathbf{z}_2\cdot\mathbf{C}_2 = \mathbf{z}'_2$.
3. $\mathbf{C}'_\sigma$ = network synthesis tensor transforming $\mathbf{z}'_1$ to $\mathbf{z}'_2$ by $\mathbf{C}^{*\prime}_{\sigma t}\cdot\mathbf{z}'_1\cdot\mathbf{C}'_\sigma = \mathbf{z}'_2$. It is assumed to be known. In general it has a singular matrix.
4. $\mathbf{C}_\sigma$ = primitive synthesis tensor (to be determined) transforming $\mathbf{z}_1$ to $\mathbf{z}_2$ by $\mathbf{C}^*_{\sigma t}\cdot\mathbf{z}_1\cdot\mathbf{C}_\sigma = \mathbf{z}_2$.

Since the two networks have different number of coils, $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *have different number of rows and columns.*

Substituting 1 and 2 into 3:

$$\mathbf{C}^{*\prime}_{\sigma t}\cdot(\mathbf{C}^*_{1t}\cdot\mathbf{z}_1\cdot\mathbf{C}_1)\cdot\mathbf{C}'_\sigma = \mathbf{C}^*_{2t}\cdot\mathbf{z}_2\cdot\mathbf{C}_2$$

Transferring $\mathbf{C}^*_{2t}$ and $\mathbf{C}_2$ to the left-hand side

$$\mathbf{C}^{*-1}_{2t}\cdot\mathbf{C}^{*\prime}_{\sigma t}\cdot\mathbf{C}^*_{1t}\cdot\mathbf{z}_1\cdot\mathbf{C}_1\cdot\mathbf{C}'_\sigma\cdot\mathbf{C}_2^{-1} = \mathbf{z}_2$$

By comparison with 4 the individual synthesis tensor $\mathbf{C}_\sigma$ *transforming* $\mathbf{z}_1$ *to* $\mathbf{z}_2$ *is found from the original synthesis tensor* $\mathbf{C}'_\sigma$ *by*

$$\boxed{\mathbf{C}_\sigma = \mathbf{C}_1\cdot\mathbf{C}'_\sigma\cdot\mathbf{C}_2^{-1}} \qquad 23.57$$

where $\mathbf{C}_1$ *is the transformation tensor of the first network and* $\mathbf{C}_2$ *is that of the new network assumed.*

Since the inverse of the transformation tensor $\mathbf{C}_2$ is needed in calculating $\mathbf{C}_\sigma$ *each network is assumed as a completely orthogonal network in establishing its* $\mathbf{C}'_\sigma$.

(*c*) In changing $\mathbf{Y}^1$ to $\mathbf{Y}^2$ each $\mathbf{C}$ is replaced by $\mathbf{C}^{*-1}_t$ so that

$$\mathbf{C}^{*-1}_{\sigma t} = \mathbf{C}^{*-1}_{1t}\cdot\mathbf{C}^{*\prime-1}_{\sigma t}\cdot\mathbf{C}^*_{2t} \quad \text{or} \quad \mathbf{A}_\sigma = \mathbf{A}_1\cdot\mathbf{A}'_\sigma\cdot\mathbf{A}_2^{-1} \qquad 23.58$$

(*d*) If it is not intended to change the network itself, but only the impedances of the individual coils, then $\mathbf{C}_2 = \mathbf{C}_1$. That is, *the impedances of the individual coils of any network* $\mathbf{C}_1$ *may be changed by the transformation tensor*

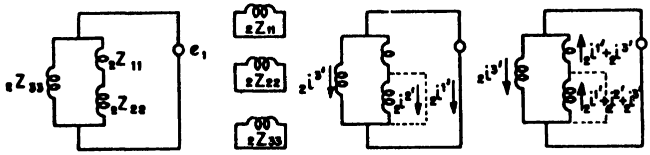
$$\boxed{\mathbf{C}_\sigma = \mathbf{C}_1\cdot\mathbf{C}'_\sigma\cdot\mathbf{C}_1^{-1}} \qquad 23.59$$

without changing the performance characteristic of the network.

Of course, each coil itself may be a *compound* coil representing a whole network.

XVIII. EXAMPLE OF TWO EQUIVALENT NETWORKS

(*a*) Let *three* coils with *asymmetrical* mutual impedances be given,



(a) (b) (c) (d)

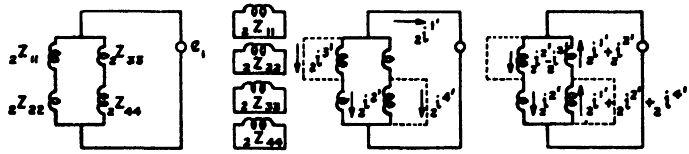
FIG. 23.8.—First Network with Three Coils

forming two meshes and one junction-pair (Fig. 23.8*a*). As a complete network its $\mathbf{C}$ and $\mathbf{C}^{-1}$ are

$$\mathbf{C}_1 = \begin{array}{c|ccc} & 1_{1'} & 2_{1'} & 3_{1'} \\ \hline 1_1 & 1 & & \\ 2_1 & & 1 & \\ 3_1 & 1 & 1 & 1 \end{array} \qquad \mathbf{C}_1^{-1} = \begin{array}{c|ccc} & 1_1 & 2_1 & 3_1 \\ \hline 1_{1'} & 1 & & \\ 2_{1'} & & 1 & \\ 3_{1'} & 1 & -1 & -1 \end{array} \qquad 23.61$$

When viewed from e_1 this network has a certain impedance Z_1'.

(*b*) Now let another network of Fig. 23.9 be given in which *four*



(a) (b) (c) (d)

FIG. 23.9.—Second Network with Four Coils

coils are arranged arbitrarily to form two meshes and two junction-pairs. (Each coil of course may represent the equivalent impedance of a whole network). As a complete network its $\mathbf{C}$ and $\mathbf{C}^{-1}$ are

$$\mathbf{C}_2 = \begin{array}{c|cccc} & 1_{2'} & 2_{2'} & 3_{2'} & 4_{2'} \\ \hline 1_2 & & 1 & -1 & \\ 2_2 & & 1 & & \\ 3_2 & 1 & 1 & & \\ 4_2 & 1 & 1 & & 1 \end{array} \qquad \mathbf{C}_2^{-1} = \begin{array}{c|cccc} & 1_2 & 2_2 & 3_2 & 4_2 \\ \hline 1_2' & & -1 & 1 & \\ 2_2' & & 1 & & \\ 3_2' & -1 & 1 & & \\ 4_2' & & & -1 & 1 \end{array} \qquad 23.62$$

The problem is to find the self- and mutual impedances of the four coils of the second network $\mathbf{z}_2$ *in terms of the three coils of the original network* $\mathbf{z}_1$

$$\mathbf{z}_1 = \begin{array}{c|c|c|c|} & 1_1 & 2_1 & 3_1 \\ \hline 1_1 & Z_{11} & Z_{12} & \\ \hline 2_1 & Z_{21} & Z_{22} & \\ \hline 3_1 & & & Z_{33} \\ \hline \end{array} \qquad 23.63 \qquad \mathbf{z}_2 = \begin{array}{c|c|c|c|c|} & 1_2 & 2_2 & 3_2 & 4_2 \\ \hline 1_2 & Z'_{11} & Z'_{12} & Z'_{13} & Z'_{14} \\ \hline 2_2 & Z'_{21} & Z'_{22} & Z'_{23} & Z'_{24} \\ \hline 3_2 & Z'_{31} & Z'_{32} & Z'_{33} & Z'_{34} \\ \hline 4_2 & Z'_{41} & Z'_{42} & Z'_{43} & Z'_{44} \\ \hline \end{array} \qquad 23.64$$

so that both networks should have the same input impedance when viewed from the impressed voltage e_1.

(*c*) The impedance tensor $\mathbf{z}'_1$ of the original network (not established) may be multiplied by the synthesis tensor $\mathbf{C}_\sigma$ of equation 23.51

$$\mathbf{C}'_\sigma = \begin{array}{|c|c|c|} \hline \mathbf{I} & & \mathbf{C}^1_3 \\ \hline \mathbf{C}^2_1 & \mathbf{C}^2_2 & \mathbf{C}^2_3 \\ \hline & & \mathbf{C}^3_3 \\ \hline \end{array} = \begin{array}{c|c|c|c|c|} & 1'_2 & 2'_2 & 3'_2 & 4'_2 \\ \hline 1'_1 & 1 & & k_3 & k_6 \\ \hline 2'_1 & k_1 & k_2 & k_4 & k_7 \\ \hline 3'_1 & & & k_5 & k_8 \\ \hline \end{array} \qquad 23.65$$

where the component k's are any real or complex numbers or operators, giving the impedance tensor $\mathbf{z}'_2$ of the new network (not established).

Let the above $\mathbf{C}'_\sigma$ *be changed to an individual synthesis tensor* $\mathbf{C}_\sigma$ *changing the impedances of their primitive networks* (namely, changing $\mathbf{z}_1$ to $\mathbf{z}_2$) by equation 23.58. Hence by $\mathbf{C}_1 \cdot \mathbf{C}'_\sigma \cdot \mathbf{C}_2^{-1} =$

$$\begin{array}{|c|c|c|} \hline 1 & & \\ \hline & 1 & \\ \hline 1 & 1 & 1 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|} \hline 1 & & k_3 & k_6 \\ \hline k_1 & k_2 & k_4 & k_7 \\ \hline & & k_5 & k_8 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|} \hline & -1 & 1 & \\ \hline & 1 & & \\ \hline -1 & 1 & & \\ \hline & & -1 & 1 \\ \hline \end{array} =$$

$$= \begin{array}{|c|c|c|c|} \hline -k_3 & k_3 - 1 & 1 - k_2 & k_6 \\ \hline -k_4 & k_2 + k_4 - k_1 & k_1 - k_7 & k_7 \\ \hline -k_3-k_4-k_5 & k_3+k_4+k_5+k_2-k_1-1 & 1+k_1-k_6-k_7-k_8 & k_6+k_7+k_8 \\ \hline \end{array} =$$

$$= \mathbf{C}_\sigma = \begin{array}{c|c|c|c|c|} & 1_2 & 2_2 & 3_2 & 4_2 \\ \hline 1_1 & K_1 & K_2 & K_3 & K_4 \\ \hline 2_1 & K_5 & K_6 & K_7 & K_8 \\ \hline 3_1 & K_9 & K_{10} & K_{11} & K_{12} \\ \hline \end{array} \qquad 23.66$$

in which the k's are arbitrary quantities.

(*d*) The impedance tensor $\mathbf{z}_2$ of the primitive network of Fig. 23.9 is by $\mathbf{C}^*_{\sigma t}\cdot\mathbf{z}_1\cdot\mathbf{C}_\sigma = \mathbf{z}_2 =$

$$
\begin{array}{c|l|l|l|l|}
 & \qquad 1_2 & \qquad 2_2 & \qquad 3_2 & \qquad 4_2 \\
\hline
1_2 & \begin{array}{l} K^*_{1t}(K_1Z_{11}+K_5Z_{12})+ \\ K^*_{5t}(K_1Z_{21}+K_5Z_{22})+ \\ K^*_{9t}K_9Z_{33} \end{array} & \begin{array}{l} K^*_{1t}(K_2Z_{11}+K_6Z_{12})+ \\ K^*_{5t}(K_2Z_{21}+K_6Z_{22})+ \\ K^*_{9t}K_{10}Z_{33} \end{array} & \begin{array}{l} K^*_{1t}(K_3Z_{11}+K_7Z_{12}) \\ K^*_{5t}(K_3Z_{21}+K_7Z_{22}) \\ K^*_{9t}K_{11}Z_{33} \end{array} & \begin{array}{l} K^*_{1t}(K_4Z_{11}+K_8Z_{12}) \\ K^*_{5t}(K_4Z_{21}+K_8Z_{22}) \\ K^*_{9t}K_{12}Z_{33} \end{array} \\
\hline
2_2 & \begin{array}{l} K^*_{2t}(K_1Z_{11}+K_5Z_{12})+ \\ K^*_{6t}(K_1Z_{21}+K_5Z_{22})+ \\ K^*_{10}K_9Z_{33} \end{array} & \begin{array}{l} K^*_{2t}(K_2Z_{11}+K_6Z_{12})+ \\ K^*_{6t}(K_2Z_{21}+K_6Z_{22})+ \\ K^*_{10t}K_{10}Z_{33} \end{array} & \begin{array}{l} K^*_{2t}(K_3Z_{11}+K_7Z_{12}) \\ K^*_{6t}(K_3Z_{21}+K_7Z_{22}) \\ K^*_{10}K_{11}Z_{33} \end{array} & \begin{array}{l} K^*_{2t}(K_4Z_{11}+K_8Z_{12}) \\ K^*_{6t}(K_4Z_{21}+K_8Z_{22}) \\ K^*_{10}K_{12}Z_{33} \end{array} \\
\hline
3_2 & \begin{array}{l} K^*_{3t}(K_1Z_{11}+K_5Z_{12})+ \\ K^*_{7t}(K_1Z_{21}+K_5Z_{22})+ \\ K^*_{11t}K_9Z_{33} \end{array} & \begin{array}{l} K^*_{3t}(K_2Z_{11}+K_6Z_{12})+ \\ K^*_{7t}(K_2Z_{21}+K_6Z_{22})+ \\ K^*_{11t}K_{10}Z_{33} \end{array} & \begin{array}{l} K^*_{3t}(K_3Z_{11}+K_7Z_{12}) \\ K^*_{7t}(K_3Z_{21}+K_7Z_{22}) \\ K^*_{11t}K_{11}Z_{33} \end{array} & \begin{array}{l} K^*_{3t}(K_4Z_{11}+K_8Z_{12}) \\ K^*_{7t}(K_4Z_{21}+K_8Z_{22}) \\ K^*_{11t}K_{12}Z_{33} \end{array} \\
\hline
4_2 & \begin{array}{l} K^*_{4t}(K_1Z_{11}+K_5Z_{12})+ \\ K^*_{8t}(K_1Z_{21}+K_5Z_{22})+ \\ K^*_{12t}K_9Z_{33} \end{array} & \begin{array}{l} K^*_{4t}(K_2Z_{11}+K_6Z_{12})+ \\ K^*_{8t}(K_2Z_{21}+K_6Z_{22})+ \\ K^*_{12t}K_{10}Z_{33} \end{array} & \begin{array}{l} K^*_{4t}(K_3Z_{11}+K_7Z_{12}) \\ K^*_{8t}(K_3Z_{21}+K_7Z_{22}) \\ K^*_{12t}K_{11}Z_{33} \end{array} & \begin{array}{l} K^*_{4t}(K_4Z_{11}+K_8Z_{12}) \\ K^*_{8t}(K_4Z_{21}+K_8Z_{22}) \\ K^*_{12t}K_{12}Z_{33} \end{array} \\
\hline
\end{array}
\tag{23.67}
$$

This matrix represents the values of the self- and mutual impedances of the four coils of Fig. 23.9 *in terms of the three coils of Fig.* 23.8.

The various k's are so selected arbitrarily that the four coils should be physically realizable. Of course some of the k's in $\mathbf{C}'_\sigma$ may be assumed to be zero, while the diagonal k's may be assumed unity.

It should be expressly noted that the second network (having the same input impedance as the original network) has been established without calculating the impedance matrix or the criterion of performance or the input impedance of either network. *By simply knowing the individual synthesis tensor* $\mathbf{C}_\sigma$ *of equation* 23.66, *it can be foretold without any additional calculations that the two networks of Figs.* 23.8 *and* 23.9 *have the same input impedance when viewed from* e_1.

XIX. THE PHYSICAL REALIZABILITY OF COILS

As tentative steps in the determination of the arbitrary k's, the following may be mentioned:

1. The first condition of the realizability of coils is that the various mutual inductances in equation 23.67, namely, the non-diagonal components, should be of the form $+jX$. This condition puts several limitations upon the arbitrary values of the k's in terms of the Z's.

2. It may be required that some or all of the mutual inductances should be zero. This requirement restricts the arbitrariness of the various k's still further.

3. Another condition is that the mutual inductances must be smaller than the self-inductances. This condition puts upper and lower limits upon the values of the k's.

XX. THE THREE ARBITRARY SETS OF QUANTITIES OF NETWORK SYNTHESIS

(*a*) When the criterion of performance has been established for some desired network characteristics, but no network is as yet available whose constants satisfy the criterion, *it is just as well to assume the primitive network with any arbitrary number of coils as the starting point* and establish (on paper) the constants of the primitive network without paying any attention to its realizability. Most of its constants may be assumed arbitrarily; the rest must be so selected that they satisfy the criterion. *If the network performance is to be independent of frequency, the primitive network serving as a starting point may contain only resistances.*

For instance, to maintain constant load currents the criterion that has to be satisfied by the coils is from equation 23.3

$$\mathbf{Y}^{22} = \mathbf{Y}^{23} . \mathbf{Y}^{33\,-1} . \mathbf{Y}^{32} \qquad 23.69$$

Hence if the admittance tensor $\mathbf{Y}$ of the primitive network is divided into $4^2 = 16$ component tensors (one of the four groups being the ignored meshes), *fifteen of the tensors may be selected arbitrarily while the diagonal tensor* $\mathbf{Y}^{22}$ *has to satisfy this last equation.*

(*b*) On these arbitrary tensors two other arbitrary operations may be performed.

1. The coils may be interconnected into any arbitrarily selected network by a non-singular $\mathbf{C}$.

2. The tensors may be multiplied by the synthesis tensor $\mathbf{C}'_{\sigma}$ whose component tensors are mostly arbitrary.

(*c*) Hence, *to find a network with a prescribed performance characteristic, three arbitrary sets of quantities have to be given definite values*;

1. The arbitrary components of the *impedance tensor* of the originally assumed primitive network, satisfying the "criterion of performance."

2. The components of the *transformation tensor* of any arbitrary selected network showing its manner of interconnection, etc.

3. The arbitrary components of the *synthesis tensor*.

The value of these arbitrary quantities is fixed by the conditions of realizability of the coils and by economic factors.

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